Anomalous exponents of self-similar blowup solution to an aggregation equation in odd dimensions

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Abstract

We calculate the anomalous exponents of the self-similar solution of the second kind to an aggregation equation in odd dimensions. This solution describes the radially symmetric finite-time blowup phenomena and has been observed in numerical simulations of the dynamic problem. The nonlocal equation for the self-similar profile is transformed into a system of ODEs, whose boundary conditions are matched by a shooting method. The anomalous exponents are then retrieved from this transformed system.

1. Introduction

We consider radially symmetric, self-similar blowup solutions to the equation

$$u_t = \nabla \cdot (u \nabla K * u) \quad \text{on } \mathbb{R}^n \times [0, T), \tag{1}$$

where $K * u$ is the convolution of the kernel $K(x) = |x|$ with a mass density u. Equations of this type are used recently as a continuum model in the aggregation of biological species [1, 2, 3]. It is shown in [4] that the solution blows up in finite time for even more general kernels. Furthermore, the radially symmetric self-similar blow up solutions are of the second kind [5], in the sense that it is not possible to find exact exponents α and β in the ansatz for the solution

$$u(x,t) = (T-t)^{-\alpha} U(r), \quad r = \frac{|x|}{(T-t)^{\beta}}, \tag{2}$$

which blows up at time T, from a consideration of conservation, symmetry or scaling argument only [6]. Therefore, similarity solutions of this kind are much more difficult to study.

In this letter, we consider the special case in odd dimensions $n = 2N+1$ with $N = 1, 2, \cdots$. Using the fact that the successive Laplacians of kernel $K(x) = |x|$ is proportional to the fundamental solution of the Laplace equation, we can transform the equation (3) for the blowup profile U into an equivalent system of ordinary differential equations. Once appropriate boundary conditions of the system are matched by a shooting method, the anomalous exponents α and β are then retrieved.

2. Equivalent system and shooting methods

Substituting the ansatz (2) into the original equation (1) we get

$$\nabla \cdot (U \nabla K * U) = \alpha U + \beta r \frac{dU}{dr} \tag{3}$$

with $\alpha = (n-1)\beta + 1 = 2N\beta + 1$.

2.1. Equivalent system of ODEs

We introduce the variables $U_0 (= U), U_1, \cdots, U_{N+1}$ such that

$$-\Delta U_1 = U_0, \ \cdots, \ -\Delta U_N = U_{N-1}, \ \Delta U_{N+1} = k_N U_N, \tag{4}$$

where $\Delta = \frac{1}{r^{2N}} \frac{d}{dr} r^{2N} \frac{d}{dr}$ is the Laplacian in radial coordinate r and

$$k_N = 2^N (2N+1) N! \frac{\pi^{\frac{2N+1}{2}}}{\Gamma(N+1+\frac{1}{2})}.$$

Let

$$y_{2i} = U_i(r), \ i = 0, 1, \cdots, N, \qquad y_{2i-1} = \frac{dU_i}{dr}, \ i = 1, 2, \cdots, N+1. \tag{5}$$

The equations (3) and (4) are equivalent to the system of ODEs

$$\begin{cases} \frac{dy_0}{dr} &= -\frac{2N\beta+1-k_N y_{2N}}{\beta r - y_{2N+1}}, \\ \frac{dy_{2i-1}}{dr} &= -y_{2i-2} - \frac{2N}{r} y_{2i-1}, \qquad i = 1, 2, \cdots, N \\ \frac{dy_{2i}}{dr} &= y_{2i-1}, \qquad i = 1, 2, \cdots, N \\ \frac{dy_{2N+1}}{dr} &= k_N y_{2N} - \frac{2N}{r} y_{2N+1}. \end{cases} \tag{6}$$

The decay rate of U_0 from the numerical simulations in [5] suggest that $\Delta K * U_0 = U_N \to 0$ at infinite, implying $y_0(\infty) = y_1(\infty) = \cdots = y_{2N}(\infty) = 0$ if appropriate initial conditions $\vec{y}(0) = (y_0(0), y_1(0), \cdots, y_{2N+1}(0))$ are chosen. Since $\lambda^{2N} U(\lambda r)$ is also a solution of (3) if U is, the normalization $y_0(0) = U(0) = 1$ is fixed throughout this paper. Moreover, the initial conditions

$$y_{2i+1}(0) = 0, \ i = 1, 2, \cdots, N+1, \qquad \text{and} \qquad y_{2N} = \frac{2N\beta+1}{k_N}, \tag{7}$$

is used for the well-posedness of the system (6) near the origin $r = 0$. This leaves N shooting parameters, the exponent β and the rest of the initial conditions $(y_2(0), \cdots, y_{2N-2}(0))$.

A key observation to simply the calculation the calculation of the exponent β is that we can get a β-independent system from (6) by two additional transformations. The first is the scaling transformation

$$\tilde{y}_i(r) = (\beta - 1)^{\frac{i}{2N}} y_i((\beta-1)^{\frac{1}{2N}} r), \tag{8}$$

motivated by the $\beta - 1$ factor in the coefficients of the power series expansion of (6) near the origin. This is followed by the second transformation

$$z_i(r) = \tilde{y}_i(r), \ i = 0, \cdots, 2N-1, \tag{9}$$

$$z_{2N}(r) = \tilde{y}_{2N}(r) - \frac{2N+1}{k_N(\beta-1)}, \qquad z_{2N+1}(r) = \tilde{y}_{2N+1}(r) - \frac{1}{\beta-1} r, \tag{10}$$

to get get rid of the dependence on β in the equation for $\tilde{y}_0$, i.e

$$\frac{d\tilde{y}_0}{dr} = -\frac{2N\beta + 1 - k_N(\beta-1)\tilde{y}_{2N}}{\beta r - (\beta-1)\tilde{y}_{2N+1}}. \tag{11}$$

This gives the final β-independent system for z_i

$$\begin{cases} \frac{dz_0}{dr} &= -\frac{2N-k_N z_{2N}}{r-z_{2N+1}}, \\ \frac{dz_{2i-1}}{dr} &= -z_{2i-2} - \frac{2N}{r} z_{2i-1}, \qquad i = 1, 2, \cdots, N \\ \frac{dz_{2i}}{dr} &= z_{2i-1}, \qquad i = 1, 2, \cdots, N \\ \frac{dz_{2N+1}}{dr} &= k_N z_{2N} - \frac{2N}{r} z_{2N+1}. \end{cases} \tag{12}$$

In this way, the shooting parameters for (12) are now reduced by one and are $(z_2(0), \cdots, z_{2N-2}(0))$, such that the desired boundary condition at infinity becomes

$$z_i(\infty) = 0, \;\; i = 0, 1, \cdots, 2N-1 \tag{13}$$

and $z_{2N}(\infty)$ is a constant. This constant $z_{2N}(\infty)$ turns out to recover the anomalous exponent β according to the first equation in (10),

$$z_{2N}(\infty) = \tilde{y}_{2N}(\infty) - \frac{2N+1}{k_N(\beta-1)} = -\frac{2N+1}{k_N(\beta-1)}. \tag{14}$$

In actual computation, the system (12) is solved on the finite interval $[0, L]$ for L large enough and β is retrieved from the more stable estimation (22) below.

2.2. Shooting method

Since the system (12) has a regular singular point at the origin, it is solved with any ODE solver starting at small $r = r_0 (> 0)$ instead of at the origin. The initial condition $z_i(r)$ can be obtained as a convergent power series as follow. For r near the origin, we assume the series expansion for z_0 as

$$z_0(r) = \sum_{k=0}^{\infty} u_{2k} r^{2k}. \tag{15}$$

Only even order terms survive due to the radial symmetry and all odds terms vanish identically. Integrating the system (12) with the initial condition (or the shooting parameters) $\mathbf{s} = (s_1, \cdots, s_{N-1}) = (z_2(0), \cdots, z_{2N-2}(0))$, we have

$$\begin{aligned} z_{2i-1}(r) &= (2N-1)!! \sum_{j=1}^{i-1} (-1)^j \frac{1}{2^{j-1}(j-1)!(2N+2j-1)!!} s_{i-j} r^{2j-1} \\ &\quad + \frac{(-1)^i}{2^{i-1}} \sum_{k=0}^{\infty} \frac{k!(2k+2N-1)!!}{(k+i-1)!(2k+2N+2i-1)!!} u_{2k} r^{2k+2i-1}, i = 1, 2, \cdots, N, \end{aligned} \tag{16}$$

$$\begin{aligned} z_{2i}(r) &= (2N-1)!! \sum_{j=0}^{i-1} (-1)^j \frac{1}{2^j j!(2N+2j-1)!!} s_{i-j} r^{2j} \\ &\quad + \frac{(-1)^i}{2^i} \sum_{k=0}^{\infty} \frac{k!(2k+2N-1)!!}{(k+i)!(2k+2N+2i-1)!!} u_{2k} r^{2k+2i}, i = 1, 2, \cdots, N, \end{aligned} \tag{17}$$

$$\begin{aligned} z_{2N+1}(r) &= k_N (2N-1)!! \sum_{j=0}^{N-1} (-1)^j \frac{1}{2^j j!(2N+2j+1)!!} s_{N-j} r^{2j+1} \\ &\quad + \frac{(-1)^N k_N}{2^N} \sum_{k=0}^{\infty} \frac{k!(2k+2N-1)!! u_{2k}}{(k+N)!(2k+4N+1)!!} r^{2k+2N+1}. \end{aligned} \tag{18}$$

Substituting the expression z_0, z_{2N}, z_{2N+1} into the first equation in (12), we have

$$
\begin{aligned}
&k_N\left[(2N-1)!!\sum_{j=0}^{N-1}(-1)^j\frac{1}{2^j j!(2N+2j-1)!!}s_{N-j}r^{2j}\right.\\
&\left.+\frac{(-1)^N}{2^N}\sum_{k=0}^{\infty}\frac{k!(2k+2N-1)!!u_{2k}}{(k+N)!(2k+4N-1)!!}r^{2k+2N}\right]\sum_{k=0}^{\infty}u_{2k}r^{2k}\\
&+k_N\left[(2N-1)!!\sum_{j=0}^{N-1}(-1)^j\frac{1}{2^j j!(2N+2j+1)!!}s_{N-j}r^{2j}\right.\\
&\left.+\frac{(-1)^N}{2^N}\sum_{k=0}^{\infty}\frac{k!(2k+2N-1)!!u_{2k}}{(k+N)!(2k+4N+1)!!}r^{2k+2N}\right]\sum_{k=0}^{\infty}2ku_{2k}r^{2k}\\
&=\sum_{k=0}^{\infty}(2N+2k)u_{2k}r^{2k}.
\end{aligned}
\tag{19}
$$

The matching condition for the coefficients of r^{2k} gives the following recursive relations for u_{2k},

$$
u_{2j}=\frac{k_N(2N+1)!!(2N+2j+1)}{2j}\sum_{l=1}^{j}\frac{(-1)^l}{2^l l!(2N+2l+1)!!}u_{2j-2l}s_{N-l},\quad j=1,\cdots,N-1,
$$

$$
\begin{aligned}
u_{2N+2k}=&\frac{(-1)^N k_N(4N+2k+1)(2N+1)}{2^{N+1}(N+k)}\sum_{l=0}^{k}\frac{l!(2N+2l-1)!!}{(N+l)!(4N+2l+1)!!}u_{2l}u_{2k-2l}\\
&+\frac{k_N(2N+1)!!(4N+2k+1)}{2(N+k)}\sum_{j=1}^{N-1}\frac{(-1)^j s_{N-j}}{2^j j!(2N+2j+1)!!}u_{2N+2k-2j},\qquad k=0,1,\cdots.
\end{aligned}
$$

Numerical simulations indicates that the coefficients u_{2k} converge geometrically, and the corresponding power series has a finite radius of convergence (approximately 0.87 in dimension three for z_0). For small r_0, with a fixed shooting parameter $\mathbf{s}$, the initial condition $z_i(r_0)$ can be obtained accurately with just a few leading terms in the expansion.

3. Numerical Results

For each shooting parameter $\mathbf{s}=(s_1,\cdots,s_{N-1})$ with the corresponding solution $\mathbf{z}(L;\mathbf{s})$ computed at $r=L$, we can find a direction $\delta\mathbf{s}=(\delta s_1,\delta s_2,\cdots,\delta s_{N-1})$ to make the far field condition $\tilde{\mathbf{z}}(L;\mathbf{s}+\delta\mathbf{s})=(z_2(L),z_4(L),\cdots,z_{2N-2}(L))$ closer to zero. Assuming $\tilde{\mathbf{z}}(L)$ depends only on $\mathbf{s}$ (the second summation) in (17), its variation $\delta\tilde{\mathbf{z}}(L)$ can be written as

$$
\delta z_{2i}(L)=(2N-1)!!\sum_{j=0}^{i-1}\frac{(-1)^j}{2^j j!(2N+2j-1)!!}L^{2j}\delta s_{i-j},\quad i=1,2,\cdots,N-1.
\tag{20}
$$

This suggests the following Newton-like scheme

$$
\mathbf{s}^{m+1}=\mathbf{s}^m+\omega\delta\mathbf{s}^m,
\tag{21}
$$

where $\delta \mathbf{s}^m$ is solved from (20) with $\delta z_{2i}(L) = -z_{2i}(L)$ and $\omega(< 1)$ is a positive relaxation parameter. The triangle system (20) can be solved easily. For instance, in dimension five

$$\delta s_1 = -z_2(L)$$

and in dimension seven

$$\delta s_1 = -z_2(L), \quad \delta s_2 = -\left(z_4(L) + \frac{L^2}{14} z_2(L)\right).$$

Because of the sensitive dependence of $u_{2N}(L)$ on the shooting parameters $\mathbf{s}$ (see Figure 1 with just one shooting parameter), the anomalous exponent β is recovered not from (14) but the following equivalent and much more stable relation

$$\frac{2N\beta + 1}{\beta - 1} = z_{2N}(0) - z_{2N}(\infty) = \frac{1}{(N-1)!} \int_0^\infty r^{2N} z_0(r) dr, \tag{22}$$

from successive integrations of the system (12).

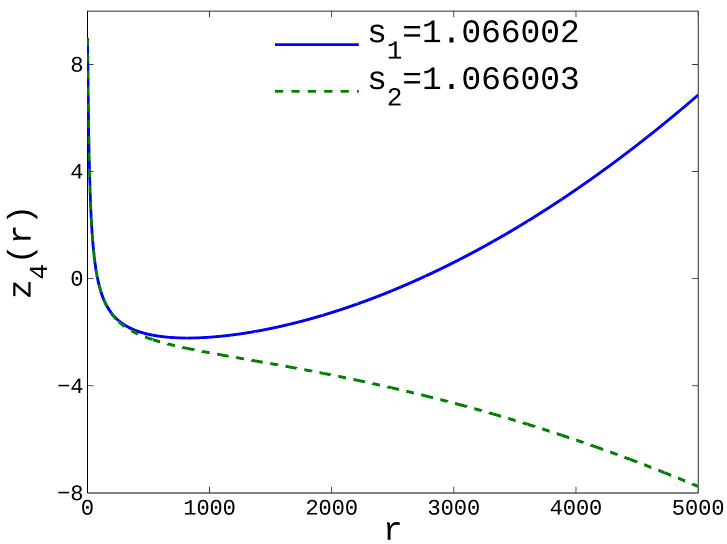

Figure 1: In dimension five, the solution $z_4(r)$ depends very sensitively on the only shooting parameter s_1 when r is large. The dependence is even more sensitive in higher dimensions.

3.1. Dimension Three ($N = 1$)

Dimension three is special in the sense that there is no shooting parameter. The anomalous exponent β is retrieved from either (14) or (22), where the accuracy depends on the length of the interval $[0, L]$ on which (12) is solved. This is also compared with those by both direct simulation of the blowup dynamic for (1) followed by data fitting and numerical renormalization group (RG) calculation performed in [5], in Table 1. The computation time is at most a few seconds for the ODE system while at least a few hours for direct simulation or numerical RG.

	L	$\beta(n=3)$	$\beta(n=5)$	$\beta(n=7)$	$\beta(n=9)$
shooting method	10^2	1.580957	1.593860	1.574476	1.602537
shooting method	10^3	1.582976	1.598702	1.596328	1.607854
shooting method	10^4	1.583092	1.602900	N/A	N/A
numerical RG [5]	400	1.582889	1.599152	1.604324	1.629743
Direct Computation [5]	N/A	1.582226	1.598044	1.606732	1.623508

Table 1: Comparison of the computed anomalous exponents β from different methods in different dimensions.

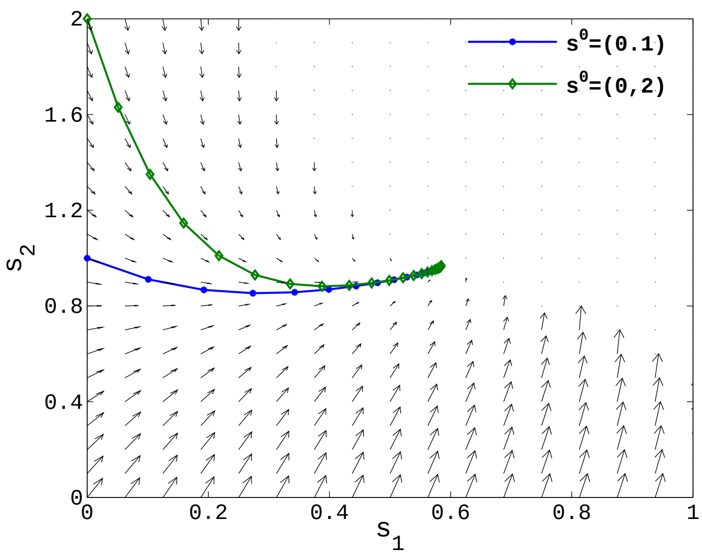

Figure 2: The gradient field and sample trajectories in dimension seven with $L = 100$. The gradient field $(\delta s_1, \delta s_2)$ is normalized by the factor of $(1 + \delta s_1^2 + \delta s_2^2)^{-1/2}$ for better visualization.

3.2. Higher Dimensions $(N \geq 2)$

The solution to the system (12) may not be defined for $r(> 0)$ with certain shooting parameters when the denominator $r - z_{2N+1}$ in the first equation in (12) changes sign. In this case the assumption of the weak dependence of $z_{2i}(L)$ with $i \geq 1$ on z_1 is not valid. Therefore the variation (20) is true only on part of the parameter space. This is is shown in Figure 2 for dimension seven. The solution ceases to exist for shooting parameter $\mathbf{s}$ on the upper right region, where the gradient field is not defined. However, once the initial guess $\mathbf{s}^0$ is in the basin of attraction, it alway converges to a neighborhood of the unique fixed point. Numerical experiments indicate that the initial guess can be chosen as alternating zeros and positive numbers, such as $\mathbf{s}^0 = (0, C, 0, C, \cdots)$, for C positive and large. The choice of $C = 2$ works for any test cases up to dimension fifteen. The numerical results are presented in Table 1, compared with those obtained from much slower computation of the full partial different equation. Because of the sensitive dependence of the solution $\mathbf{z}(L)$ on the shooting parameter $\mathbf{s}$, the exponent β calculated using this shooting method is less accurate in higher dimensions.

4. Conclusion

We find the exponent (and the profile) for the self-similar solution of the aggregation equation in odd dimensions. Evidence is clear that we have an exact second-kind similar solution. However, the shooting method proposed here relies on reducing the problem to coupled local equations in odd dimensions only. An interesting open problem is to develop a full theory for the nonlocal problem in general dimensions.

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