Math 309 Assignment 1 Solution

Problem 1. (i) Since f is convex,

$$f(x+\lambda(y-x)) = f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y), \lambda \in (0,1) \tag{1}$$

By definition,

$$\begin{aligned}(\nabla f(x), y-x) &= \frac{d}{d\lambda} f(x+\lambda(y-x)) \\ &= \lim_{\lambda \searrow 0} \frac{f(x+\lambda(y-x))-f(x)}{\lambda} \\ &\leq \lim_{\lambda \searrow 0} \frac{(1-\lambda)f(x)+\lambda f(y)-f(x)}{\lambda} \\ &= f(y)-f(x)\end{aligned} \tag{2}$$

Therefore $f(y) \geq f(x) + (\nabla f(x), y-x)$.

(ii) Switching x and y we have $f(x) \geq f(y) + (\nabla f(y), x-y)$. Adding them together,

$$f(y)+f(x) \geq f(y)+f(x)+(\nabla f(x), y-x)+(\nabla f(y), x-y),$$

which is exactly $(\nabla f(y) - \nabla f(x), y-x)$.

Problem 2. (i)

$$\nabla f(x) = \frac{1}{2}Ax + \frac{1}{2}A^t x - b, \qquad \nabla^2 f(x) = \frac{1}{2}(A+A^t).$$

(ii)

$$\begin{aligned}f(x)-f(x^*) &= \frac{1}{2}x^t Ax - \frac{1}{2}x^{*t}Ax^* - b^t x + b^t x^* \\ &= \frac{1}{2}(x-x^*)^t A(x-x^*) + \frac{1}{2}x^t Ax^* + \frac{1}{2}x^{*t}Ax - x^{*t}Ax^* - b^t x + b^t x^* \\ &= \frac{1}{2}(x-x^*)^t A(x-x^*) + \frac{1}{2}x^t b + \frac{1}{2}b^t x - b^t x^* - b^t x + b^t x^* \\ &= \frac{1}{2}(x-x^*)^t A(x-x^*) \geq 0\end{aligned} \tag{3}$$

since A is positive definite.

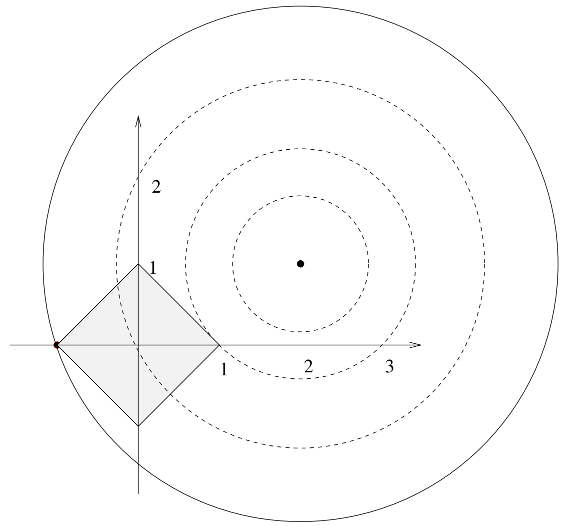

Figure 1: The plot for the feasible region and the contours (or level sets) of the objective function.

Problem 3. (i)

(ii) From the plot, the maximizer is obtained as $(x_1^*, x_2^*) = (-1, 0)$ and the maximal value is

$$(2 - (-1))^2 + (1 - 0)^2 = 10.$$

Problem 4. (i) >> `fminunc(@mypeaks,[0 -2])`

Instead of of the initial point $x_1 = [0, -2]$, you may try others like $[-1, -2], [1, -2], \cdots$. The returned (global) minizer is close to $x^*[0.2283, -1.6255]$.

(ii) >> `fminunc(@mypeaks,[-2 0])`

The initial point x_1 can be $[-2, 1], [-2, -1], \cdots$. The returned (local) minizer is close to $x^* = [-1.347, 0.2045]$.

(iii) >> `fminunc(@mypeaks,[0 2])`

In general, the build-in algorithm stops at some point that can not be predicted.

Remark: This function is **not convex**, and the returned result depends critically on the starting point. For convex problem, the minimizer is unique (for strictly convex object function).