

Dynamical Systems

Week 1 Review session (Introduction and basic ODEs)

Introductory Examples

September 2024

Today's Agenda

1. Different resources like pp1ane and lecture notes in webpage formats (Blackboard)
2. Introductory examples
3. Review on matrix exponential
4. Review on methods for basic ODEs

Notice that the review sessions are recorded, and will be available on Blackboard.

Different approaches to understand dynamical systems

$$\dot{x} = xy, \quad \dot{y} = \frac{1 - x^2 + y^2}{2}.$$

- ▶ **Analytical:** find the closed form solution $(x(t), y(t))$
- ▶ **Numerical:** find the numerical value at particular time(s)
- ▶ **Qualitative:** find properties that the solution $(x(t), y(t))$ satisfies

Closed Form Solution

$$\dot{x} = xy, \quad \dot{y} = \frac{1 - x^2 + y^2}{2}.$$

The general solution with initial condition $x(0) = x_0, y(0) = y_0$ is given by

$$x(t) = \frac{2x_0}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2) \cos t - 2y_0 \sin t},$$
$$y(t) = \frac{(1 - x_0^2 - y_0^2) \sin t + 2y_0 \cos t}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2) \cos t - 2y_0 \sin t}.$$

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Can you say anything about the solution?

Numerical Solution

$$\dot{x} = xy, \quad \dot{y} = \frac{1 - x^2 + y^2}{2}, \quad x(0) = 1, \quad y(0) = 0.$$

t	x	y	t	x	y
0.0	1.00000	0.00000			
0.1	1.00250	0.04992	1.1	1.30949	0.42650
0.2	1.01002	0.09933	1.2	1.36735	0.43695
0.3	1.02258	0.14773	1.3	1.42869	0.43925
0.4	1.04025	0.19458	1.4	1.49247	0.43260
0.5	1.06309	0.23932	1.5	1.55730	0.41632
0.6	1.09115	0.28136	1.6	1.62150	0.38997
0.7	1.12450	0.32004	1.7	1.68304	0.35340
0.8	1.16312	0.35467	1.8	1.73970	0.30686
0.9	1.20697	0.38449	1.9	1.78904	0.25109
1.0	1.25586	0.40872	2.0	1.82878	0.18732

Any insights from the numbers?

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Any insights from the numbers?

The solution is better visualised (in 3D or less) with a software package, and may be inaccurate in long time.

Qualitative Properties

$$\dot{x} = xy, \quad \dot{y} = \frac{1 - x^2 + y^2}{2}, \quad x(0) = x_0, \quad y(0) = y_0.$$

The solution $(x(t), y(t))$ lies on the circle¹

$$x^2 + y^2 - \frac{1 + x_0^2 + y_0^2}{x_0}x + 1 = 0.$$

¹See the simulation on the course webpage or using pplane

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Qualitative description could be more convenient; consider

$$\frac{d^2x}{dt^2} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}x, \quad \frac{d^2y}{dt^2} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}y, \quad \frac{d^2z}{dt^2} = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}z.$$

and Kepler's first law *the orbit of a planet is an ellipse with the Sun at one of the two foci.*

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Quantitative or Qualitative?

$$\frac{dx}{dt} = -\frac{x^3 - Tx + y}{\epsilon}, \quad \frac{dy}{dt} = x.$$

x : the length of a muscle fibre in the heart; y : an electrochemical potential; ϵ and T are two constants.

Normal Heartbeat



Irregular Heartbeat



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Are we more interested in the numerical value ($x(t), y(t)$) at a time t (say 60 seconds) or the number of oscillations within one minute?

Application in Partial Differential Equations

Question: when (for which q and N) the elliptic PDE

$$-\Delta u = u^q \quad \text{on } \Omega \subset \mathbb{R}^N$$

with $u = 0$ on the boundary $\partial\Omega$ has a solution?

When Ω is the unit ball $B = \{x \in \mathbb{R}^N \mid |x| < 1\}$, the radial solution governed by the equivalent 2nd order ODE

$$u''(r) + \frac{N-1}{r}u'(r) + u(r)^q = 0$$

can be characterised completely by the techniques in this module, which turns out to be true for most general domain Ω .

Application to systems of planetary motion

$$m \frac{d^2 \mathbf{x}}{dt^2} = -\nabla_{\mathbf{x}} \phi(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^3} \mathbf{x}, \quad \mathbf{x}(0) \text{ and } \dot{\mathbf{x}}(0) \text{ is given.}$$

We can think of the following simplified scenario: the sun is fixed at the origin with mass M , and the earth is positioned at \mathbf{x} with mass m .

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- (a) Analytically, can we get/confirm qualitative information about the solution, such as Kepler's three laws (all of which turns out to be related to *invariant set*)?
- (b) There are many methods available for solving the second order differential equations, but most of them produce incorrect numerical solutions in the long run (the earth collapses to or escapes from the sun). How do we distinguish those qualitatively correct ones from the rest?

Intended Learning Outcomes

- ▶ solve linear or decoupled system of ODEs or maps, and deduce their **long term behaviours** of the solutions
- ▶ derive **qualitative properties** of solutions to system of ODEs or maps by semi-group property, conserved quantities or change of variables
- ▶ calculate **fixed points** of system of ODEs, determine their **linear types** and sketch the **phase portrait**
- ▶ construct Lyapunov function to show the **stability** of the solutions
- ▶ apply the Poincare-Bendixson theorem to show the existence of **periodic solution** and apply Floquet theory to periodic linear system
- ▶ calculate the **stable or unstable manifold**
- ▶ calculate the **centre manifold** and classify the **bifurcation** type from the reduced dynamics
- ▶ calculate **fixed points** or **periodic orbits** of maps, locate and classify **bifurcation points**

Matrix exponential (with the parameter t)

$$\exp(tA) = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$$

Questions:

- ▶ Do we expect $\exp(tA)\exp(tB) = \exp(t(A+B))$?
- ▶ Do we expect $\exp(tA)\exp(sA) = \exp((t+s)A)$?
- ▶ Find the matrix exponential $\exp(tA)$ of the following matrices:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Three main types of scalar ODES

- ▶ Linear: $y' = a(x)y + b$
- ▶ Separable: $y' = f(x)g(y)$
- ▶ Homogeneous: $y' = f(x, y)$, $f(\lambda x, \lambda y) = f(x, y)$

Question: for which type of ODE, the equation $dy/dx = g(x, y)$ can not be solved, but $dx/dy = 1/g(x, y)$ (by reversing the role of x and y) can be solved (because it falls into this type)?