Math 309 Assignment 1

Due on January 17

Problem 1. (i) Let f be a smooth convex function defined on a convex domain $\Omega \subset \mathbb{R}^n$. For any $x, y \in \Omega$, using the definition of the convexity and directional derivative, show that

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \tag{1}$$

(ii) Using (**??**), show that

$$(\nabla f(y) - \nabla f(x)) \cdot (y - x) \geq 0.$$

Hint: Notice the symmetric role of x and y.

Problem 2. (i) Calculate the gradient and the Hessian matrix for the function

$$f(x) = \frac{1}{2} x^t A x - b^t x = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j - \sum_{i=1}^{n} b_i x_i, \qquad x \in \mathbb{R}^n$$

where A is a matrix of size $n \times n$ and b is a vector in $\mathbb{R}^n$. Write your answer in matrix (and vector) form.

(ii) If A is symmetric and positive definite, and x^* satisfies $Ax^* = b$. Show that $f(x) \geq f(x^*)$. (Hint: write $f(x) - f(x^*)$ in terms of some function of $x - x^*$).

Problem 3. For the problem

$$\begin{array}{ll} \max & (x_1 - 2)^2 + (x_2 - 1)^2 \\ \text{subject to} & \\ & \|x\|_1 \leq 1. \end{array}$$

Here $\|x\|_1 = |x_1| + |x_2|$.

(i) Plot the feasible region (the set of points (x_1, x_2) that satisfies the constrain $\|x\|_1 \leq 1$) and plot a few contour lines (or level sets) of the objective function.

(ii) Find the maximizer (x_1^*, x_2^*) from the plot and the maximal value at that point.

Problem 4. (Try the MATLAB build-in unconstrained functions `fminunc` or `fminsearch`). The peak function is defined as

$$f(x, y) = 3(1 - x)^2 e^{-x^2 - (y+1)^2} - 10 \left(\frac{x}{5} - x^3 - y^5 \right) e^{-x^2 - y^2} - \frac{1}{3} e^{-(x+1)^2 - y^2}.$$