Math 309 Assignment 2

Due on February 2

Problem 1. Find the *Maximum Likelihood Estimator* $(\bar{\mu}, \bar{\sigma})$ of the function

$$f(\mu, \sigma) = \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_j-\mu)^2}{2\sigma^2}},$$

where x_j's are the observed or sampled values, assumed to be normally distributed. Show that the estimator $(\bar{\mu}, \bar{\sigma})$ you found is a local maximizer, satisfying the second order sufficient condition.

Problem 2. Find the global minimizer of

$$f(x) = \max(|x-2|, x^2), x \in \mathbb{R}.$$

Problem 3. For the 2×2 symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Show that if the trace $\mathrm{tr}(M) = a + c > 0$ and the determinant $\det(M) = ac - b^2 > 0$, then M is positive definite.

Problem 4. Let the matrices $A \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times k}, C \in \mathbb{R}^{k\times k}, V \in \mathbb{R}^{k\times n}$. If A, C and $C^{-1} + VA^{-1}C$ are invertible, show that $A + UCV$ has a inverse given by

$$A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

This is a generalization of Sherman-Morrison formula in the class.

Problem 5 (Compare the convergence rate of Steepest Decent and Conjugate Gradient). The solution of the linear equation $Ax = b$ ($A \in \mathbb{R}^{n\times n}$ is positive definite and $b \in \mathbb{R}^n$ is a column vector) can be regarded as the minimizer of the quadratic equation

$$f(x) = \frac{1}{2}x^t Ax - b^t x.$$

For a given size $n = 40$, generate the symmetric random marix A and vector b and compute the exact solution x_ext by direct matrix inversion:

```
clear all;
n = 40;                         % system size
rand('seed',sum(100*clock));  % update the rand seed
A = rand(n);                    % generate the random matrix
A = A'*A+5*eye(n);              % generate the symmetric matrix
b = rand(n,1);                  % generate the right hand side
x_ext = A\b;                    % The exact solution (minimizer)
```

Then find the decay of the A-norm of the Steepest Descent Method for 10 iterations

```
x = zeros(n,1);                              % Starting point
err_sd = sqrt((x-x_ext)'*A*(x-x_ext));     % Initial error in A-norm
for k=1:10
     p = b-A*x;
     alpha = p'*p/(p'*A*p);
     x = x + alpha*p;
     err_sd = [err_sd sqrt((x-x_ext)'*A*(x-x_ext))];
end
```

Complete the code using Conjugate Gradient Method for 10 iterations to record `err_cg` and then compare their performance by plotting the error (uncomment the last line).

```
x = zeros(n,1);
r = A*x-b;
p = -r;
err_cg = sqrt((x-x_ext)'*A*(x-x_ext));
for k=1:10
     % Complete this block
end
% semilogy(0:10,err_sd,'-*',0:10,err_cg,'-+');
```

You can download the incomplete code `sdcgcomp.m` from webct and finish it. You should oberve that the conjugate gradient method should converges much faster than the steepest descent method.

Please hand in: the code and the plot of the convergence of these two methods.