Imperial College London
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Second year project **Group 24**

Ising Model for Gang and Graffiti

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Abstract

This project analyses behaviours of gangs in relation to graffiti. The 2D square-lattice Ising model is utilised, and suitable modifications are made to simulate a gang situation. Patterns of how variations in parameter values lead to phase transitions, which symbolises big change in gang presence in real life, are observed by setting up a set of codes to simulate the situation. The results can aid efforts in gang control and raise public understanding in gang behaviours.

Contents

Chapter 1

Introduction

The idea that most things, whether humans or atoms, have certain tendencies to point towards certain directions intrigues the development of physical and social sciences. Sociology have been looking into the development of gangs [14], and in a seemingly unrelated manner, the study of orientations of atoms in ferromagnetism during the early 20th century led to the development of lattice models in statistical mechanics [6].

Gradually, scientists realised that lattice models can be applied to analyse different phenomena in many areas of life. Medical sciences benefited from research in DNA and gene modelling as lattices [13]; physical chemistry developed exotic polymers and alloys by studying behaviour of atoms as a structure [8]. And we are thinking, it should be possible to analyse patterns of gangs and graffiti using a lattice model.

Gangs and graffiti are part of the world. Though the gang culture in the UK and in London is not as prevalent as in the US and the British population has a high degree of safety concerning gang crimes, British gangs are still real and observable. For example, the 2011 riots in the UK was triggered by the death of a gang member in Tottenham, and gangs played a part in magnifying the riots [7]. Not only does it highlight the presence of gangs especially in London, but also exposes the massive danger when individual gang crimes lead to extensive public unrest.

Figure 1.1: The 2011 England riots

To look at behaviours of gangs, we look at the graffiti they put on, which is used to mark their territories. Gang graffiti may be sophisticated like artwork or simple like scribbling of

words. They are visible in many places in a city.

Figure 1.2: A simple gang graffiti in Aberdeen

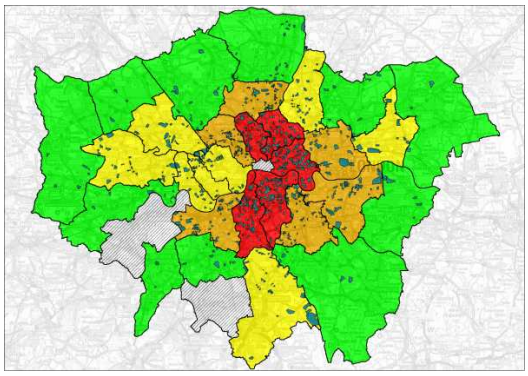

Figure 1.3: Known turfs of gangs in London (the blue bits)

Despite the seriousness of the gang problem, there was little research that deals with gang distribution and behaviours in a mathematical sense. In this project we analyse a model that can simulate the gang situation in London or in the UK, where stability is largely observed and direct gang wars rarely happen, but gang interactions still happen indirectly and affect many aspects of the society.

We will base our project on the Ising model, which is one the most fundamental lattice models in statistical mechanics. The Ising model suits the situation above, as it considers local but not long-range interactions. Through setting up appropriate Hamiltonian expressions, we hope to find results to analyse the situation.

In this project, we will introduce the Ising model in Chapter 2 and discuss the modifications to the standard model in Chapter 3. Then, we will set up a set of codes with the aid of algorithms in Chapter 4 and analyse the simulation results under different parameters in Chapter 5.

Chapter 2

The Ising Model

2.1 Background and significance

The Ising model was introduced by German physicist Wilhelm Lenz in 1920 in his research in ferromagnetism. However, its foundations were actually built by his student Ernst Ising after Lenz asked him to look into the behaviours, especially phase transitions, of magnetic dipole moments of atomic spins in a 1-dimension structure, where each atom has a dipole of either -1 or +1 and only interacts with its most adjacent neighbours [6].

Realising there was no phase transition in a 1-dimension model, Ising incorrectly concluded that no phase transition would exist in models of any dimensions [3]. Thankfully, research in later years by the likes of Rudolf Peierls [12], Lars Onsager [10] and Chen-Ning Yang [15] corrected Ising's initial mistake. Nowadays, the Ising model includes lattice structures of various shapes, and its study has already exceeded the third dimension with the help of mean field theory [1].

The Ising model puts an emphasis on phase transitions under different parameters. The original purpose of the model lies on magnetisation, which is an example of phase transition. Magnetisation is basically a display of alignment where each individual atom carries an orientation. This is a nice analogy to various phenomena in this world, and that of course includes behaviours of gangs and patterns of graffiti as shown later in this research project.

One point that makes the Ising model fascinating is that interactions within the shortest range, namely interactions between neighbouring molecules, can hugely influence the big picture including the layout of the general structure, as individual molecules relate to each other through interactions with each of their own neighbours. The changes in the big picture relate to predictions of the existence and properties of phase transitions, the most basic objective of the model.

2.2 The standard 2D Ising model

Consider a 2-dimensional square-lattice Λ. For each lattice site $j \in \Lambda$, it has agent spin $\sigma_j \in \{-1, +1\}$ and external magnetic field h_j. For any two adjacent sites $i, j \in \Lambda$, there is

an interaction J_{ij}. The Hamiltoninan is

$$-\mathcal{H}(\sigma) = \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j + \mu \sum_j h_j\sigma_j \tag{2.1}$$

where $\sigma = (\sigma_j)_{j\in\Lambda}$ is the spin configuration, and μ is the magnetic moment. $\langle i, j\rangle$ denotes neighbouring lattice sites i, j (see [1] for more information).

Note that each lattice site only interacts with its four closest neighbours and the external magnetic field.

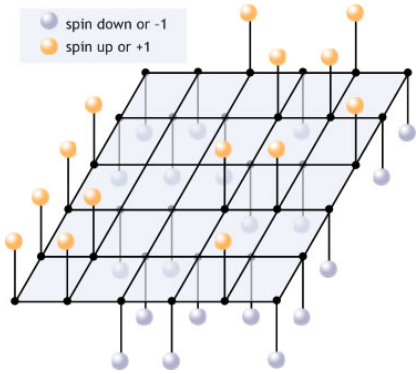

Figure 2.1: A 2D square-lattice Ising model showing spins

Parameters J and μh corresponds to influences of neighbours and external magnetic field respectively. When J is positive, the site is likely to align its spin parallel to the spins of its neighbouring sites, and vice versa. Similarly, when μh is positive, the site is likely to align its spin parralel to the external magnetic field, and vice versa.

For a ferromagnetic lattice where all sites show parralel spins, the Hamiltonian is small, indicating a low 'energy level'. The lattice in Figure 2.2 is probably not very magnetic.

2.3 Application in gang and graffiti

In this project we consider a community with only two rival gangs, namely 'red and 'blue'. Both gangs run their own underworld business, and are affected by external social, cultural and economical factors, such as economic performance, unemployment, education, police intervention etc. However, they try to maintain 'external peace' within the region and avoid direct gang wars. Still, they compete indirectly by marking graffiti to claim their own turfs (different 'redness' or 'blueness' of graffiti at different sites) and influence the general population to join (or leave) their gangs.

Even though it is easy to imagine the agent spin at each site σ_j as the gang affiliation (red or blue), gangs do not behave exactly like atoms in magnetic lattices. We have to consider cases where sites are (temporarily) neutral in gang affiliation, and cases where gangs are trying to claim these sites. We also have to look into how sites align to each other in the sense of gang affiliation, and we need to maintain the consistency between gang affiliation

and 'colour' of graffiti markings at each site. The modifications to the standard 2D Ising model will be explained in the next chapter.

Note: The report by A. Barbaro on a similar topic is used as reference. Barbaro's report used one single lattice model and mean-field rendition to study the formation of gang clusters and phase transitions [2]. In this project, however, we will produce a 'reduced' model from a 'full' model so as to emulate the Ising model even more closely. We focus on how the parameters affect the behaviour of gangs, such as phase transition and gang dominance, and spot the differences and similarities in both models.

Chapter 3

Ising model for gang and graffiti

In order to study the indirect interaction between red and blue gangs via graffiti markings we need to adopt some important variations to the classical Ising model [4].

The spins at each site i, s_i, are composed of a discrete variable, the agent spin η_i, representing the gang member - which can only take values amongst $\{-1, 0, 1\}$, - and of a continuous variable, the graffiti field g_i - which can take any value on the real line.

$\eta_i = -1$ indicates the presence of a blue-gang member at site i, $\eta_i = +1$ the presence of a red-gang member at site i, and $\eta_i = 0$ the absence of agents at site i. On the other hand, $g_i > 0$ and $g_i < 0$ represent an excess of red and blue graffiti respectively [2].

3.1 The 'graffiti-interaction' GI-Hamiltonian

Consider the Hamiltonian,

$$-\mathcal{H}(\underline{s}) = J \sum_{\langle i,j \rangle} \eta_j g_i + K \sum_i \eta_i g_i + \alpha \sum_i \eta_i^2 - \lambda \sum_i g_i^2. \tag{3.1}$$

The meaning of the parameters can be explained as follows:

1. The parameter J indicates the tendency of nearest-neighbouring agent spins to align with the spin at site i. We require $J > 0$ in order for the model to be consistent with the gang-graffiti scenario we want to investigate;

2. The parameter K represents the external magnetic field of the classical Ising model: the bigger the K, the more aligned the spins, i.e. the more induced will be the members of each gang to be close to a member of the same gang. We require $K > 0$ to avoid a check-board configuration;

3. The parameter $\alpha \in \mathbb{R}$ stands for the tendency of a gang member to occupy unclaimed territories, regardless of the colour: if $\alpha \ll -1$, then we have a restraint in occupying an unclaimed turf by any of the two gangs; on the other hand if $\alpha \gg 1$, then gang members have a strong tendency to occupy unclaimed turf;

4. The parameter λ resembles the influence of community and/or police: as it increases, the presence of graffiti decreases. Due to consistency, $\lambda > 0$. [2]

The reason why we require J and K to be positive can be visually shown by generating territory configurations of the following two scenarios:

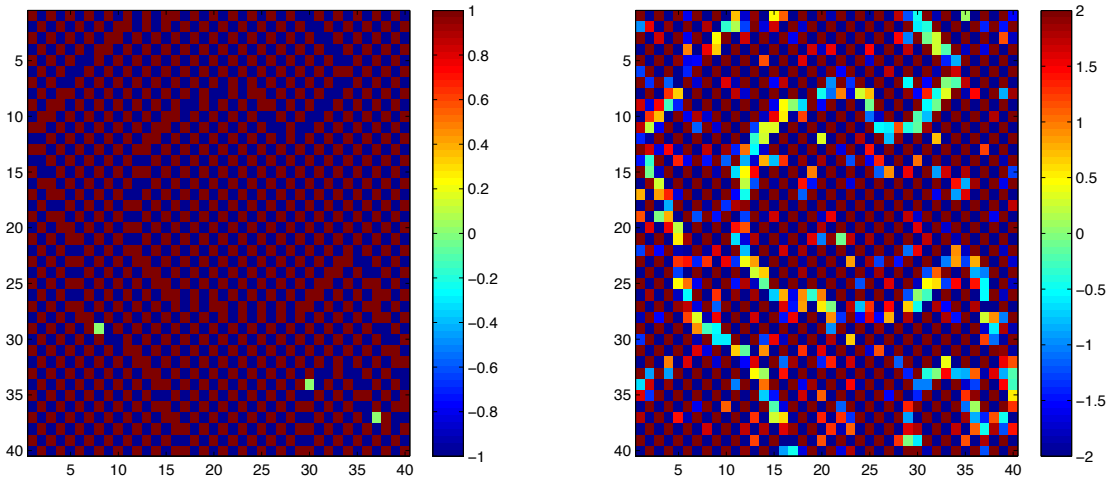

Figure 3.1: Configuration of the full model with $J < 0$.

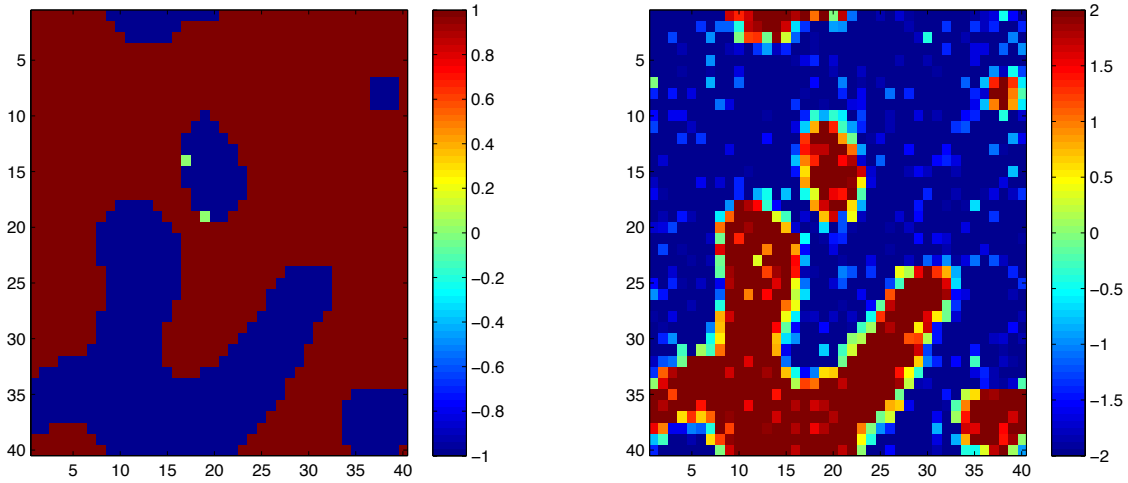

Figure 3.2: Configuration of the full model with $J, K < 0$.

Figure 3.1 displays an unrealistic scenario as far as the study of indirect, gang-to-gang interactions is concerned because the check-board configuration corresponds to a picture of the territory where a member of each gang has nearest-neighbours belonging to the opposite gang (apart from regions on the lattice where we have a blue or a red segment). As a result, we can not analyse how the two gangs interact with each other via graffiti markings, since such scenario can be interpreted in reality as a peaceful living between their members.

Figure 3.2 instead appears very similar to the allowed cases at first glance. However, if we look more carefully at the gang-members graph and graffiti graph, we clearly observe that

$J, K < 0$ output an unrealistic scenario as gang members occupy turfs which are marked by graffiti of the rival gang.

3.2 The reduced Hamiltonian

We are interested in reducing (3.1) into a simpler form which resembles the Hamiltonian of the standard Ising model.
To do so, we differentiate the above equation with respect to g_i in order to express the simplified Hamiltonian in terms of η_i only.
To simplify the calculation, we can rewrite (3.1) as

$$-\mathcal{H}(\underline{s}) = J\sum_i g_i \sum_{j\in N(i)} \eta_j + K\sum_i \eta_i g_i + \alpha\sum_i \eta_i^2 - \lambda\sum_i g_i^2, \tag{3.2}$$

where $\sum_{j\in N(i)} \eta_j$ represent the sum of the nearest-neighbours to the site i. Differentiating (3.2) in g_i yields

$$-\frac{\partial}{\partial g_i}\mathcal{H}(\underline{s}) = J\sum_{j\in N(i)} \eta_j + K\eta_i - 2\lambda g_i$$

which is equal to 0 for

$$g_i = \frac{J\sum_{j\in N(i)} \eta_j + K\eta_i}{2\lambda}. \tag{3.3}$$

Moreover, note that

$$\frac{\partial^2}{\partial g_i^2}\mathcal{H}(\underline{s}) = 2\lambda > 0.$$

Hence (3.3) minimizes (3.1) and if we plug it into (3.2) we obtain the reduced Hamiltonian

$$-\mathcal{H}_r(\underline{s}) = \alpha\sum_i \eta_i^2 + \frac{1}{4\lambda}\sum_i \Big[J\sum_{j\in N(i)} \eta_j + K\eta_i\Big]^2. \tag{3.4}$$

Next, we need to expand the square of the second term on the RHS of (3.4) and, in order to fully understand the simplification, we consider the 1D case first:

$$\sum_i \Big[J\sum_{j\in N(i)}\eta_j + K\eta_i\Big]^2 = \sum_i \Big[J(\eta_{i-1}+\eta_{i+1}) + K\eta_i\Big]^2$$
$$= \sum_i \Big[J^2(\eta_{i-1}^2+\eta_{i+1}^2) + K^2\eta_i^2 + 2J^2\eta_{i-1}\eta_{i+1} + 2JK\eta_i\eta_{i+1} + 2JK\eta_i\eta_{i-1}\Big]$$
$$= \sum_i (2J^2+K^2)\eta_i^2 + J^2\sum_{\langle\langle i,j\rangle\rangle}\eta_j\eta_i + 2JK\sum_i \eta_i\sum_{j\in N(i)}\eta_j, \tag{3.5}$$

grouping terms after having expanded out the sums.

Remark: the notation $\langle\langle i,j\rangle\rangle$ indicates the second-nearest neighbours, i.e. neighbours at twice the unit distance.

Therefore, (3.5) into (3.4) yields the following reduced Hamiltonian in 1D:

$$-\mathcal{H}_r(\underline{s}) = \left(\alpha + \frac{2J^2+K^2}{4\lambda}\right)\sum_i \eta_i^2 + \frac{JK}{2\lambda}\sum_{\langle i,j\rangle}\eta_i\eta_j + \frac{J^2}{4\lambda}\sum_{\langle\langle i,j\rangle\rangle}\eta_j\eta_i.$$

On the other hand, for the 2D-square lattice it is necessary to introduce double indices, namely $i=(i_v,i_h)$ and $j=(j_v.j_h)$, where the subscripts 'h' and 'v' indicate the horizontal and the vertical nearest-neighbours respectively.

Now we can rewrite the sum over nearest-neighbours in (3.4) as

$$\sum_{j\in N(i)}\eta_j = \eta_{(i_v,i_{h-1})} + \eta_{(i_v,i_{h+1})} + \eta_{(i_{v-1},i_h)} + \eta_{(i_{v+1},i_h)},$$

and its square as

$$\left(\sum_{j\in N(i)}\eta_j\right)^2 = \left(\eta^2_{(i_v,i_{h-1})} + \eta^2_{(i_v,i_{h+1})} + \eta^2_{(i_{v-1},i_h)} + \eta^2_{(i_{v+1},i_h)}\right)$$
$$+ 2\eta_{(i_v,i_{h-1})}\eta_{(i_v,i_{h+1})} + 2\eta_{(i_{v-1},i_h)}\eta_{(i_{v+1},i_h)}$$
$$+ 2\eta_{(i_v,i_{h-1})}\eta_{(i_{v-1},i_h)} + 2\eta_{(i_v,i_{h-1})}\eta_{(i_{v+1},i_h)} + 2\eta_{(i_v,i_{h+1})}\eta_{(i_{v-1},i_h)} + 2\eta_{(i_v,i_{h+1})}\eta_{(i_{v+1},i_h)}.$$

Then,

$$\sum_i\left(\sum_{j\in N(i)}\eta_j\right)^2 = \sum_i\left(\eta_{(i_v,i_{h-1})} + \eta_{(i_v,i_{h+1})} + \eta_{(i_{v-1},i_h)} + \eta_{(i_{v+1},i_h)}\right)^2$$
$$= \sum_i\left(\eta^2_{(i_v,i_{h-1})} + \eta^2_{(i_v,i_{h+1})} + \eta^2_{(i_{v-1},i_h)} + \eta^2_{(i_{v+1},i_h)}\right) \tag{3.6}$$
$$+ 2\sum_i\left(\eta_{(i_v,i_{h-1})}\eta_{(i_v,i_{h+1})} + \eta_{(i_{v-1},i_h)}\eta_{(i_{v+1},i_h)}\right) \tag{3.7}$$
$$+ 2\sum_i\left(\eta_{(i_v,i_{h-1})}\eta_{(i_{v-1},i_h)} + \eta_{(i_v,i_{h-1})}\eta_{(i_{v+1},i_h)} + \eta_{(i_v,i_{h+1})}\eta_{(i_{v-1},i_h)} + \eta_{(i_v,i_{h+1})}\eta_{(i_{v+1},i_h)}\right) \tag{3.8}$$

Note that:

$$\begin{aligned}(3.6) =& 4\sum_i \eta_i^2 \\ (3.7) =& \sum_i \eta_{(i_v,i_h)}\left(\eta_{(i_v,i_{h-2})} + \eta_{(i_v,i_{h+2})}\right) + \sum_i \eta_{(i_v,i_h)}\left(\eta_{(i_{v-2},i_h)} + \eta_{(i_{v+2},i_h)}\right) \\ =& 2\sum_{\langle\langle i,j\rangle\rangle_1} \eta_j\eta_i,\end{aligned}$$

after having expanded the sums and having grouped terms with the same indices. Similarly,

$$(3.8) = 4\sum_{\langle\langle i,j\rangle\rangle_2} \eta_j\eta_i.$$

Hence, the second term on the RHS of can be written as

$$\sum_i \left[J\sum_{j\in N(i)}\eta_j + K\eta_i\right]^2 = \sum_i (4J^2+K^2)\eta_i^2 + 2J^2\sum_{\langle\langle i,j\rangle\rangle_1}\eta_j\eta_i + 4J^2\sum_{\langle\langle i,j\rangle\rangle_2}\eta_j\eta_i + 2JK\sum_i \eta_i\sum_{j\in N(i)}\eta_j, \tag{3.9}$$

and the reduced Hamiltonian in 2D as

$$-\mathcal{H}_r(\underline{s}) = \left(\alpha + \frac{4J^2+K^2}{4\lambda}\right)\sum_i \eta_i^2 + \frac{JK}{\lambda}\sum_{\langle i,j\rangle}\eta_i\eta_j + \frac{J^2}{2\lambda}\sum_{\langle\langle i,j\rangle\rangle_1}\eta_j\eta_i + \frac{J^2}{\lambda}\sum_{\langle\langle i,j\rangle\rangle_2}\eta_j\eta_i.$$

Remark: in the 2D-case, we need to make a distinction between nearest-neighbouring sites which are at an in-line, twice-the-unit distance and at a "diagonal" distance from the site i, as shown in Figure 3.3 below. For that matter, we index their sums separately with $\langle\langle i,j\rangle\rangle_1$ and $\langle\langle i,j\rangle\rangle_2$ respectively.

To make it more concise, let

$$J_1 = \alpha + \frac{4J^2+K^2}{4\lambda}, J_2 = \frac{JK}{\lambda}, J_3 = \frac{J^2}{2\lambda}, J_4 = \frac{J^2}{\lambda}$$

to get

$$-\mathcal{H}_r(\underline{s}) = J_1\sum_i \eta_i^2 + J_2\sum_{\langle i,j\rangle}\eta_i\eta_j + J_3\sum_{\langle\langle i,j\rangle\rangle_1}\eta_j\eta_i + J_4\sum_{\langle\langle i,j\rangle\rangle_2}\eta_j\eta_i.$$

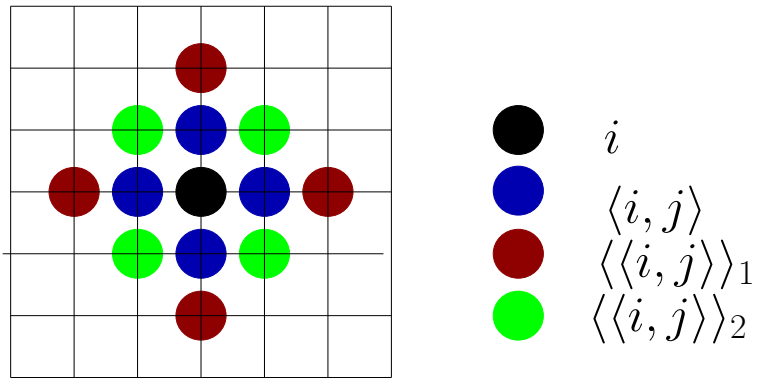

Figure 3.3: Different types of nearest-neighbours

More comparisons of the two models will be covered in Chapter 5 with simulation results.

Chapter 4

Monte Carlo simulation of the lattice model

The Metropolis algorithm, a well known Monte Carlo Method to estimate the Ising model, is an algorithm that can be used to simulate a large variety of real life situations.
In this project, we will use it to simulate our gangs and graffiti model, and its derivation follows from the constraints on the parameters J, K, α and λ we have outlined in Chapter 3.

4.1 Metropolis algorithm

The Metropolis Algorithm [5] starts from an arbitrary point and generates the sequence by repeating the following cycle, and the probability of jumping to one point to another depends only on the previous point but not on the entire history (this is one of the properties of the Markov Chain):

1. Select a new point x^*, chosen according to a symmetric proposal PDF (we will be using the uniform distribution here);
2. Calculate the acceptance probability $A(x^* \mid x_t) = min[1, \frac{p(x^*)}{p(x_t)}]$;
3. Accept x^* with probability $A(x^* \mid x_t)$, i.e.
   - if $p(x^*) > p(x_t)$, we accept it;
   - if $p(x^*) < p(x_t)$, extract a uniform random number between 0 and 1 and accept x^* if the random number is less then $\frac{p(x^*)}{p(x_t)}$.
     If the point is accepted, then $x_{t+1} = x^*$. Otherwise $x_{t+1} = x_t$.

When we calculate the acceptance probability, we have to choose it such that it satisfies the condition of detailed balance as shown below:

$p_u P(u \to v) = p_v P(v \to u)$

$\Rightarrow \frac{P(u \to v)}{P(v \to u)} = \frac{p_v}{p_u} = e^{-(E_v - E_u)}$

Remark: $p_u = \frac{1}{z}e^{-E_u/kT}$, this was shown by Gibbs(1902) that it is the equilibrium occupation probabilities at Temperature T. k is the Boltzmann's constant and E_u is the energy at state u [9].

4.2 Metropolis algorithm on the Ising model

To use the metropolis algorithm on the Ising model, we can perform the following procedures [11]:

1. We choose a set of selection probabilities P randomly, one for each possible transition (e.g. one for each $n \times n$ point in the lattice) and choose the starting point randomly;
2. Since the probability of u going to v must satisfy $e^{-(H_v - H_u)}$, we set this as the acceptance probability and calculate the difference in energy according to the Hamiltonian equation.
   To calculate the change in energy, we have to first consider the Hamiltonian equation for the Ising model,

$$E = -J \sum_{\langle i,j \rangle} s_i s_j - \mu \sum_i s_i. \tag{4.1}$$

   By extracting the energy in the (i, j) and by changing $s(i, j)$ to $-s(i, j)$, we get that the change in energy from

$$-Js(i,j)[s(i,j+1) + s(i,j-1) + s(i-1,j) + s(i+1,j)] - \mu s(i,j) \tag{4.2}$$

   to

$$Js(i,j)(s(i,j+1) + s(i,j-1) + s(i-1,j) + s(i+1,j)) + \mu s(i,j) \tag{4.3}$$

   is

$$2Js(i,j)(s(i,j+1) + s(i,j-1) + s(i-1,j) + s(i+1,j)) + 2\mu s(i,j) \tag{4.4}$$

3. If the difference in energy is smaller than 0 or the selection probability is smaller than the acceptance probability, then we flip the value of that spin.
4. Repeat with the desired number of iterations.

Alternatively, we can view the Ising model as a Markov Chain. First we pick a random site on the square lattice and we determine the route by the acceptance probability and the assumptions.
In this case, the matrix in the Markov Chain would be

$$\begin{pmatrix} 1 - e^{-(H_2 - H_1)} & 1 \\ e^{-(H_2 - H_1)} & 0 \end{pmatrix}$$

where the (i,j)-th entry would be the probability of j going to i. Here, we have assumed that higher energy must jump to the lower energy with probability 1.
Observe that the sum of each entries in the columns of the above Markov matrix must equal to one because of the axioms of probability.
The eigenvector corresponding to eigenvalue 1 is

$$\begin{pmatrix} e^{-H_1} \\ e^{-H_2} \end{pmatrix}.$$

4.3 Metropolis algorithm on gang and graffiti

For the full gangs and graffiti model, we will be using the GI-Hamiltonian equation (3.1). We can apply the process to η_i and calculate g_i by completing the square for the GI-Hamiltonian equation (3.1) (a quadratic equation). In particular when λ is large, the reduced Hamiltonian will tend to zero and result in minimal gang activity. For the reduced gangs and graffiti model, we will be using the reduced Hamiltonian equation and optimize g,

$$-\mathcal{H}_r(\underline{s}) = J_1 \sum_i \eta_i^2 + J_2 \sum_{\langle i,j \rangle} \eta_i \eta_j + J_3 \sum_{\langle\langle i,j \rangle\rangle_1} \eta_j \eta_i + J_4 \sum_{\langle\langle i,j \rangle\rangle_2} \eta_j \eta_i.$$

Even though the process would be similar we have to modify the codes to suit the gangs and graffiti model and to optimise the Hamiltonian equation for g. We will have three states $\{-1, 0, 1\}$ in stead of two $\{-1, 1\}$, and the acceptance probability would still have the same calculation process, but we will add some assumptions to make the matrix in the Markov Chain simpler, in order to speed up the process.

We define E_i to be the energy at stage i, and we will assume that if $E_1 < E_2 < E_3$, then:

(1) the the probability of E_3 jumping to E_1 is 1.(e.g. the probability of E_3 jumping to with E_1or E_2 is zero.)

(2) the probability of E_1 jumping to E_3 is zero.

(3) the probability of E_2 staying at E_2 is zero.

With the above assumptions we can obtain the matrix in the Markov chain using the same principle. From (1), we know that the the last column of the Markov Matrix is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

From (2) and the condition of detailed balance, we get that the probability of E_2 jumping to E_3 is $(e^{-E_3+E_2})$ whilst the probability of E_2 jumping to E_1 is $(1-e^{-E_3+E_2})$, so that the second column of the Marcov Matrix is

$$\begin{pmatrix} 1-e^{-E_3+E_2} \\ 0 \\ e^{-E_3+E_2} \end{pmatrix}$$

From (3) and the condition of detailed balance, we get that the probability of E_1 jumping to E_2 is $(e^{-E_2+E_1})$ whilst the probability of E_1 staying at E_1 is $(1-e^{-E_2+E_1})$, so that the first column of the Markov Matrix is

$$\begin{pmatrix} 1-e^{-E_2+E_1} \\ e^{-E_2+E_1} \\ 0 \end{pmatrix}$$

Combining the above results, we can construct the Markov matrix

$$\begin{pmatrix} 1-e^{-E_2+E_1} & 1-e^{-E_3+E_2} & 1 \\ e^{-E_2+E_1} & 0 & 0 \\ 0 & e^{-E_3+E_2} & 0 \end{pmatrix}$$

Keep in mind that this matrix is not unique, we just choose this matrix because it contains the highest number of zero-entries that it possibly can. A consequence of this construction is that the codes will speed up the iterating process quite remarkably.

We can also observe that this matrix has eigenvector

$$\begin{pmatrix} e^{-E_1} \\ e^{-E_2} \\ e^{-E_3} \end{pmatrix}$$

with eigenvalue 1. So the system will eventually come to equilibrium given that the number of iterations is large enough.

4.4 Implementation of the Metropolis algorithm on the reduced model

First of all, we have to define the parameters J, K, α, λ, the size of lattice, and the number of iterations. We will be setting the number of iterations to be 400, $J = 1$ and $K = 1$ throughout the study. Since we need an actual lattice to work with, we will create an $(n+4) \times (n+4)$ matrix where the entries of the central $n \times n$ matrix are either ± 1 or 0.

The reason why we choose an $(n+4) \times (n+4)$ matrix is because we have to apply periodic boundary condition to the matrix. That is, we specify the neighbour of the spins on one edge of the lattice to be equal to the opposite edge of the lattice. In particular, this ensures the spins on the edge, as well as the other spins, to have the same number of neighbours and local geometry.

Next we just need to set an array of random probabilities (e.g. the selection probability) and a $2 \times n^2$ matrix to select the spin site randomly. Then we calculate the difference in energy of the chosen site with the neighbour sites according to the reduced Hamiltonian equation.

The next step is change the value of that site $-1, 0, 1$ with the acceptance probability.(e.g. Allow the jump in energy if the corresponding selection probabilities is smaller than probability in the entries of the Markov matrix)

Then we need to update the boundary condition and store the data. The last step is to iterate the above process for a desired number of iterations until the oscillation in the result is minimised.

We will store the data about the total energy of the system at each iteration to see the change in activity of the gangs. We will also keep an eye on the fraction of the two gang members at each iteration to monitor the dominance of a particular gang or the growth of the gangs.

To see the results better, we will be plotting the iteration figure and the number of iteration-fraction of gang member graph. We will be plotting different parameters against the fraction of gangs graphs also to look for any phase transition as defined in the previous chapters.

Chapter 5

Simulation results

In this section, we will simulate different possible situations with different values for the parameters given before. To simplify the observations, we will only study the influence of two parameters from our Hamiltonian function, namely α (tendency of a gang member to occupy unclaimed territories) and λ (influence of community and/or police), on the value of the fraction of gang members (spots occupied by gang members/total number of spots). We will be particularly interested in phase transitions for its useful application: how to decrease or increase the fraction of gang members using these two parameters.

5.1 Phase transitions

5.1.1 Phase transition for fixed values of α

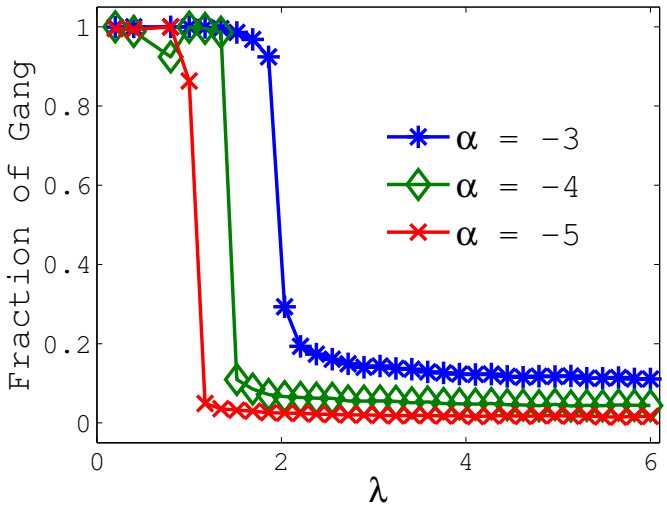

Figure 5.1: Fraction of gang members for fixed α using the full model .

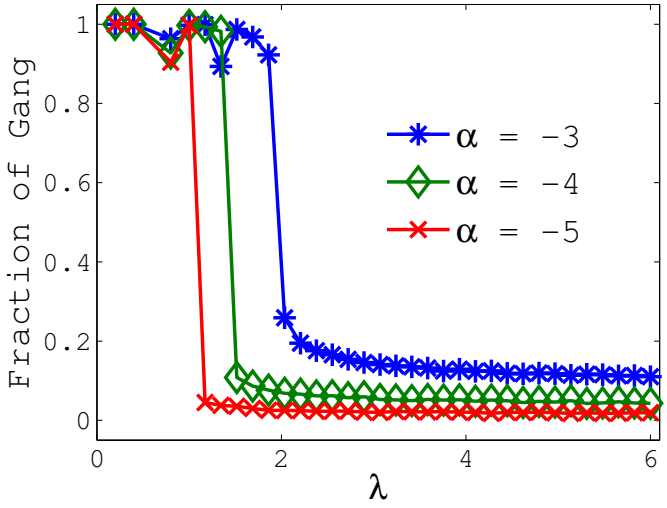

Figure 5.2: Fraction of gang members for fixed α using the reduced model .

From Figure 5.1 and Figure 5.2, we can observe a phase transition from a clustered configuration with a strong occupancy of the territory by the two gangs to a configuration where the presence of gang members is strongly decimated. Moreover as α decreases the phase transition becomes sharper and sharper and it makes the fraction of gang members go from 1 to 0 for a very small value of λ.
Also we can see that the smaller is the lambda, the bigger is the fraction of gang members and, inversely, the bigger is lambda, the smaller is the fraction of gang members.
The two graphs show how a phase transition may occur for multiple pairs of parameter-values (α, λ) both in the full model and in the reduce one.

A couple of important upshots we need to highlight are that:

- The bigger is the fixed alpha, the smoother is the phase transition;
- As we increase the fixed value of alpha, the phase transition occurs for bigger values of lambda.

5.1.2 Phase transition for fixed values of λ

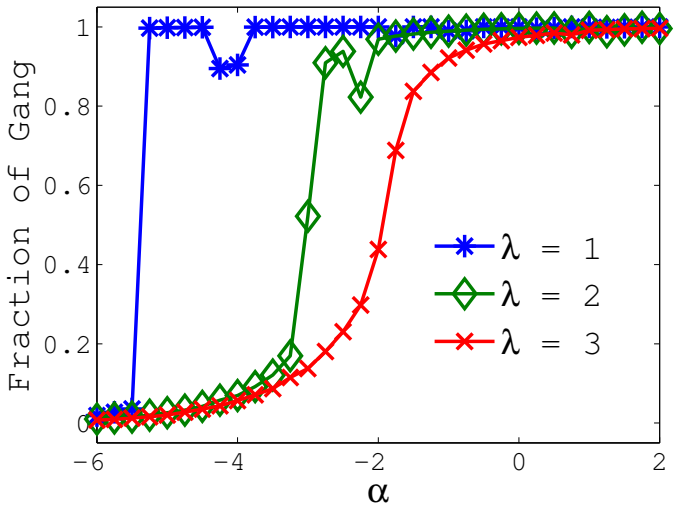

Figure 5.3: Fraction of gang members for fixed λ using the full model .

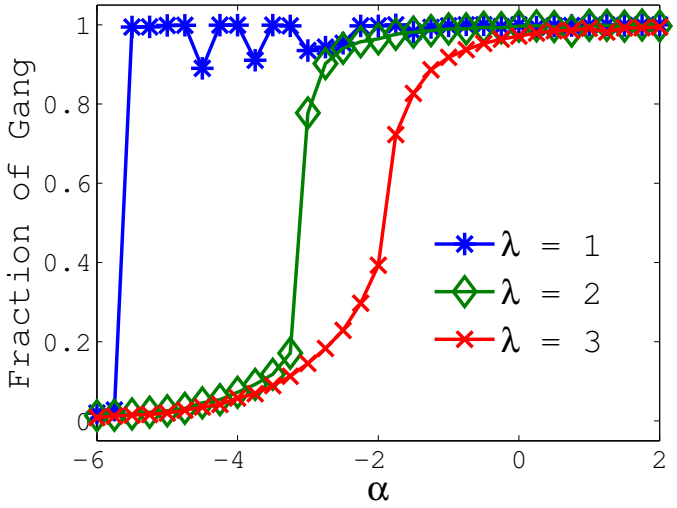

Figure 5.4: Fraction of gang members for fixed λ using the reduced model .

On the other hand, from the fraction of gang as a function of α for fixed λ (from Figure 5.3 and Figure 5.4), we can observe an inverse phase transition, namely from a configuration with a very small presence of gang members to one where the territory is totally occupied by the 'reds' and the 'blues'. Even though they are opposite, the related analysis of the graphs and the phase transitions is very similar to the one applied in the previous section:

- As λ decreases the phase transition is more steep;
- The smaller is α, the smaller is the fraction of gang members, and inversely, the bigger is α, the bigger is the fraction of gang members;
- The phase transition can occur for several critical values of α and λ;

- As we increase the value of the fixed lambda, the phase transition occurs for bigger values of lambda, which is coherent with reality.

We can see that the simulation results are conform with what we can think intuitively. Indeed, as we increase the tendency of a gang member to occupy unclaimed territories or decrease the influence of community and/or police, the fraction of gang members increases. Conversely, if we decrease gangs' propensity to territorial occupancy or increase the intervention of community and/or police, the presence of gang members on the territory decreases.

Also, as the graphs are almost similar for the two different models (full and reduced), we can say that the full GI-model resembles with a high level of accuracy the behaviour of the system if we were to make use of the standard Ising model instead, which we have tried to emulate with the reduced model.

5.2 Gang dominance

Another interesting phenomenon we observed is the appearance of gang dominance in certain parameter regimes, for both the full model and reduced model. We discuss this dominance for both fixed α and λ below for the reduced model, but we could not give a heuristic explanation or a rigorous justification.

5.2.1 Gang dominance for fixed α

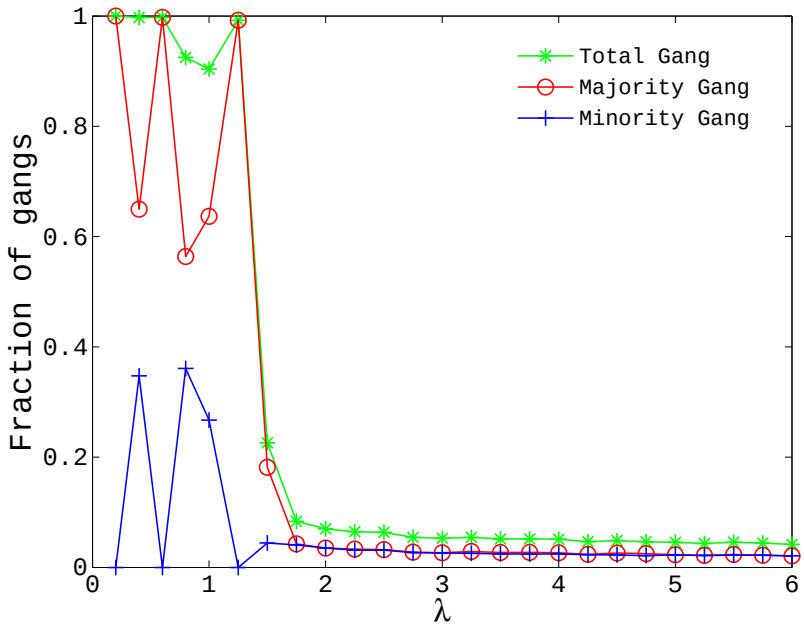

Figure 5.5: Gang dominancy for fixed $\alpha = -4$ using the reduced model .

On figure 5.5 , we can see that when lambda is relatively small, there is a dominance of one of the gang, and when lambda gets bigger and after the phase transition has occurred, we can see that there is no more dominance and that the two gangs occupy equally the territory.

5.2.2 Gang dominance for fixed λ

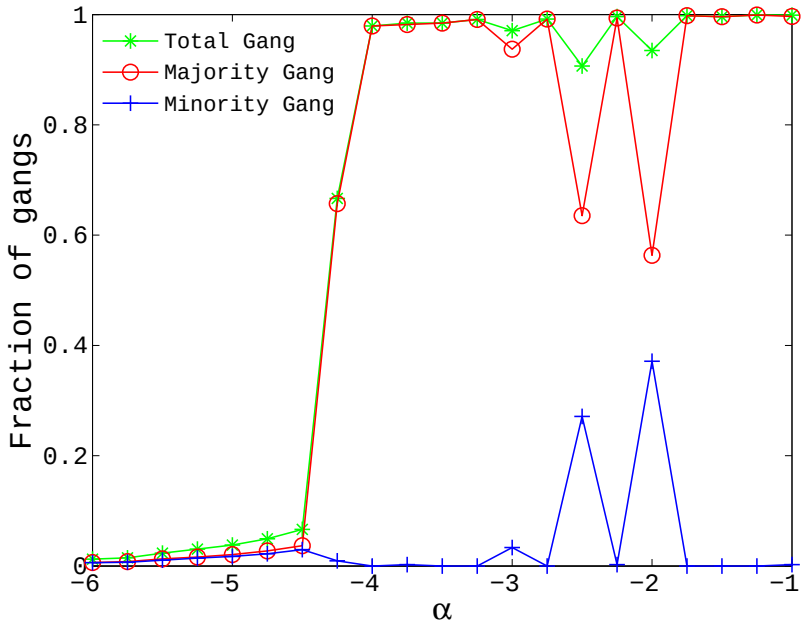

Figure 5.6: Gang dominancy for fixed $\lambda = 1.4$ using the reduced model .

On figure 5.6, we can see that when alpha is relatively small there is no dominance and that the two gangs occupy equally the territory, and when alpha gets bigger and after the phase transition has occurred, we can see that there is a dominance.

5.3 Energy of the two models

Since the reduced model is derived by minimising the GI-Hamiltonian with respect to g_i, it follows that the total energy is also minimised. This can be illustrated by Figure 5.7, where the total energy of the full model and the total energy of the reduced model are plotted against the number of iterations and the values of the parameters J, K, α and λ are the same.

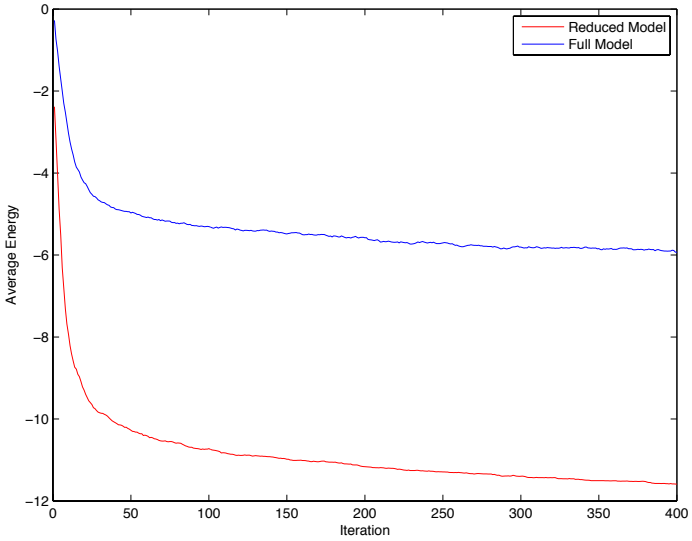

Figure 5.7: Energy comparison between full model and reduced model

Chapter 6

Conclusion

In this project, we have tried to analyse indirect gangs interactions through graffiti markings using the Ising model, which first we have adapted to our situation basing our work on the research paper we were given [2]. Then we have simplified the GI-model from the paper using mathematical methods in order to spot the link with the standard Ising model and to have a deeper understanding of the problem. Using Matlab, we have created codes to simulate the evolution of various gangs and graffiti configurations on the territory by varying the parameters in search for phase transitions. Finally, we have observed the mathematical results and we have related the obtained phase transitions to the corresponding real-life scenarios to prove how the model in question is a strong, effective and well-defined tool to reproduce on computer machines the dynamics of gangs interactions. Indeed, we have proved that there exist phase transitions for certain values of the parameters which consist of going from a large number of gang members to a really low number of gang members and vice versa.

The mathematical model discussed in this project can have a wide number of concrete applications where territorial predominance or conflicts between populations is observed via markings but no direct interactions are observed. As far as our studying is concerned, the Ising model for gang and graffiti could be useful to the police to approach gang crimes differently. Even though it may appear obsolete and ineffective in the first place, the model is actually innovative and original as it allows to explore and apply investigative methods which are by far less dangerous than on-territory investigations. In fact, because our model simulates the situation without direct gang interactions, the results can be used to analyse the effects of head-to-head gang wars by comparing the patterns between the simulations and real-life crime, and to see where and how police can act to have the most effective impact.

In conjunction with police intervention, the simulation findings may also be useful to enhance community intervention (have people to get rid of graffiti, educate children against vandalism, not let an area be too affected by poverty, unemployment, etc.) which could have a strong impact on diminishing the presence and influence of gangs.

Also, while these kinds of mathematical models are powerful, they raise some ethical problems. For example, concerns can be raised on abusive control of the population, which is an issue that is becoming more and more important in view of the technologies and algorithms using personal data.

Appendix A

Matlab Code

A.1 Main program to simulate the full model

```
%% Main program to simulate Gang and Graffiti interaction

clf; clear all;
N = 40;  % size of the lattice N by N
RN = 20; % number of simulations RN*RN per iteration

% parameters of the full model
J = -1.0;
K = -1.0;
alpha =  -2;
lambda = 1;
RealSimu = 0;
MaxIter = 400;

% initialize the gang and graffiti
s = zeros(N+2,N+2);  % gang
g = zeros(N+2,N+2);  % graffiti

% initial configuration
s(2:N+1,2:N+1) = floor(3*rand(N,N))-1;
g(2:N+1,2:N+1) = randn(N,N);
s = PeriodComp(s);
g = PeriodComp(g);

% data to save at each iteration
myE = [];  % energy
myM = [];  % the average gang fraction


for iter=1:MaxIter
    ri = ceil(N*rand(2,RN*RN)); % the random site
    pi = rand(1,RN*RN);         % random probability
    pn = randn(1,RN*RN);        % random normal for graffiti
    for k=1:RN*RN
        % choose the random site
        i = ri(1,k)+1;
        j = ri(2,k)+1;
```

```
38          % calculate the three energies for the three gang states {-1,0,1}
39          dE = [-alpha+K*g(i,j)+J*(g(i,j+1)+g(i,j-1)+g(i+1,j)+g(i-1,j)) ...
40             0 -alpha-(K*g(i,j)+J*(g(i,j+1)+g(i,j-1)+g(i+1,j)+g(i-1,j)))];
41          % sort the three energies
42          [newdE,I] = sort(dE);
43          if find(I==s(i,j)+2)==1
44              % at the lowest energy state, jump to the second lowest energy
45              % state with probability exp(newdE(1)-newdE(2));
46              if pi(k)<exp(newdE(1)-newdE(2))
47                  %jump to energy newdE(2)
48                  s(i,j) = I(2)-2;
49              end
50          elseif find(I==s(i,j)+2)==2
51              % at the middle energy state, jump to the highest energy with
52              % probability exp(newdE(2)-newdE(3)), otherwise to the lowest
53              % energy state
54              if pi(k)<exp(newdE(2)-newdE(3))
55                  s(i,j) = I(3)-2;
56              else
57                  s(i,j) = I(1)-2;
58              end
59          elseif find(I==s(i,j)+2)==3
60              % alway jump to the lowest energy
61              s(i,j) = I(1)-2;
62          else
63              error('Something Wrong!');
64          end
65          % make the periodic update for s
66          s = PeriodUpdate(s,i,j,N);
67          % update g(i,j) using normal random number
68          mg = J*(s(i,j+1)+s(i,j-1)+s(i+1,j)+s(i-1,j))+K*s(i,j);
69          g(i,j) = mg/2/lambda+pn(k)/sqrt(2*lambda);
70          % make the periodic update for g
71          g = PeriodUpdate(g,i,j,N);
72      end
73      % plot the simulation results at each iteration
74      if (RealSimu==1)
75          figure(3)
76          subplot(1,2,1)
77          imagesc(s(2:N+1,2:N+1),[-1 1]);
78          colorbar;
79          set(gca,'XTick',0:5:20,'YTick',0:5:20)
80          subplot(1,2,2)
81          imagesc(g(2:N+1,2:N+1));
82          colorbar;
83         set(gca,'XTick',0:5:20,'YTick',0:5:20)
84         getframe;
85      end
86      % save the data per iteration
87      E = J*sum(sum(g(2:N+1,2:N+1).*(s(1:N,2:N+1)+s(3:N+2,2:N+1)+s(2:N+1,1:N
            )+s(2:N+1,3:N+2)))) ...
88          + K*sum(sum(s(2:N+1,2:N+1).*g(2:N+1,2:N+1)))+alpha*sum(sum(s(2:N
                +1,2:N+1).^2))-lambda*sum(sum(g(2:N+1,2:N+1).^2));
89      myE = [myE -E/N^2];
90      myM = [myM; sum(sum(abs(s(2:N+1,2:N+1))))/N^2 sum(sum(s(2:N+1,2:N+1)
            >0.5))/N^2 sum(sum(s(2:N+1,2:N+1)<-0.5))/N^2];
91  end
```

```
92
93 %%
94 figure(1) % plot the energy per iteration
95 plot(myE);
96 xlabel('Iteration');
97 ylabel('Average Energy');
98
99 figure(2) % plot the fraction of gang
100 plot(1:MaxIter,myM(:,1),'k-',1:MaxIter,myM(:,2),'r--',1:MaxIter,myM(:,3),'
    b-.','LineWidth',1);
101 xlabel('Iteration');
102 ylabel('Fractional of gang member');
103 axis([1 MaxIter 0 1]);
```

A.2 Main program to simulate the reduced model

```
1 %% Main program to simulate Gang and Graffiti interaction, reduced model
2
3 clf; clear all;
4 N = 40;  % size of the lattice N by N
5 RN = 20; % number of simulations RN*RN per iteration
6
7 % parameters
8 J = 1.0;
9 K = 1.0;
10 alpha =  -2;
11 lambda = 2;
12 RealSimu = 1;
13 MaxIter = 100;
14
15 % Combined parameters in the reduced model
16 J1 = alpha+J^2/lambda+K^2/4/lambda;
17 J2 = J*K/lambda;
18 J3 = J^2/2/lambda;
19 J4 = J^2/lambda;
20
21 % initial configuration for gang
22 s = zeros(N+4,N+4);
23 s(3:N+2,3:N+2) = floor(3*rand(N,N))-1;
24 s = RedPeriodComp(s,N);
25
26 % data to save at each iteration
27 myE = [];  % energy
28 myM = [];  % the average gang fraction
29
30
31 for iter=1:MaxIter
32     ri = ceil(N*rand(2,RN*RN)); % the random site
33     pi = rand(1,RN*RN);         % random probability
34     pn = randn(1,RN*RN);        % random normal for graffiti
35     for k=1:RN*RN
36         % choose the random site
37         i = ri(1,k)+2;
38         j = ri(2,k)+2;
39         % calculate the three energies for the three gang states {-1,0,1}
40         temp = J2*(s(i,j+1)+s(i,j-1)+s(i+1,j)+s(i-1,j)) ...
```

```
            + J3*(s(i,j+2)+s(i,j-2)+s(i+2,j)+s(i-2,j)) ...
            + J4*(s(i+1,j+1)+s(i-1,j+1)+s(i+1,j-1)+s(i-1,j-1));
        dE = [-J1+temp 0 -J1-temp];
        % sort the three energies
        [newdE,I] = sort(dE);
        if find(I==s(i,j)+2)==1
            % at the lowest energy state, jump to the second lowest energy
            % state with probability exp(newdE(1)-newdE(2));
            if pi(k)<exp(newdE(1)-newdE(2))
                %jump to energy newdE(2)
                s(i,j) = I(2)-2;
            end
        elseif find(I==s(i,j)+2)==2
            % at the middle energy state, jump to the highest energy with
            % probability exp(newdE(2)-newdE(3)), otherwise to the lowest
            % energy state
            if pi(k)<exp(newdE(2)-newdE(3))
                s(i,j) = I(3)-2;
            else
                s(i,j) = I(1)-2;
            end
        elseif find(I==s(i,j)+2)==3
            % alway jump to the lowest energy
            s(i,j) = I(1)-2;
        else
            error('Something Wrong!');
        end
        % make the periodic update for s
        s = RedPeriodUpdate(s,i,j,N);
    end
    % plot the simulation results at each iteration
    if (RealSimu==1)
        figure(1)
        imagesc(s(3:N+2,3:N+2),[-1 1]);
        colorbar;
        set(gca,'XTick',0:10:40,'YTick',0:10:40)
        mov(iter) = getframe(gcf);
    end
    % save the data per iteration
    E = J1*sum(sum(s(3:N+2,3:N+2).^2)) ...
        +J2*sum(sum(s(3:N+2,3:N+2).*(s(2:N+1,3:N+2)+s(4:N+3,3:N+2)+s(3:N
           +2,2:N+1)+s(3:N+2,4:N+3)))) ...
        +J3*sum(sum(s(3:N+2,3:N+2).*(s(1:N,3:N+2)+s(5:N+4,3:N+2)+s(3:N
           +2,1:N)+s(3:N+2,5:N+4)))) ...
        +J4*sum(sum(s(3:N+2,3:N+2).*(s(2:N+1,2:N+1)+s(2:N+1,4:N+3)+s(4:N
           +3,2:N+1)+s(4:N+3,4:N+3))));
    myE = [myE -E/N^2];
    myM = [myM; sum(sum(abs(s(2:N+1,2:N+1))))/N^2 sum(sum(s(2:N+1,2:N+1)
       >0.5))/N^2 sum(sum(s(2:N+1,2:N+1)<-0.5))/N^2];
end
```

A.3 Functions to setup the periodic boundary condition

```
function sout = PeriodComp(sin)

sout = sin;
```

```
sout(1,2:end-1)=sin(end-1,2:end-1);
sout(end,2:end-1) = sin(2,end-1);
sout(2:end-1,1) = sin(2:end-1,end-1);
sout(2:end-1,end) = sin(2:end-1,2);

sout(1,1) = sout(end-1,end-1);
sout(1,end) = sout(end-1,2);
sout(end,end) = sout(2,2);
sout(end,1) = sout(2,end-1);
```

```
function s = PeriodUpdate(s,i,j,N)

if i==2
    if j==2
        s(i+N,j+N) = s(i,j);
        s(i,j+N)   = s(i,j);
        s(i+N,j)   = s(i,j);
    elseif (j==N+1)
        s(i,j-N)   = s(i,j);
        s(i+N,j)   = s(i,j);
        s(i+N,j-N) = s(i,j);
    else
        s(i+N,j) = s(i,j);
    end
elseif i==N+1
    if j==2
        s(i-N,j+N) = s(i,j);
        s(i,j+N)   = s(i,j);
        s(i-N,j)   = s(i,j);
    elseif (j==N+1)
        s(i,j-N)   = s(i,j);
        s(i-N,j)   = s(i,j);
        s(i-N,j-N) = s(i,j);
    else
        s(i-N,j) = s(i,j);
    end
else
    if j==2
        s(i,j+N) = s(i,j);
    elseif (j==N+1)
        s(i,j-N) = s(i,j);
    end
end
```

```
function s_out = RedPeriodComp(s_in,N)

s_out = s_in;

s_out(1,3:N+2) = s_in(N+1,3:N+2);
s_out(2,3:N+2) = s_in(N+2,3:N+2);
s_out(N+3,3:N+2) = s_in(3,3:N+2);
s_out(N+4,3:N+2) = s_in(4,3:N+2);
```

```
9
10 s_out(3:N+2,1) = s_in(3:N+2,N+1);
11 s_out(3:N+2,2) = s_in(3:N+2,N+2);
12 s_out(3:N+2,N+3) = s_in(3:N+2,3);
13 s_out(3:N+2,N+4) = s_in(3:N+2,4);
14
15
16 s_out(1,1) = s_out(N+1,N+1);
17 s_out(1,2) = s_out(N+1,N+2);
18 s_out(2,1) = s_out(N+2,N+1);
19 s_out(2,2) = s_out(N+2,N+2);
20
21 s_out(N+3:N+4,1:2) = s_out(1:2,1:2);
22 s_out(1:2,N+3:N+4) = s_out(1:2,1:2);
23 s_out(N+3:N+4,N+3:N+4) = s_out(1:2,1:2);
```

```
1  function s = RedPeriodUpdate(s,i,j,N)
2
3  if (i==3 | i==4)
4      if (j==3 | i==4)
5          s(i+N,j+N) = s(i,j);
6          s(i,j+N)   = s(i,j);
7          s(i+N,j)   = s(i,j);
8      elseif (j==N+1 | j==N+2)
9          s(i,j-N)   = s(i,j);
10         s(i+N,j)   = s(i,j);
11         s(i+N,j-N) = s(i,j);
12     else
13         s(i+N,j) = s(i,j);
14     end
15 elseif (i==N+1 | i==N+2)
16     if (j==3 | j==4)
17         s(i-N,j+N) = s(i,j);
18         s(i,j+N)   = s(i,j);
19         s(i-N,j)   = s(i,j);
20     elseif (j==N+1 | j==N+2)
21         s(i,j-N)   = s(i,j);
22         s(i-N,j)   = s(i,j);
23         s(i-N,j-N) = s(i,j);
24     else
25         s(i-N,j) = s(i,j);
26     end
27 else
28     if (j==3 | j==4)
29         s(i,j+N) = s(i,j);
30     elseif (j==N+1 | j==N+2)
31         s(i,j-N) = s(i,j);
32     end
33 end
```

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