MATH 44041/64041
Applied Dynamical Systems

Yanghong Huang

20$^{\text{th}}$ September 2019

Recommended Reading:

(1) Morris W. Hirsch, Stephen Smale, and Robert L. Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Elsevier/Academic Press, Amsterdam, 3rd edition, 2013.

(2) James D. Meiss. *Differential dynamical systems*, volume 14 of *Mathematical Modeling and Computation*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

(3) Steven H. Strogatz. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. Westview press, 2014.

(4) Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*. Springer-Verlag, New York, second edition, 2003.

Table of Contents

1 Introduction

1.1 Problems modelled with differential or difference equations

Applied mathematicians create mathematical models to describe the world. These may involve physics (mechanics), chemistry (reaction kinetics), economics (stock movements, supply and demand), social sciences (voter preferences, opinion formation) or any number of different disciplines and problems. The common thread though is that the model is only useful if it can be used to obtain more insights into the problem being addressed. The methods that can be brought to bear depend on the nature of the model.

Models used to simulate and predict weather or climate could be very complicated, because various processes like heat transfer (both vertically and horizontally) are coupled together on the surfaces of land, ocean and ice. For the fantastically detailed climate models used to assess the probability of climate change the techniques are essentially computational, but mathematics is important in the design of the schemes and the analysis of the data. Climate scientists will also use much cruder models to provide insights into the relative importance of different effects. These models are designed so that more detailed mathematical analysis is possible, and longer, more varied computer simulation as well because the time spent on the computation is so much smaller.

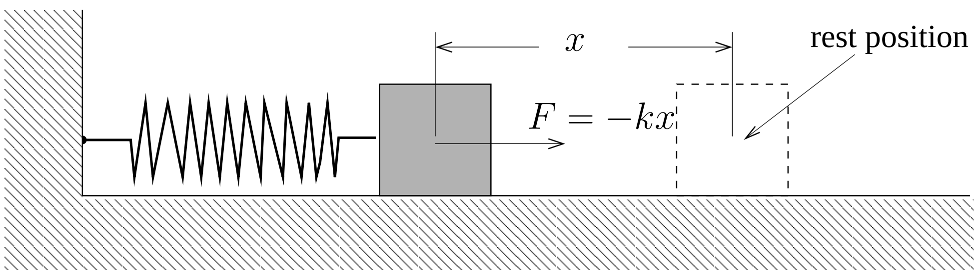


Figure 1.1: A harmonic oscillator only under the spring force $F = -kx$.

The aim of this course is to describe some of the mathematical techniques that can be used to analyse differential or difference equations that arise frequently in models. Differential equations are used to describe how quantities vary in time (or space). If there is only one independent variable then the model is an ordinary differential equation (ODE) such as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{1.1}$$

with solutions $x(t)$ that is a function of the continuous, independent variable t and the initial conditions (if they are specified). This equation describes the motion of an object under the sole force of spring force (see Figure 1.1), governed by the Newton's equation $m\ddot{x} = F = -kx$. It is sometimes useful to consider time as a discrete variable, leading to difference equations such as the logistic equation

$$x_{n+1} = \mu x_n(1 - x_n) \quad \text{with} \quad \mu \in [0, 4]. \tag{1.2}$$

This generates a sequence $\{x_0, x_1, x_2, \dots\}$ rather than a function of a continuous variable. We assume you are familiar with basic linear differential equations and difference equations.

There are two features that may be new to you (and will be our focus later) in this course: **nonlinearity** and **parameter variation**.

Nonlinearity refers to the existence of terms like x^2 in the equation – terms that are not linear in the *dependent* variable you are seeking — for example, the logistic equation (1.2) is nonlinear whilst equations (1.1) and are linear in x and u respectively. In general, nonlinear equations cannot be solved in terms of simple functions, and new techniques are needed to obtain information about solution.

In many models these are **parameters** — quantities which are constant in any single realisation of the experiment, but which can be changed (like the interest rate set by Bank of England to regulate the economy). In fluid mechanics an example is the Reynolds number of a flow, in chemistry reaction rates depend on ambient temperatures, in social sciences behaviour may be influenced by the average number of friends a person has (and in epidemiology the average number of contacts). Often these parameters are not known accurately and so it is important to know how sensitive any conclusions are to parameter variation. This is described by *bifurcation theory*: the study of how quantitative changes occur as parameters are varied. The quantity μ in the logistic equation is an example of a parameter. 'Tipping points' are another.

Finally, nonlinearity can lead to behaviour that is more complicated than the obvious static and periodic solutions (or more general quasi-periodic solutions). This is called *chaos*, and one of the interesting features of chaos is that it has its own version of bifurcation theory — there are a number of common *routes to chaos* describing how chaotic solutions are created as parameters change. We will discuss some of these too.

1.2 What is this course about?

In this course, we will be focus on *qualitative properties* of continuous and discrete dynamical systems, complementing other common methods, like explicit solutions and numerical approximations. Explicit solutions, even available in certain cases, may not be useful. For example, the general solution of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1 - x^2 + y^2}{2}$$

with initial condition $x(0) = x_0, y(0) = y_0$ is

$$\begin{aligned} x(t) &= \frac{2x_0}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t}, \\ y(t) &= \frac{2y_0 \cos t + (1 - x_0^2 - y_0^2)\sin t}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t}. \end{aligned}$$

Can you get any information from this explicit solution, without plotting any sample trajectories? Numerical approximations may also not be so effective for high dimensional systems or long time behaviours.

Before considering complicated nonlinear systems, we start with a few basic notations and concepts in the next section.

2 Notation and basic concepts

2.1 Ordinary differential equations (ODEs)

We only consider coupled *first order* equations: *autonomous*(time-independent) equations

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n,$$

or occasionally *non-autonomous* (time-dependent) ones

$$\dot{x} = f(x,t), \qquad x \in \mathbb{R}^n,\ f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n.$$

Here the n-dimensional Euclidean space $\mathbb{R}^n$ is called the *phase space* of the system.

Remark (Conventions about notations).

(a) In the rest of the course, we use variables like x for both vector and scalar, and there is usually no confusion. For instance, x in the system $\dot{x} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x$ is a column vector of length two, but x in the system

$$\dot{x} = x - y, \quad \dot{y} = -x + y$$

is a scalar.

(b) Explicit time dependence is also omitted, that is, we write x instead of $x(t)$, and similarly $\dot{x}, \ddot{x}$ for the time derivatives $\frac{\mathrm{d}}{\mathrm{d}t}x(t)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$.

(c) The solution to the system $\dot{x} = f(x)$ is usually written as x or $x(t)$, and sometime $x(x_0, t)$ or $\varphi_t(x_0)$, if the dependence on the initial condition x_0 is emphasized.

There is no need to consider higher order equations, because they can always be converted into first order systems by introducing new variables for the derivatives, as in the following example.

Example 2.1 (Coupled first order equations)**.** Take the simple harmonic oscillator (1.1),

$$\ddot{x} + \omega^2 x = 0. \tag{2.1}$$

This is a second order equation, and can be recast in the form of (2.1) by setting $y = \dot{x}$ (hence $\dot{y} = \ddot{x} = -\omega^2 x$). That is, we get the coupled first order equations

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that this is an example of a *linear* system of differential equations, the general form of which is

$$\dot{v} = Av. \qquad v \in \mathbb{R}^n$$

where A is a $n \times n$ matrix (possibly time dependent).

Exercise. Change the fourth order equation $\frac{\mathrm{d}^4}{\mathrm{d}t^4}x - \omega^4 x = 0$ describing the vibration of a beam into a system of first order equations, by introducing $y = \dot{x}, z = \ddot{x}, w = \dddot{x}$. What does the coefficient matrix look like? How about the equivalent first order system for the n-th order autonomous ODE

$$x^{(n)} = F\big(x, x', \cdots, x^{(n-1)}\big),$$

by introducing the variables $x_k = x^{(k)}$, $k = 0, 1, 2, \cdots, n-1$ for the k-th order derivatives.

We can always consider autonomous equations (with no explicit dependence on "time"), because non-autonomous system like $\dot{x} = f(x,t)$ can be converted into autonomous system by introducing a new independent variable τ as "time", while taking t as dependent variable. That is, $\dot{x} = f(x,t)$ is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x = f(x,t), \quad \frac{\mathrm{d}}{\mathrm{d}\tau}t = 1,$$

which is an autonomous system for $\tilde{x} = (x,t)$ that depends on τ.

Example 2.2 (The simplest scalar linear equation)**.** The differential equation $\dot{x} = ax$ for $x \in \mathbb{R}$ and with initial condition $x(0) = x_0$ is simple but illustrates features of stability and instability to which we will return. If $x_0 = 0$ then $\dot{x} = 0$ and so $x(t) = 0$ for all time (it is a *stationary point*). If $x_0 \neq 0$, it can be solved using separation of variables, that is,

$$\int_{x_0}^{x} \frac{\mathrm{d}x}{x} = \int_0^t a\mathrm{d}t,$$

which gives $x = x_0 e^{at}$. Alternatively, we can use the *integrating factor* e^{-at}. Taking the derivative of $e^{-at}x$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-at}x) = -ae^{-at}x + e^{-at}\dot{x} = e^{-at}(\dot{x} - ax) = 0.$$

Therefore, $e^{-at}x$ is a constant and equal its value at $t = 0$. That is $e^{-at}x = x_0$, or $x = e^{at}x_0$. The behaviour of solutions depends on the sign of a (the real part of a if it is complex):

$$\text{if } a < 0 \text{ then } \ |x(t)| \to 0; \qquad \text{if } a > 0 \text{ then } \ |x(t)| \to \infty.$$

A radioactive material contains unstable nuclei whose atomic nucleus loses energy and decays into another nuclide. Let N_A be the number [1] of atoms in a sample, then N_A is usually governed by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}N_A = -\lambda_A N_A$$

where λ_A is the decay constant. The solution is $N_A(t) = N_A(0)e^{-\lambda_A t}$. The time $T_{1/2} = \frac{\ln 2}{\lambda_A}$ is called *half-life*, is the time taken for the radioactive substance to decay to half of the initial value, i.e., $N_A(T_{1/2}) = N_A(0)/2$.

[1] This number is so large in practice that it can be treated as a continuous quantity to be differentiated.

Example 2.3 (Chain of two radioactive decays)**.** If one nuclide A decays into B by one process, and then B decays into C by a second process, then the amounts of A and B are governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}N_A = -\lambda_A N_A, \qquad \frac{\mathrm{d}}{\mathrm{d}t}N_B = -\lambda_B N_B + \lambda_A N_A,$$

with the initial condition $N_B(0) = 0$ (no B at the very beginning). From the solution $N_A(t) = N_A(0)e^{-\lambda_A t}$, the second equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}N_B(t) = -\lambda_B N_B(t) + \lambda_A N_A(0)e^{-\lambda_A t}.$$

If $\lambda_B \neq \lambda_A$, then this ODE can be integrated with the integrating factor $e^{\lambda_B t}$ to give

$$\lambda_B(t) = \frac{N_A(0)\lambda_A}{\lambda_B - \lambda_A}\big(e^{-\lambda_A t} - e^{-\lambda_B t}\big).$$

Exercise. (1) Find the time T when $N_B(t)$ reaches its maximum; (2) Find the solution when $\lambda_B = \lambda_A$.

Example 2.4 (Linear matrix equations)**.** The previous system can be written as

$$\dot{x} = Ax \tag{2.2}$$

with $x \in \mathbb{R}^n$, and where A is an $n \times n$ constant matrix. Solutions can be written as

$$x(t) = e^{tA}\, x_0 \tag{2.3}$$

where the exponential matrix is defined by (exactly the same as in the scalar case)

$$e^B \;=\; I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}B^n,$$

where I is the identity matrix. With this definition of matrix exponential, the expression (2.3) is a solution to the linear matrix equation (2.2) can be proved by differentiating term by term. In practice, the matrix exponential e^B is not calculated from above series expansion, but by transforming B into Jordan blocks, using eigenvectors of B. If $B = S\Lambda S^{-1}$, where the columns of S are the (generalised) eigenvalues of B, and Λ consists of Jordan blocks:

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_m \end{pmatrix}, \qquad \Lambda_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix}.$$

Then $e^B = Se^{\Lambda}S^{-1}$, while e^{Λ} can be computed easily. In general to find e^B, it is easier to find the eigenvectors and eigenvalues (or equivalently the decomposition $B = S\Lambda S^{-1}$) than to calculate the series with powers B^n (but there are exceptions as in the following example).

Example 2.5. If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, using the fact that $A^{4n} = I, A^{4n+1} = A, A^{4n+2} = -I, A^{4n+3} = -A$ (n is an integer) and the above definition for matrix exponential, we get

$$\exp(tA) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Exercise. What is $\exp(tA)$ for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

Review on different ways to solve differential equations:

(i) **Linear ODEs with constant coefficients**:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x(t) + \cdots + a_1\frac{\mathrm{d}}{\mathrm{d}t}x(t) + a_0 x(t) = 0.$$

Looking for solution of the form $x(t) = e^{\omega t}$, where ω are the roots of the n-th degree polynomial

$$\omega^n + a_{n-1}\omega^{n-1} + a_1\omega + a_0 = 0.$$

(ii) **Linear first order scalar equation $\dot{x} = a(t)x + b(t)$:** Multiply both sides by the *integrating factor* $\exp\left(-\int^t a(\tau)d\tau\right)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x\exp\left(-\int^t a(\tau)d\tau\right)\right] = [\dot{x} - a(t)x]\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right) = b(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right).$$

followed by integrating on both sides,

$$x(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right) = C + \int b(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right)\mathrm{d}t.$$

(iii) **Separable first order equation $\dot{x} = f(x)g(t)$:** Integrate $\dfrac{\mathrm{d}x}{f(x)} = g(t)\mathrm{d}t$ to get

$$\int^x \frac{\mathrm{d}x}{f(x)} = \int^t g(t)\mathrm{d}t,$$

where the integration constant is determined by the initial condition (if it is given).

(iv) **First order homogeneous ODEs $\dot{x} = f(t,x)$, where $f(\lambda x, \lambda t) = f(x,t)$ for any λ:** The trick is to introduce $z = x/t$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \frac{\mathrm{d}}{\mathrm{d}t}(zt) = z + t\frac{\mathrm{d}z}{\mathrm{d}t}$$

and $f(x,t) = f(zt,t) = f(z,1)$, the original ODE becomes $z + t\dot{z} = f(z,1)$, which is separable, and the solution is given by

$$\int \frac{\mathrm{d}t}{t} = \int \frac{\mathrm{d}z}{f(z,1) - z}.$$

System of equations, especially nonlinear ones, are much more difficult to solve analytically, if not impossible. Nevertheless, we can still have a good understanding of the qualitative properties, using different techniques that will be developed in the rest of the course.

2.2 Trajectories, phase portrait and flow on the phase space

In many situations, although explicit solutions of the underlying equations may not be available, qualitative properties and long time behaviours can still be obtained using various techniques. For example, we can understand solutions of the logistic ODE

$$\dot{x} = x(1 - x)$$

with different initial conditions $x(0)$. If $x(0) < 0$, then $x(t)$ decreases, and $x(t) \to -\infty$ as $t \to \infty$. If $x(0) \in (0, 1)$, then $x(t)$ increases to 1 and finally if $x(0) > 1$, $x(t)$ decreases to 1. In general, for the one dimension equation $\dot{x} = f(x)$, although we can get the solution from

$$\int_{x(0)}^{x(t)} \frac{\mathrm{d}x}{f(x)} = t,$$

the qualitative properties can be understood better using a *phase portrait* as Figure 2.1: x increases on regions where $f(x) > 0$ and decreases where $f(x) < 0$.

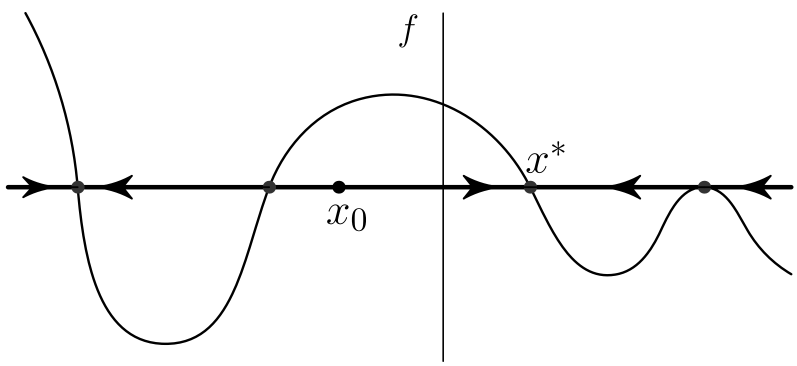


Figure 2.1: Phase portrait of the one dimensional autonomous equation $\dot{x} = f(x)$.

This picture can be extended to higher dimensions. Consider the equation $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. If we plot the trajectory $\{x(t) \mid t_1 \leq t \leq t_2\}$ in $\mathbb{R}^n$ for some time t_1 and t_2, then $f(x(t))$ is exactly the tangent vector of $x(t)$ (the definition of the ODE $\dot{x} = f(x)$!). In other words, once we have the *vector field* $f(x)$ at any points x, then we can "integrate" along the vector field to get the solution trajectory, as in Figure 2.2. A sketch of the different trajectories in phase space is called a *phase portrait*. Indicate the direction of time on phase portraits by an arrow denoting the direction of increasing time along the trajectory. In some cases, some trajectories can be obtained explicitly by solving ODEs with time t eliminated; otherwise, general behaviour of the underlying system can be inferred by "connecting" the vector field given by $f(x)$.

Example 2.6 (Equations governed by trajectories)**.** Consider the system

$$\dot{x} = -y, \quad \dot{y} = x.$$

The trajectory is governed by the different equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = -\frac{x}{y},$$

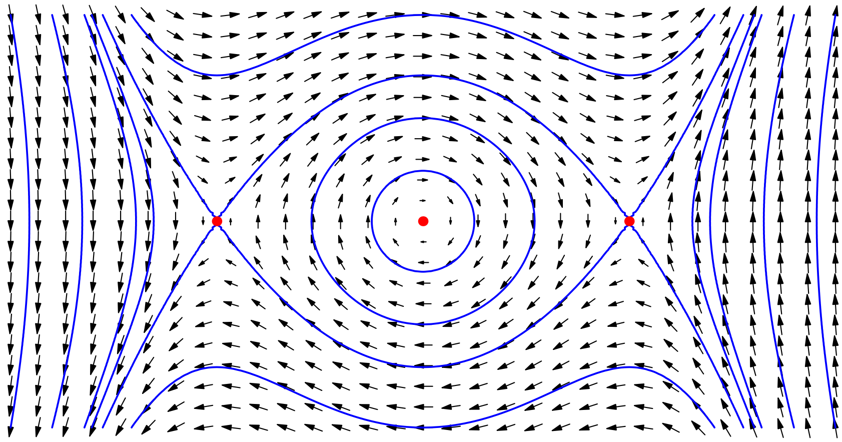


Figure 2.2: Vector fields and phase portrait for the system $\dot{x} = y, \dot{y} = -x + x^3$.

which is separable. Rewriting this ODE as $y\mathrm{d}y + x\mathrm{d}x = 0$ and integrating both sides, we get

$$x^2 + y^2 = C$$

for some constant $C > 0$.

Example 2.7 (Newtonian dynamics in one dimension)**.** Consider the Newtonian dynamics $m\ddot{x} = -U'(x)$ in one dimension (so x is a scalar, m is the mass and U is called the *potential*). By introducing the momentum $p = m\dot{x}$, then the original second order scalar ODE is equivalent to the first order system

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -U'(x).$$

The trajectory, governed by the ODE $\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{mU'(x)}{p}$ is separable, is

$$\frac{p^2}{2m} + U(x) = E$$

for some constant E, called the total *energy*.

Example 2.8. For the ODE $\dot{x} = y, \dot{y} = -x + x^3$, the ODE governing the trajectory is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x + x^3}{y}.$$

Therefore the trajectories are $x^2 + y^2 - x^4/2 = C$ for some constant C (not necessarily positive).

Remark. For two dimensional system $\dot{x} = f(x, y), \dot{y} = g(x, y)$, the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$ can not always be solved explicitly. For instance, if the system in the previous example is changed to $\dot{x} = y + x, \dot{y} = -x + x^3$, there seems no expressions for general trajectories. But whenever there is a solution that written in the form $F(x, y) = C$, the function $F(x, y)$ is called a *conserved quantity* (because the time derivative $\frac{d}{dt}F(x, y)$ is zero), containing important information about the underlying system.

Remark. In most cases, it is easier to write the trajectories *implicitly* as $x^2 + y^2 = C$ or $x^2 + y^2 - x^4/2 = C$ in previous two examples. There is no need to write y as a function of x or x as a function of y.

Semi-group property for autonomous ODEs

Another way of representing solutions is via the *flow*: $x(t) = \varphi_t(x_0)$ represents the solution to $\dot{x} = f(x)$ at time t with initial condition x_0 at $t = 0$, i.e. $\varphi_0(x_0) = x_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(x_0) = f(\varphi_t(x_0)).$$

For example, the solution to the system of ODEs

$$\dot{x} = -x + y, \quad \dot{y} = -y$$

with initial condition (x_0, y_0) is given by

$$\varphi_t(x_0, y_0) = (x_0 e^{-t} + te^{-t} y_0, e^{-t} y_0).$$

For autonomous equations $\dot{x} = f(x)$, where f has no explicit dependence on t, the solution $\varphi_t(x)$ satisfies the semi-group property,

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x)) = \varphi_s(\varphi_t(x)).$$

This fact can be verified by the uniqueness of the solution to the system $\dot{x} = f(x)$, by defining two functions $\psi_1(t) = \varphi_{t+s}(x), \psi_2(t) = \varphi_t(\varphi_s(x))$. Then $\psi_1(t)$ is a solution to $\dot{x} = f(x)$ with initial condition $\psi_1(0) = \varphi_s(x)$ and $\psi_2(t)$ is also a solution to $\dot{x} = f(x)$ with initial condition $\psi_2(0) = \varphi_0(\varphi_s(x)) = \varphi_s(x)$. Since $\psi_1(0) = \psi_2(0)$, by the uniqueness of solutions to ODEs, $\psi_1(t) = \psi_2(t)$, or $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$. Similarly, we can show $\varphi_{t+s}(x) = \varphi_s(\varphi_t(x))$.

Example 2.9. The solution to the ODE $\dot{x} = x^2, x(0) = x_0$ satisfies the semi-group property. In fact, this is an separable ODE. Integrating both sides of $x^{-2}\mathrm{d}x = dt$, we get

$$t = \int_0^t \mathrm{d}t = \int_{x_0}^{x(t)} \frac{\mathrm{d}x}{x^2} = \frac{1}{x_0} - \frac{1}{x(t)}.$$

That is $\varphi_t(x_0) = \frac{x_0}{1-tx_0}$ and $\varphi_s(\varphi_t(x_0)) = \varphi_s\left(\frac{x_0}{1-tx_0}\right) = \frac{\frac{x_0}{1-tx_0}}{1-s\frac{x_0}{1-tx_0}} = \frac{x_0}{1-(t+s)x_0} = \varphi_{t+s}(x_0)$.

Remark. The reason we use *semi-group* instead of *group* here is that some dynamical systems can not be defined backward in time, or lose the uniqueness of solution when solving backward in time (common for infinite dimensional systems, like partial differential equations).

Remark. The solution of any *autonomous* system always satisfies the semi-group property (the law of dynamics does not dependent on "time"); on the other hand, if a function $\varphi_t(x_0)$ satisfies the semi-group property, then it is the solution to the first order autonomous system $\dot{x} = f(x), x(0) = x_0$. The function f, or the "law of dynamics" can be actually determined by writing $\frac{d}{dt}\varphi_t(x_0)$ as a function $f(\varphi_t(x_0))$. For instant, if $\varphi_t(x_0) = x_0/(1 - tx_0)$, then $\frac{d}{dt}\varphi_t(x_0) = x_0^2/(1-tx_0)^2 = (\varphi_t(x_0))^2 = f(\varphi_t(x_0))$ with $f(x) = x^2$, the same as in Example 2.9 (there is no explicit t dependence). Alternatively, $f(x)$ can be determined at the initial time (the law can be inferred from any instance of time). That is, $f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\big|_{t=0}$.

2.3 Special solutions: fixed points and periodic orbits

Special solutions, if they exists, usually give a lot of information about the general behaviour of the underlying system. There are two obvious special solutions for $\dot{x} = f(x)$ arising in practice:

Stationary (or fixed) points: A *stationary point* x^* satisfies

$$x(t) \equiv x^*, \tag{2.4}$$

i.e. the trajectory is a single point and the solution does not change in time. Thus

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = f\big(x(t)\big) = f(x^*),$$

and stationary points can be found by solving the algebraic equation $f(x^*) = 0$.

Periodic Orbits: if there exists $T > 0$ such that

$$x(t+T) = x(t) \quad \text{for all } t \in \mathbb{R}.$$

then the trajectory is called a periodic orbit and T is called the period of the periodic orbit. Note that kT is also a period for any positive integer k because $x(t+kT) = x(t)$, and sometimes T is referred as the *minimal period*). A *periodic orbit* with $T = 0$, which is not allowed in the definition, would be a stationary point. Periodic orbits are much harder to find, and they form *closed curves* in phase space.

Example 2.10 (Fixed points of linear constant coefficient ODEs)**.** If A is a non-singular $n \times n$ matrix, then the only fixed point is the origin. In other words, the only solution to $Ax = 0$ is $x = 0$.

Exercise. What if the coefficient matrix A is singular as in

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Find the solution starting from (x_0, y_0). What are the fixed points?

Example 2.11 (Fixed points of potential dynamics)**.** Consider the Newton's equation $m\ddot{x} = -\nabla U(x)$ in n-dimensional space (the force is derived from the potential U), which is equivalent to the first order system of $2n$ equations ($p = m\dot{x}$ is the linear momentum):

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = -\nabla U(x).$$

Then any fixed point takes the form $(x^*, p^* = 0)$, where $\nabla U(x^*) = 0$. For those who took courses in mechanics, the fixed point is stable if x^* at the local minimum (bottom of the potential well), and unstable if x^* is at a saddle point.

Example 2.12 (Harmonic oscillation)**.** The simplest example of periodic phenomenon is the motion of a harmonic oscillator, $\ddot{x} + \omega^2 x = 0$, or the equivalent first order system

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x.$$

The only fixed point is the origin, but there are many periodic orbits around the origin. In fact, the solution can be written as

$$x(t) = A\cos\omega t + B\sin\omega t.$$

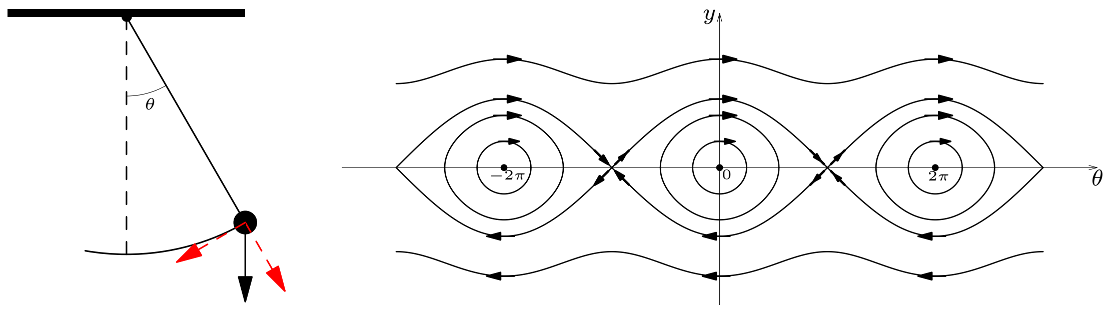


Figure 2.3: The simple pendulum and associated phase portrait.

Example 2.13 (Simple Pendulum)**.** Consider the pendulum in Figure 2.3. By taking components of the force in the radial direction, the equation of motion is

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

or the first order system (by introducing $y = \dot{\theta}$)

$$\dot{\theta} = y, \qquad \dot{y} = -\frac{g}{\ell}\sin\theta.$$

So phase space is $\mathbb{R}^2$, or more precisely the cylinder $\mathbb{T} \times \mathbb{R}$ with $\theta \in [0, 2\pi)$ (here θ is taken modulo 2π). The solution can not be represented using elementary functions, but can be given in terms of more special ones called *elliptic functions*.

Stationary points are given by solving $\dot{\theta} = \dot{y} = 0$, i.e. $y = 0$ and $\sin\theta = 0$, so the stationary points are (see Figure 2.3)

$$(k\pi, 0) \quad k \in \mathbb{Z}.$$

The only fixed point is the origin, but there are many periodic orbits around the origin.

The simple pendulum equation has special properties that make it easier to sketch the phase portrait than for more general systems: the energy (also called *Hamiltonian*)

$$E = \frac{1}{2}y^2 - \frac{g}{\ell}\cos\theta$$

is constant on solutions, which is determined from the initial condition $(\theta_0, y_0 = \dot{\theta}(0))$. This can be seen by differentiating both sides with respect to time (using the chain rule on the right hand side):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = y\,\dot{y} + \dot{\theta}\frac{g}{\ell}\sin\theta = \frac{g}{\ell}(-y\ \sin\theta + y\ \sin\theta) = 0.$$

Example 2.14 (Prey-predator system)**.** Let x and y be the population number of prey (for example, rabbits) and predator (for example, foxes), then the simplest system of ODEs is

$$\dot{x} = (A - By)x, \qquad \dot{y} = (Cx - D)y, \tag{2.5}$$

where A, B, C, D are all positive constants. The fixed points are

$$(0,0), \qquad \left(\frac{D}{C}, \frac{A}{B}\right).$$

2.4 Invariant sets

For the ODE $\dot{x} = f(x)$, a set $\mathcal{S} \subseteq \mathbb{R}^n$ is called *invariant*, if $x_0 \in \mathcal{S}$ implies the solution $x(t) = \varphi_t(x_0) \in \mathcal{S}$ with initial condition $x(0) = \varphi_0(x_0) = x_0$ for all $t \geq 0$. The basic idea behind invariant sets is: if you start in the set, you stay in the set. Common invariant sets include:

(1) Single/multiple stationary points

(2) Periodic orbits

(3) Trajectory passing one or more specific points

$$\mathcal{S}_+(x_0) = \{\varphi_t(x_0) \mid t \geq 0\} \qquad \text{or} \qquad \mathcal{S}(x_0) = \{\varphi_t(x_0) \mid t \in \mathbb{R}\}.$$

Example 2.15. The unit circle $x^2 + y^2 = 1$ is invariant for the system

$$\dot{x} = -x + y + x(x^2 + y^2), \qquad \dot{y} = -x - y + y(x^2 + y^2).$$

In other words, if (x_0, y_0) is on the unit circle, then the solution is also on the unit circle for any time $t > 0$. Therefore, we only need to show that $x^2 + y^2$ does not change (always unit), for all time. Taking the time derivative of $x^2 + y^2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 2x\big(-x + y + x(x^2 + y^2)\big)&\\ + 2y\big(-x - y + y(x^2 + y^2)\big) = 2(x^2 + y^2)(x^2 + y^2 - 1) = 0.&\end{aligned}$$

That is, $x^2 + y^2$ does not change in time if (x, y) is on the circle, and $x^2 + y^2 = 1$ for all time.

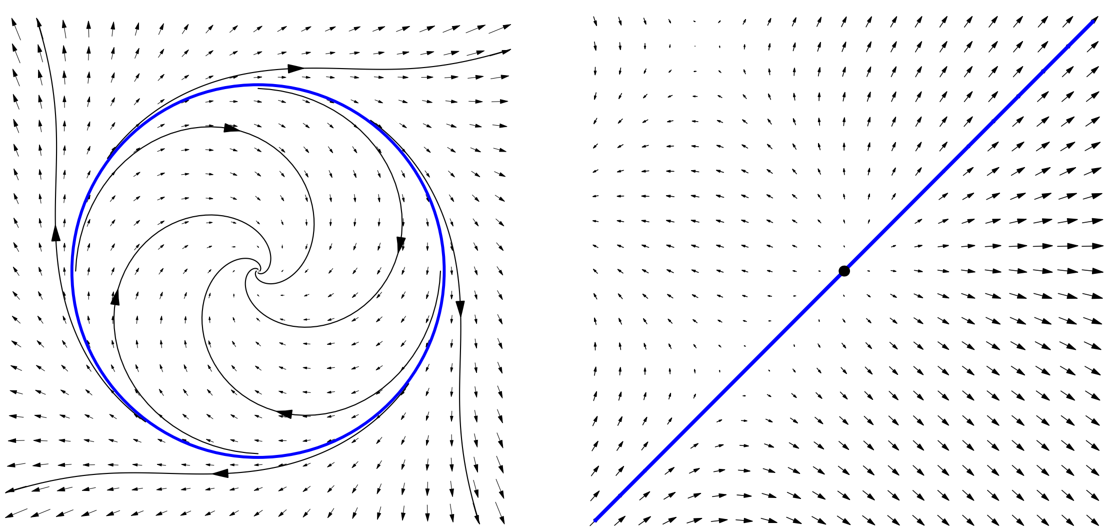


Figure 2.4: Invariant circle for Example 2.15 and invariant straight line for Example 2.17.

This example shows an important fact: a set $\mathcal{S} = \{x \mid G(x) = 0\}$ is *invariant* iff

$$\frac{\mathrm{d}G}{\mathrm{d}t} = f \cdot \nabla G = 0 \qquad \text{on} \quad G(x) = 0.$$

Geometrically, ∇G is the normal to the curve $\mathcal{S}$, and $f \cdot \nabla G = 0$ means that the vector field defining the ODE is orthogonal to the normal.

Example 2.16 (Prey-predator system)**.** The two coordinate axis $\mathcal{S}_x = \{(x, 0)\}, \mathcal{S}_y = \{(0, y)\}$ are two invariant sets of the system

$$\dot{x} = (A - By)x, \qquad \dot{y} = (Cx - D)y.$$

If the initial condition (x_0, y_0) is on the x-axis, then $y_0 = 0$, then the solution $(x(t), y(t))$ stays on the x-axis, or $y(t) \equiv 0$, because $\dot{y} \equiv 0$. Alternatively, from the second ODE $\dot{y}(t) = (Cx(t) - D)y(t)$, we can "solve" $y(t)$ (assuming $x(t)$ is known, this is a linear ODE)

$$y(t) = y_0 \exp\left[\int_0^t (Cx(\tau) - D)\mathrm{d}\tau\right] = 0.$$

Therefore, the x-axis is invariant. Similarly, we can show y-axis is invariant.

Example 2.17. We can show that the line $y = 2x$ is invariant under the system

$$\dot{x} = \frac{5}{2}x - \frac{1}{2}y + 2x^2 + \frac{1}{2}y^2, \qquad \dot{y} = -x + 2y + 4xy.$$

Geometrically, the line $y = 2x$ is a trajectory on the phase portrait.

Define $G(x, y) = y - 2x$, so the line is $\mathcal{S} = \{(x, y) \mid G(x, y) = 0\}$. We can look at the evolution of the function G under the system,

$$\begin{aligned}\dot{G}(x, y) &= \dot{y} - 2\dot{x} \\ &= (-x + 2y + 4xy) - (5x - y + 4x^2 + y^2) \\ &= -6x + 3y - 4x^2 + 4xy - y^2\end{aligned}$$

If $(x, y) \in \mathcal{S}$, $G = 0$ and $y = 2x$, which implies that

$$\dot{G}|_{G=0} = -6x + 6x - 4x^2 + 8x^2 - 4x^2 = 0.$$

In physics, the invariance of a set is generally related to the conservation of some quantities, as shown in the following three examples.

Example 2.18 (Conservation of energy for Newtonian potential dynamics)**.** If the force F of an particle with mass m is derived from a potential (gravitational potential or electric potential), that is $F(x) = \nabla U(x)$ for some U, then the Newton's equation becomes $\ddot{x} = F(x) = -\nabla U(x)$. Introduce the (linear) momentum $p = m\dot{x}$, then the dynamics is governed by the equivalent first order system

$$\dot{x} = p/m, \qquad \dot{p} = -\nabla U(x).$$

Then the total energy (also call *Hamiltonian*) $E(x, p) = \dfrac{p^2}{2m} + U(x)$ is conserved, and the dynamics is on the *constant energy surface.*

Example 2.19. We can show that the (open) unit disk $\{(x, y) \mid x^2 + y^2 < 1\}$ is invariant for the system

$$\dot{x} = -x + y, \quad \dot{y} = -x - y.$$

By the definition, we need to show that if the initial condition (x_0, y_0) is on the unit disk (i.e. $x_0^2 + y_0^2 < 1$), then $x(t)^2 + y(t)^2 < 1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2 + y^2\right) = 2x\dot{x} + 2y\dot{y} = 2x(-x+y) + 2y(-x-y) = -2(x^2+y^2) \leq 0,$$

the quantity $x(t)^2 + y(t)^2$ is non-increasing. In other words, $x(t)^2 + y(t)^2 \leq x_0^2 + y_0^2 < 1$. Therefore, the point $(x(t), y(t))$ stays on the unit disk.

Example 2.20 (Bounding functions)**.** The previous example can be generalised into the concept of bounding functions. Let $V(x, y) = c$ be a set of nested regions with c increasing outwards, that is $\{x \in \mathbb{R}^n \mid V(x) \leq c_1\} \subset \{x \in \mathbb{R}^n \mid V(x) \leq c_2\}$ for $c_1 \leq c_2$. If

$$f \cdot \nabla V < 0 \qquad \text{on } V(x) = c$$

for some c, then the set $\{x \in \mathbb{R}^n | V(x) \leq c\}$ is invariant. The idea of the proof is very simple (we will cover it in more detail later): if $f \cdot \nabla V < 0$ then f must point inwards along the surface and so no solutions can leave the region $V < c$ across the surface.

2.5 Existence and uniqueness

We have been assuming the existence of solutions of dynamical systems without comment. However this is not necessarily straightforward and needs to be examined in more depth. As the next set of examples show, solutions for ODEs may be difficult to pin down!

Different phenomena in ODEs: We will give a sequence of examples showing how complications can arise in ODEs.

Example 2.21 (Non-uniqueness with continuous right hand side)**.** Consider the ODE $\dot{x} = \sqrt{|x|}, \quad x_0 = 0$. By observation $x(t) = 0$ is a solution (a stationary point). On the other hand, using separation of variables, we get

$$\int^x \frac{dx}{\sqrt{|x|}} = \int^t dt.$$

If $x \geq 0$, both sides of above equation become $2(\sqrt{x} - \sqrt{x_0}) = t$. Therefore $x(t) = t^2/4$ is a different solution other than the trivial one $x(t) \equiv 0$! Even worse, we have a family of functions $x_\tau(t)$ for $\tau \geq 0$, defined by

$$x_\tau(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \tau, \\ (t-\tau)^2/4, & \text{if } t > \tau, \end{cases}$$

as can be verified by direct substitution (check it!). The main issue responsible for the non-uniqueness here is that $f(x) = \sqrt{|x|}$ is *not* Liptchitz continuous.

Example 2.22 (Finite time blow up)**.** Consider the differential equation

$$\dot{x} = x^2$$

with solutions $\int \frac{dx}{x^2} = t$ or $x = \frac{x_0}{1 - x_0 t}$. Thus if $x_0 > 0$ then solutions tend to infinity as $t \to x_0^{-1}$.

These examples show that we need a better understanding of existence of solutions. To show the existence, we first convert the ODE $\dot{x} = f(x,t),\ x(0) = x_0$ into an integral equation

$$x(x_0, t) = x_0 + \int_0^t f(x(s), s) ds, \tag{2.6}$$

which can be verified by differentiating and using the Fundamental Theorem of Calculus. Of course, if we do not know $x(s)$, $0 \le s < t$, then this does not help as we cannot evaluate the integral. Instead we consider the iteration

$$x^{(n+1)}(t) = T[x^{(n)}](t), \qquad \text{where } T[x](t) = x_0 + \int_0^t f(x(s), s) ds \tag{2.7}$$

with the initial condition $x^{(0)}(t) = x_0$. If the sequence of functions $\{x^{(n)}(t)\}$ converges to some function $\bar{x}(t)$, then taking the limit of both sides of (2.7), we get $\bar{x}(T) = T[\bar{x}](t)$, or $\bar{x}$ is a fixed point of the operator T. Taking derivative of both sides of $\bar{x}(T) = T[\bar{x}](t)$, we can show that $\bar{x}$ is a solution of the ODE $\dot{x} = f(x,t)$ with initial condition $\bar{x}(0) = x_0$. This is called *Picard Iteration*, and so if we can show that T defined in (2.7) is a contraction mapping then we have an existence theorem. On the assumption that this can be done Picard iteration also provides a way of constructing approximate solutions locally (See Figure 2.5).

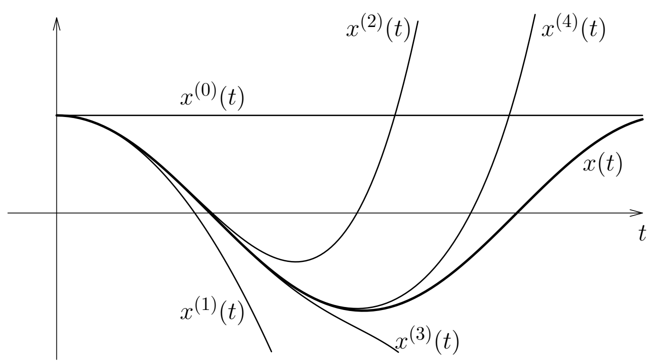


Figure 2.5: The sequence of function $x^{(n)}(t)$ converges to the exact solution $x(t)$.

Example 2.23. For the ODE $\dot{x} = ax$ with initial condition $x(0) = 1$. The Picard iteration is

$$x^{(n+1)}(t) = 1 + \int_0^t a x^{(n)}(s) ds$$

with $x^{(0)}(t) = x_0 = 1$. Therefore,

$$\begin{aligned}
x^{(1)}(t) &= x_0 + \int_0^t a x^{(0)}(s) ds = 1 + \int_0^t a ds = 1 + at, \\
x^{(2)}(t) &= x_0 + \int_0^t a x^{(1)}(s) ds = 1 + \int_0^t a(1 + as) ds = 1 + at + \frac{a^2}{2} t^2, \\
x^{(3)}(t) &= x_0 + \int_0^t a x^{(2)}(s) ds = 1 + \int_0^t a \left(1 + as + \frac{a^2}{2} s^2\right) ds = 1 + at + \frac{a^2}{2} t^2 + \frac{a^3}{3!} t^3.
\end{aligned}$$

In fact, we can show that

$$x^{(n)}(t) = 1 + at + \cdots + \frac{a^n}{n!}t^n$$

which converges to the exact solution $x(t) = e^{at}$. The n-th iteration is exactly the n-th Taylor series expansion at the start time $t = 0$. In general, higher order (greater than n) terms may appear in $x^{(n)}(t)$, which may not agree with the Taylor expansion with more than n terms, as shown in the following example.

Example 2.24. Find a power series expansion for solutions to

$$\dot{x} = x - x^2, \quad x_0 = 2$$

correct up to and including cubic terms. Set $x^{(0)}(t) = 2$. Then

$$x^{(1)}(t) = 2 + \int_0^t (2 - 2^2)\mathrm{d}s = 2 - 2t.$$

Continuing

$$x^{(2)}(t) = 2 + \int_0^t \left[(2-2s) - (2-2s)^2\right]\mathrm{d}s = 2 + \int_0^t (-2 + 6s - 4s^2)\mathrm{d}s = 2 - 2t + 3t^2 - \tfrac{4}{3}t^3.$$

Although the cubic term appears in $x^{(2)}(t)$, its coefficient is not that in the Taylor series, and will be correct in the next iteration. That is,

$$\begin{aligned} x^{(3)}(t) &= 2 + \int_0^t (2 - 2s - 3s^2 + \dots) - (2 - 2s - 3s^2 + \dots)^2 \mathrm{d}s \\ &= 2 + \int_0^t \left(-2 + 6s - 13s^2 + 16s^3 - \frac{43}{3}s^4 + 8s^5 - \frac{16}{9}s^6\right)\mathrm{d}s \\ &= 2 - 2t + 3t^2 - \tfrac{13}{3}t^3 + 4t^4 + \cdots, \end{aligned}$$

which is correct to the cubic term. This ODE can be integrated explicitly (a separable ODE) to give

$$x(t) = \frac{2}{2 - e^{-t}} = 2 - 2t - \frac{13}{3}t^3 + \frac{25}{4}t^4 - \frac{541}{60}t^5 + O(t^6).$$

Remark. After some technical work, the existence of solution can be established using the above Picard Iteration scheme $x^{(n+1)}(t) = T[x^{(n)}](t)$ by taking the limit as n goes to infinity. We will focus on qualitative properties in the rest of the course[2], and the interested readers may consult Chapter 3 in Meiss's book *differential dynamical systems*.

[2] You can safely ignore related questions about the existence and uniqueness of ODEs in the past papers.

3 Linearisation and equilibria

In this section, we will study mainly the properties of linear systems around fixed points, as a first step towards the understanding of more complicated behaviours of nonlinear systems. The local, linear part can be obtained from the full nonlinear counterpart via Taylor expansion around the fixed points.

3.1 Taylor's theorem

Suppose $x^* \in \mathbb{R}^n$ is a stationary point of $\dot{x} = f(x)$, that is $f(x^*) = 0$. If $z = x - x^*$ is small, we can use Taylor's Theorem to expand $f(x)$ around x^*, that is,

$$\dot{x} = \dot{z} = f(x) = f(x^* + z) = \cancel{f(x^*)} + Df(x^*)z + O(|z|^2) = Df(x^*)z + O(|z|^2).$$

Here $Df(x^*)$ is the $n \times n$ *Jacobian matrix* with entries $[Df(x)]_{ij} = \frac{\partial f_i}{\partial x_j}(x)$. The 'big-O' notation means that if a function $g(z) = O(|z|^2)$ then $\frac{|g(z)|}{|z|^2} < C$, for some $C < \infty$, on a neighbourhood of $z = 0$. If z is small then we can hope to ignore the small $O(|z|^2)$ terms and consider the linearisation about x^*: $\dot{z} = Az$ with $A = Df(x^*)$ or $A_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$.

Example 3.1. Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x, y) = \begin{pmatrix} \frac{5}{2}x - \frac{1}{2}y + 2x^2 + \frac{1}{2}y^2 \\ -x + 2y + 4xy \end{pmatrix}, \tag{3.1}$$

for which $(0, 0)$ is a stationary point. Since

$$Df(x, y) = \begin{pmatrix} \frac{5}{2} + 4x & -\frac{1}{2} + y \\ -1 + 4y & 2 + 4x \end{pmatrix},$$

the system can be approximated by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad A = Df(0, 0) = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}.$$

which could have been read off directly from the linear part of the equation (3.1).

Key questions in the next few subsections

1. How can we characterize solutions of linear equations $\dot{x} = Ax$?
2. (Harder) How/when does information about the linearisation provide useful local information about the original (nonlinear) problem?

Example 3.2. Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}. \tag{3.2}$$

We start with the stationary points, by looking for zeros of the right hand side of the system: the second equation implies that $x = 0$; substituting it back into the first equation, we get $y = -1$. Therefore, the only stationary point is $(0, -1)$.

From $f(x, y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$, $Df(x, y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$ and $Df(0, -1) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$. The linearisation in coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$ is $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ or

$$\dot{u} = v, \qquad \dot{v} = bu.$$

This linsear system can be solved by eliminating v (or alternatively using matrix exponential): $\ddot{u} = \dot{v} = bu$, or $\ddot{u} - bu = 0$. Using elementary methods in ODEs, if $b > 0$ the solutions are

$$u = Ae^{\sqrt{b}t} + Be^{-\sqrt{b}t} \quad \text{with} \quad v = \dot{u} = \sqrt{b}(Ae^{\sqrt{b}t} - Be^{-\sqrt{b}t}),$$

for constants A and B determined from the initial condition. Most solutions are unbounded (with general A and B). But the special solution with $A = 0$ converges to the origin.

If $b < 0$ then

$$u = A\cos\sqrt{|b|}t + B\sin\sqrt{|b|}t \quad \text{and} \quad v = \sqrt{|b|}\big(-A\sin\sqrt{|b|}t + B\cos\sqrt{|b|}t\big)$$

i.e. solutions of the linearisation oscillate in time.

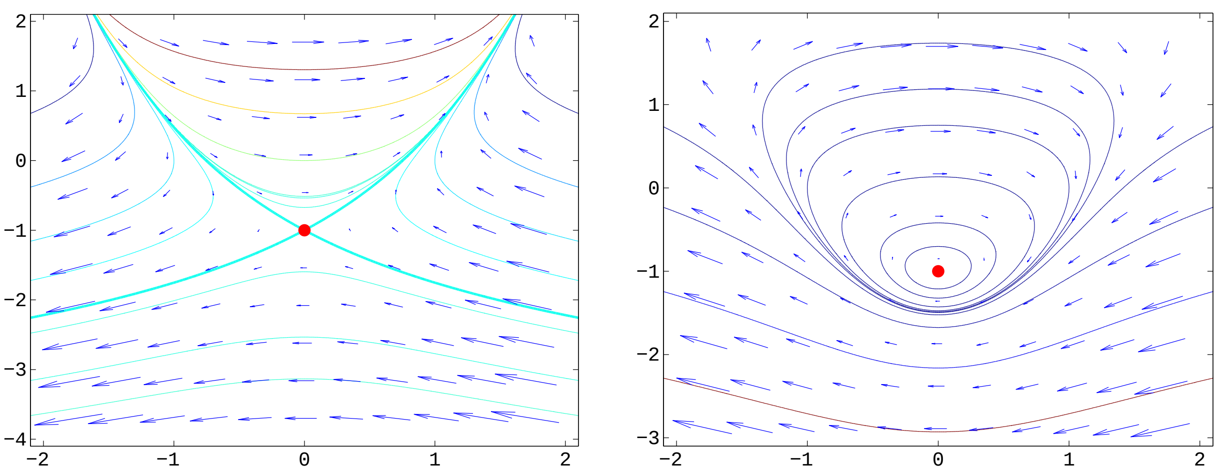


Figure 3.1: The phase portrait for two systems: (left figure) $\dot{x} = 1 - x^2 + y, \dot{y} = x$ and (right figure) $\dot{x} = 1 - x^2 + y, \dot{y} = -x$, with common fixed point $(0, -1)$.

Question: When does this linear analysis give accurate information about the behaviour of the full nonlinear problem? It will turn out that the behaviour if $b > 0$ is a good sense of the general behaviour (locally) whilst this may not be the case if $b < 0$. For instance, the trajectories of the system $\dot{x} = 1 - x^2 + y, \dot{y} = -x - x^2$ around the fixed point $(0, -1)$ are spirals.

For the above system $\dot{x} = 1 - ax^2 + y, \dot{y} = bx$, you can show that

$$V(x, y) = (2a^2x^2 - 2a + b - 2ay)\exp(2ay/b)$$

is conserved under the full system (3.2), and the trajectories are governed by $V(x, y) = C$ for different constants C.

3.2 Linear systems

Suppose $x(t)$ satisfies the linear ODEs $\dot{x} = Ax$, where A is a constant $n \times n$ matrix and $x \in \mathbb{R}^n$. If A has distinct eigenvalues λ_i with corresponding eigenvectors e_i, then the general solution is a superposition of the eigenmodes:

$$x(t) = \sum C_k e^{\lambda_k t} e_k$$

where the C_k are constants determined from the initial condition $x(0) = \sum C_k e_k$. This shows that eigenvalues and eigenvectors of A will be crucial to the understanding of the dynamics.

The eigen-pairs are also closely related to a particular coordinate transformation that simplifies the dynamics: $x = [e_1 \ e_2 \ \cdots \ e_n]y$, or $y = Ux$ with $U = [e_1 \ e_2 \ \cdots \ e_n]^{-1}$, the inverse of the matrix formed by the eigenvectors. The the ODE for y becomes

$$\dot{y} = U\dot{x} = UAx = UAU^{-1}y,$$

i.e. y satisfies a linear ODE $\dot{y} = UAU^{-1}y$.

The above choice of $U = [e_1 \ e_2 \ \cdots \ e_n]^{-1}$ is particular in that $UAU{-}1$ is *diagonal*, and the system $\dot{y} = UAU{-}1y$ is essential n decoupled ODEs:

$$\dot{y}_1 = \lambda_1 y_1, \quad \dot{y}_2 = \lambda_2 y_2, \quad \cdots \quad \dot{y}_n = \lambda_n y_n.$$

With this 'natural' choice of transformation $y = Ux$, the resulting system for y is called the *normal form* (depending only on the eigenvalues of A). We will work in the plane $\mathbb{R}^2$ with real matrix A, though extension to $\mathbb{R}^n$ is not hard.

a) <u>eigenvalues real and distinct:</u> Suppose the eigenvalues λ_1 and λ_2 of A are real and distinct, then their corresponding eigenvectors e_1, e_2 (assumed to be column vectors) are real and linearly independent. With the matrix $U = [e_1, e_2]^{-1}$, we get

$$AU^{-1} = A[e_1 \ e_2] = [Ae_1 \ Ae_2] = [\lambda_1 e_1 \ \lambda_2 e_2] = [e_1 \ e_2]\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix} = U^{-1}\mathrm{diag}(\lambda_1, \lambda_2).$$

Left multiplying both sides with U, we have $UAU^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$ as expected.

As we shall see shortly, this transformation into normal form also makes it easier to understand the structure of the solutions, which depends on the signs of λ_1 and λ_2.

a i) $\lambda_1 < \lambda_2 < 0$: stable node In this case the linearisation in the normal form coordinates $y^t = (u, v)$ is

$$\dot{u} = \lambda_1 u, \quad \dot{v} = \lambda_2 v$$

with solutions

$$u = u_0 e^{\lambda_1 t}, \quad v = v_0 e^{\lambda_2 t}.$$

Thus $(u, v) \to (0, 0)$ as $t \to \infty$ and both coordinate axis ($u = 0$ and $v = 0$) are invariant.

Moreover, if $u_0, v_0 \neq 0$ (i.e. off the coordinate axes)

$$\frac{u}{u_0} = e^{\lambda_1 t} = (e^{\lambda_2 t})^{\frac{\lambda_1}{\lambda_2}}; \qquad \frac{v}{v_0} = e^{\lambda_2 t}$$

and so

$$\frac{u}{u_0} = (\frac{v}{v_0})^{\frac{\lambda_1}{\lambda_2}}, \qquad \frac{\lambda_1}{\lambda_2} > 1, \tag{3.3}$$

or equivalently $uv^{-\lambda_1/\lambda_2}$ is a constant for points on the same trajectory. These are generalized parabolas, tangential to the v-axis at $(u,v) = (0,0)$.

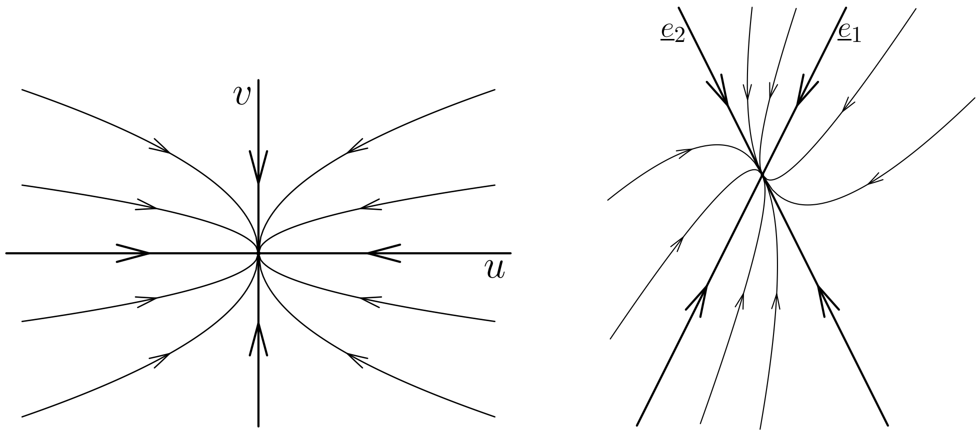


Figure 3.2: Stable node in transformed (u,v)-coordinates and in the original (x_1,x_2)-coordinates.

This is called a *stable node*. In the original coordinates the u-axis corresponds to $\underline{e}_1$ and the v-axis to the $\underline{e}_2$ direction (you can see this from the transformation $y = Ux$), so the phase portrait is as shown in Figure 3.2.

Thus in the original coordinates, lines corresponding to eigenvectors are invariant. Moreover almost all trajectories are tangential to $\underline{e}_2$ at $(0,0)$, i.e. tangential to eigenvector of eigenvalue with smallest modulus.

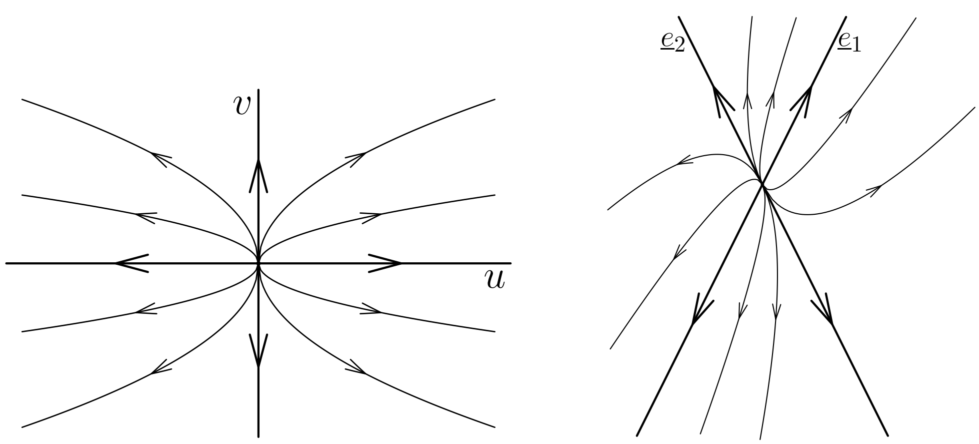


Figure 3.3: Unstable node in transformed (u,v)-coordinates and in the original (x_1,x_2)-coordinates.

a ii) $\lambda_1 > \lambda_2 > 0$**: unstable node** The phase portrait can be obtained using the same techniques as in the previous section. Indeed the manipulations are the same and the generalized parabola is also the same as changing the signs of both eigenvalues does not change the sign of their ratio. Another way of seeing the direct correspondence with the previous case is by reversing time. Set $\tau = -t$ so

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = -\frac{\mathrm{d}}{\mathrm{d}\tau}$$

and so if

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \lambda_1 u, \quad \frac{\mathrm{d}}{\mathrm{d}t}v = \lambda_2 v$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u = -\lambda_1 u, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}v = -\lambda_2 v$$

which is the same as in case ai). Thus all we need to do is to change the direction of time, i.e. the arrows on the phase portraits to get the new phase portrait. This is called an *unstable node*, as shown in Figure 3.3.

a iii) $\lambda_1 < 0 < \lambda_2$: saddle The analysis is as before but now $\frac{u}{u_0} = (\frac{v}{v_0})^{\frac{\lambda_1}{\lambda_2}}$ is a generalized hyperbola as $\lambda_1/\lambda_2 < 0$, as shown in Figure 3.4.

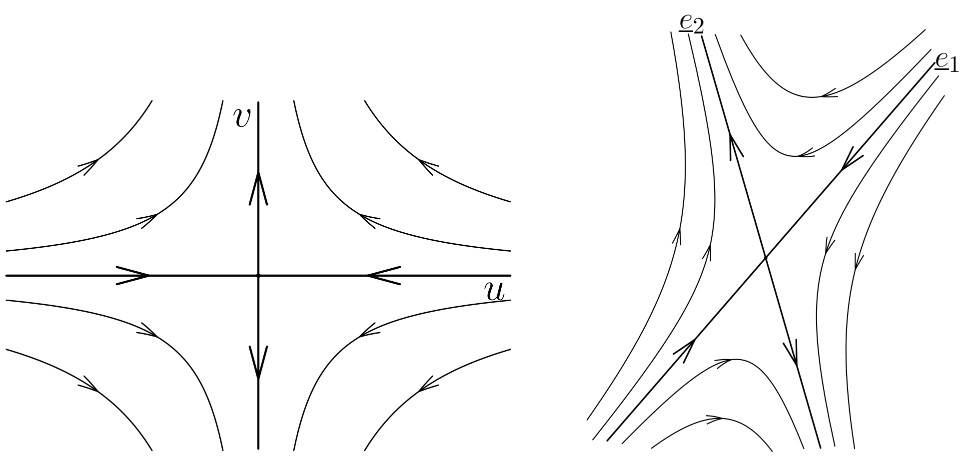


Figure 3.4: Saddle node in transformed (u, v)-coordinates and in the original (x_1, x_2)-coordinates.

b) Complex conjugate eigenvalues $\rho \pm i\omega, \omega \neq 0$. The eigenvectors $z_\pm$ are *complex*, and satisfy

$$Az_\pm = (\rho \pm i\omega)z_\pm. \tag{3.4}$$

But we prefer to work with *real* quantities, and the first step is to take the real and imaginary parts of both sides of (3.4) (only with z_+),

$$A(\mathrm{Re}z_+ + i\mathrm{Im}z_+) = (\rho + i\omega)(\mathrm{Re}z_+ + i\mathrm{Im}z_+),$$

or equivalently

$$A\mathrm{Re}z_+ = \rho\mathrm{Re}z_+ - \omega\mathrm{Im}z_+, \qquad A\mathrm{Im}z_+ = \rho\mathrm{Im}z_+ + \omega\mathrm{Re}z_+.$$

To proceed, we take real and imaginary parts of the above eigenvector $z_\pm$ (remember that the real parts of $z_\pm$ are the same, and the imaginary parts only differ in their signs), forming the matrix

$$U = [\mathrm{Re}z_+ \ , \ \mathrm{Im}z_+]^{-1} \qquad \text{or} \qquad U^{-1} = [\mathrm{Re}z_+ \ , \ \mathrm{Im}z_+] \, .$$

Then

$$\begin{aligned} AU^{-1} &= [A\,\mathrm{Re}z_+, \ A\,\mathrm{Im}z_+] \\ &= [\mathrm{Re}(Az_+) \ , \ \mathrm{Im}(Az_+)] \, , \qquad\qquad\qquad (\text{ as } A \text{ is real}) \\ &= [\rho\mathrm{Re}(z_+) - \omega\mathrm{Im}(z_+), \ \rho\mathrm{Im}(z_+) + \omega\mathrm{Im}(z_+)] \\ &= [\mathrm{Re}z_+, \mathrm{Im}z_-] \begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix} . \end{aligned}$$

Thus we end up with

$$\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix} = UAU^{-1},$$

where $\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}$ is the *complex normal form.*

In the new coordinates $y = (u, v)^t = Ux$, the system becomes $\dot{u} = \rho u + \omega v, \dot{v} = -\omega u + \rho v$. It is much easier to look at this system in the polar coordinates $u = r\cos\theta, v = r\sin\theta$. Differentiating this new transform gives

$$\begin{aligned} \dot{u} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta = \rho r\cos\theta + \omega r\sin\theta \\ \dot{v} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -\omega r\cos\theta + \rho r\sin\theta. \end{aligned}$$

To eliminate $\dot{\theta}$ to obtain an equation for $\dot{r}$, multiply the first of these by $\cos\theta$ and the second by $\sin\theta$ and add to get

$$\dot{r} = \rho r, \quad \text{i.e.} \quad r = r_0 e^{\rho t}. \tag{3.5}$$

Similarly to get the equation for $\dot{\theta}$, multiply the first by $\sin\theta$ and the second by $\cos\theta$ and take the difference:

$$\dot{\theta} = -\omega \quad \text{i.e.} \quad \theta = \theta_0 - \omega t,$$

which represents a clockwise rotation at constant angular velocity if $\omega > 0$. Using this to eliminate t from the equation for r shows that trajectories lie on spiral $r = r_0 e^{\rho(\theta-\theta_0)/w}$.

b i) $\rho > 0$**: unstable focus.** In this case (3.5) shows that solutions grow with time so trajectories spiral out of the origin. This is called a *unstable focus* clockwise if $\omega > 0$ (counter-clockwise if $\omega < 0$). In the original coordinates, the phase portrait is a distorted spiral. To determine the direction of spiralling, we can consider the sign of $\dot{x}_2$ on a horizontal line (where $x_2 = 0$) through the stationary point or the sign of $\dot{x}_1$ on a vertical line through the stationary point. If more detail is required the nullclines (see c(ii) below) indicate where solutions are flat or vertical as they move around the stationary point. This is called an *unstable focus* (see Figure 3.5).

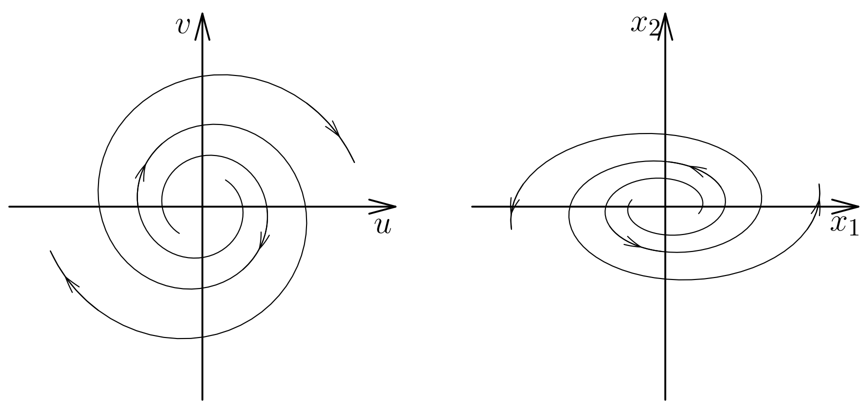


Figure 3.5: Unstable focus in transformed (u, v)-coordinates and in the original (x_1, x_2)-coordinates.

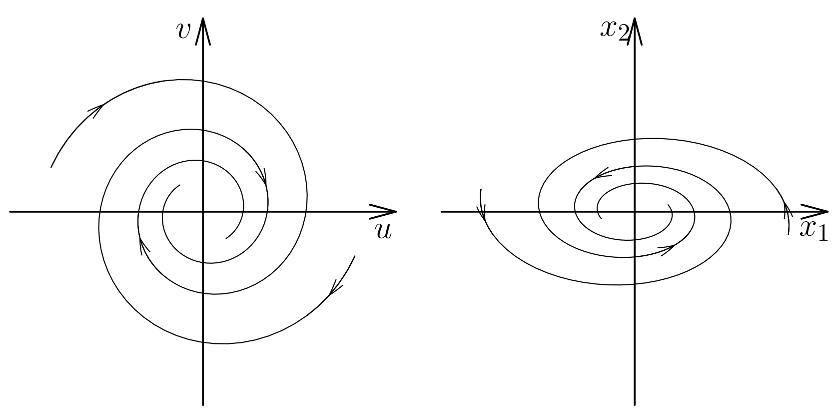


Figure 3.6: Stable focus in transformed (u, v)-coordinates and in the original (x_1, x_2)-coordinates.

b ii) $\rho < 0$ **: stable focus** In this case the θ behaviour is the same as in the previous case but the radial velocity is towards zero. Solutions tend to the origin spiralling clockwise (if $\omega > 0$) as shown in Figure 3.6. In the original coordinates the solutions spiral inwards, with the direction given by consideration of the sign of $\dot{x}_2$ (or $\dot{x}_1$) on the horizontal line (resp. vertical line) through the stationary point. If more detail is required the nullclines (see c(ii) below) indicate where solutions are flat or vertical as they move around the stationary point. This is called a *stable focus*.

b iii) $\rho = 0$**: centre** If $\rho = 0$ then $\dot{r} = 0$ and so r is constant – solutions lie on circles in the transformed (u, v)-coordinates with θ changing at a constant rate, i.e. if $r_0 \neq 0$ then solutions are periodic with period $\frac{2\pi}{|\omega|}$. This is called a *centre*, see Figure 3.7.

Clearly if nonlinear terms are added then $\dot{r}$ may no longer vanish, so we do not expect this type of behaviour to persist in typical nonlinear systems.

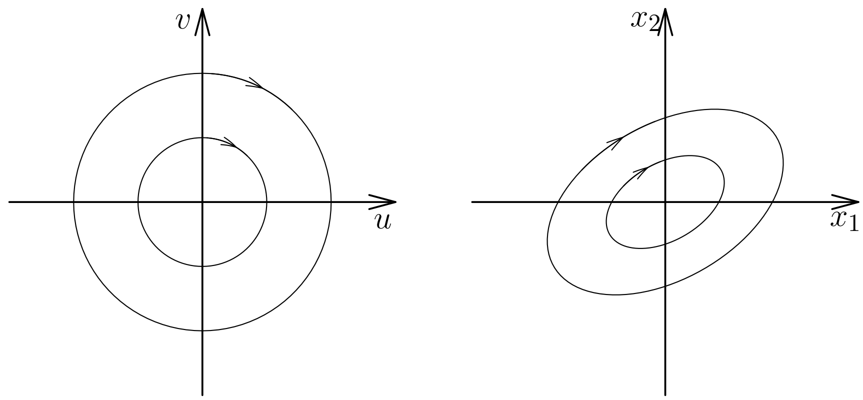


Figure 3.7: Centre in transformed (u, v)-coordinates and in the original (x_1, x_2)-coordinates.

c) Repeated real roots $\lambda \neq 0$. If the characteristic equation has two repeated roots $\lambda = \lambda_1 = \lambda_2$, then by Cayley-Hamiltonian Theorem (a matrix satisfies its own characteristic equation) $(A - \lambda I)^2 = 0$. Depending on the number of eigenvectors to the equation $(A - \lambda I)e = 0$, we have two cases (the equivalent two cases are $A = \lambda I$ and $A \neq \lambda I$).

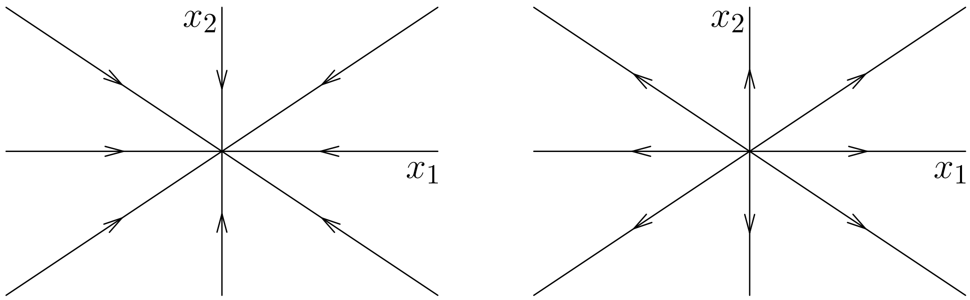
c i) Repeated real roots $\lambda \neq 0$**: star.** Suppose there are two (linearly independent) eigenvectors e_1 and e_2 to the equation $(A - \lambda I)e = 0$, then

$$A[e_1, e_2] = [\lambda e_1, \lambda e_2] = \lambda[e_1, e_2].$$

Since $[e_1, e_2]$ is non-singular, we can right multiply $[e_1, e_2]^{-1}$ to the previous equation to get

$$A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

i.e. A is diagonal in *any* basis! Then using (3.3) with $\lambda_1 = \lambda_2$, $\frac{x}{x_0} = \frac{y}{y_0}$ and so solutions lie on straight lines through the origin as shown in Figure 3.8. This is called a *stable star* if $\lambda < 0$ (so solutions tend to the origin) and an *unstable star* if $\lambda > 0$ (so solutions grow).



Stable star ($\lambda < 0$) Unstable star ($\lambda > 0$)

Figure 3.8: Stable stars and unstable stars

c ii): repeated roots $\lambda \neq 0$: degenerate node. Suppose that there is only one eigenvalue e_1 to the eigenvalue λ (although it is repeated), then we can find another vector e_2 such that $(A - \lambda I)e_2 = e_1$, or $Ae_2 = \lambda e_2 + e_1$. Let $U^{-1} = [e_1; e_2]$, the matrix with columns are the two vectors defined above. Then

$$AU^{-1} = [Ae_1, Ae_2] = [\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Hence we get the

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = UAU^{-1},$$

where the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is the *normal form* for this case of repeated roots.

In the transformed coordinates (u, v) defined as above,

$$\dot{u} = \lambda u + v, \quad \dot{v} = \lambda v. \tag{3.6}$$

The second equation is easily solved to give $v = v_0 e^{\lambda t}$. Substituting this into the first equation gives $\dot{u} = \lambda u + v_0 e^{\lambda t}$. Using the integrating factor $e^{-\lambda t}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(ue^{-\lambda t}) = v_0.$$

The integration of both sides lead to $ue^{-\lambda t} - u_0 = v_0 t$ or $u = u_0 e^{\lambda t} + v_0 t e^{\lambda t}$. It is hard to analyse solutions directly from (3.6). The phase portrait is given in Figure 3.9. They divide phase space into four regions according to the different combinations of the signs of $\dot{u}$ and $\dot{v}$. Bu considering the behaviour in each of these regions, we arrive at the phase portrait sketched. Of course, a rigorous justification takes more work, but this is enough to give a basic idea of the behaviour. The phase portrait of Figure 3.9 is an *unstable degenerate node*. For the case $\lambda < 0$ (a *stable degenerate node*), although the solution eventually converges to the origin, it may take a long excursion to finally move towards it.

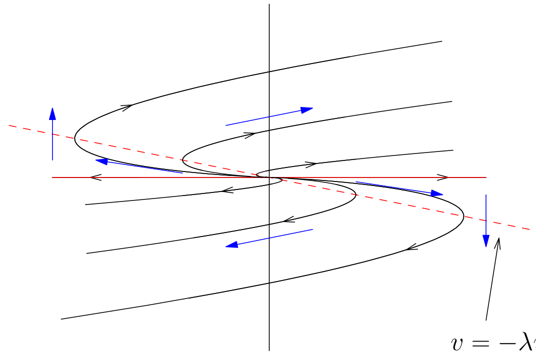


Figure 3.9: Phase portrait for $\dot{u} = \lambda u + v, \dot{v} = \lambda v$ with degenerate node ($\lambda > 0$).

Direction of rotation for foci and centres

The sign of ω in the normal form can always be chosen to be positive, but this might use a transformation that reverses the orientation of the plane, i.e. counter-clockwise rotation can be transformed into clockwise rotation. To determine the actual direction in an example either calculate the nullclines and the direction of the flow across the nullclines, or (and this is often easier) consider the direction of the flow on coordinate axes.

Example 3.3. Suppose that

$$\dot{x} = -x - 4y, \quad \dot{y} = x - y$$

so the Jacobian matrix is

$$\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

with characteristic equation $(s+1)^2 + 4 = 0$ so the eigenvalues are $-1 \pm i2$. The origin is therefore a stable node, but in which way does it rotate?

Set $x = 0$ (the y-axis) and consider how solutions move across this line. On $x = 0$ $\dot{x} = -4y$ and so if $y > 0$ then the motion is from right to left (as $\dot{x} < 0$) and if $y < 0$ (so $\dot{x} > 0$) the motion is from left to right. Thus the motion is counter-clockwise.

It is often useful to indicate this on a diagram to make sure the figure is drawn appropriately.

Summary on the relation between the signs of eigenvalues and the behaviour of the underlying linear system

Given $\dot{x} = Ax$, $x \in \mathbb{R}^2$. Find the eigenvalues of A to characterise all the behaviours of the solution summarised as below.

real, distinct: *both positive*: unstable node, almost all trajectories tangential to eigenvector of eigenvalue with smaller modulus at $(0, 0)$

both negative: unstable node, almost all trajectories tangential to eigenvector of eigenvalue with smaller modulus at $(0, 0)$

one negative one positive: saddle with both eigenvectors invariant

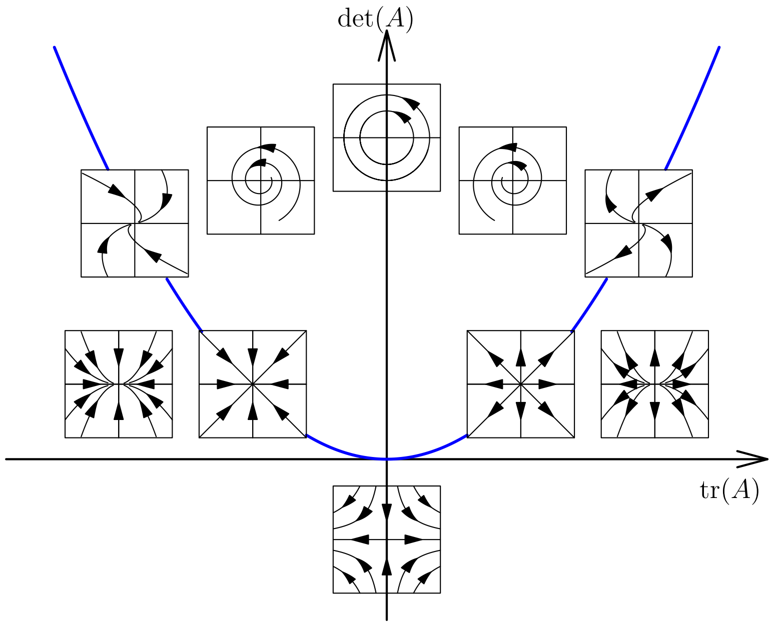


Figure 3.10: Different behaviours of the system $\dot{x} = Ax$, depending on the determinant and the trace of A.

complex conjugate pair: *positive real parts*: unstable focus

negative real parts: stable focus

zero real parts: centre

other special cases: *equal (real) eigenvalues with two eigenvectors:* star

equal (real) eigenvalues with only one eigenvectors: degenerate node otherwise, almost all trajectories tangential to eigenvector at $(0, 0)$.

For 2D systems, since the behaviours of the solutions, or equivalently of the roots depend only on the determinant and the trace of A, they can be equally summarised as in Figure 3.10, where the parabola is the curve $4\det A = (\mathrm{tr}A)^2$.

Remark. The linearised systems for higher dimensions can be analysed in a similar way.

3.3 Planar ODEs

Recall that if $x = a$ is a stationary point then we use $x = a + y$, $|y| \ll 1$ to change the coordinate to y, such that

$$\dot{y} = Df(a)y + O(|y|^2)$$

ad linearisation is $\dot{y} = Ay$, $A = Df(a)$ (Jacobian matrix of partial derivatives).

It turns out that nodes, foci and degenerate nodes retain their basic properties under small nonlinear perturbations, so this makes it possible to obtain approximate phase portraits for some systems (ignoring periodic orbits for the moment).

Example 3.4 (ODEs for competitive populations)**.** Imagine a colony of rabbits (r) and sheep (s) with $r, s \geq 0$ denoting the population size in normalized coordinates so that one unit represents many animals and we are justified in approximating the population size as a continuous variable. A model of the birth/death rates is

$$\begin{aligned} \dot{r} &= r(3 - r - s), && \text{rabbits} \\ \dot{s} &= s(4 - 2r - s), && \text{sheep} \end{aligned}$$

where $s, r \geq 0$.

We can sketch the phase portrait in three stages: first find the stationary points, then determine their types and the local phase portrait assuming the linear approximation is valid, and then put this information together to create a consistent global phase portrait.

Stationary Points: $\dot{r} = 0$ if $r = 0$ or $r + s = 3$ whilst $\dot{s} = 0$ if $s = 0$ or $2r + s = 4$. Hence the stationary points are

$$\begin{aligned} r &= 0, \quad s = 0, && (0, 0) \\ r &= 0, \quad s = 4, && (0, 4) \\ s &= 0, \quad r = 3, && (3, 0) \end{aligned}$$

together with the solution of the simultaneous equations

$$r + s = 3, \qquad 2r + s = 4$$

if they exist. Solving gives a fourth stationary point, $r = 1$ and $s = 2$, i.e. $(1, 2)$.

Note that the r-axis and the s-axis are invariant.

Linearisation:

$$Df(\underline{x}) = \begin{pmatrix} 3 - 2r - s & -r \\ -2s & 4 - 2r - 2s \end{pmatrix}$$

At $(0, 0)$:

$$Df(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

The eigenvalues are 3 and 4 with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, so it is an unstable node, with almost all solutions tangential to the r-axis and the local solution is as sketched in Figure 3.11).

At $(0, 4)$:

$$Df(0, 4) = \begin{pmatrix} -1 & 0 \\ -8 & -4 \end{pmatrix}$$

The eigenvalues are $-1, -4$ so it is a stable node. The eigenvectors are $e_{-1} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, so almost all solutions are tangential to e_{-1} at the stationary point. See Figure 3.11.

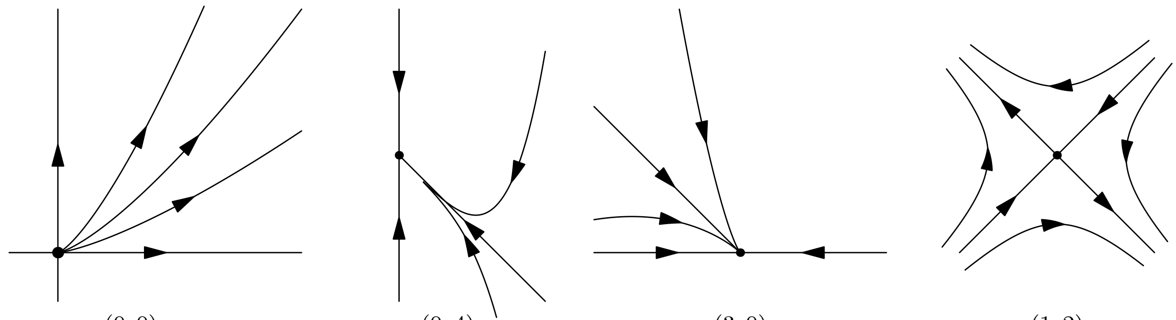


Figure 3.11: Local phase portraits near each of the stationary points.

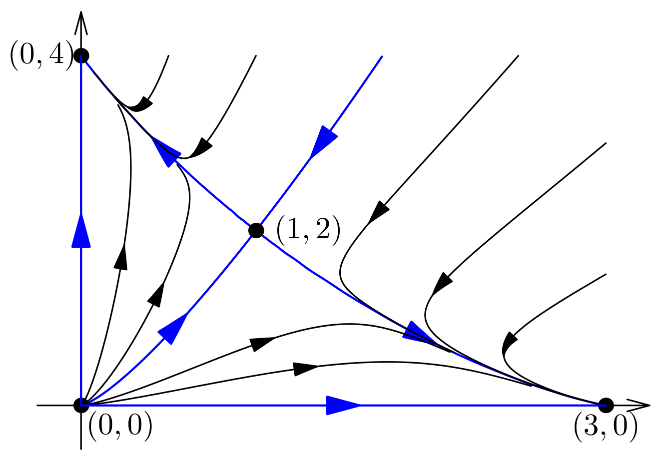


Figure 3.12: Consistent global phase portrait in $r, s \geq 0$. Note the role of the separatrices in separating regions of initial conditions tending to each of the two stable nodes.

At $(3, 0)$:

$$Df(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -2 \end{pmatrix}$$

so the eigenvalues are -3 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and -2 with eigenvector $e_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. So it is a stable node and almost all solutions are tangential to e_2 at the stationary point.

At $(1, 2)$:

$$Df(1,2) = \begin{pmatrix} -1 & -1 \\ -4 & -2 \end{pmatrix}$$

so the characteristic equation is $(s+1)(s+2) - 4 = 0$ or $s^2 + 3s - 2 = 0$, i.e. $s_\pm = \frac{-3 \pm \sqrt{17}}{2}$. Since $s_+ > 0$ and $s_- < 0$ is a saddle and the eigenvectors are $e_\pm = \begin{pmatrix} -1 \\ s_\pm + 1 \end{pmatrix}$, so e_+ slopes downwards and e_- slopes upwards.

Putting the information together suggests the global phase portrait of Figure 3.12. The important features are the separatrices which separates solutions tending to $(0, 4)$ from those approaching $(3, 0)$.and the tangential approach to approximate stationary points.

Example 3.5 (ODEs for mutualistic interactions)**.** Imagine a colony of bees (b) and flower

(f) with $b, f \geq 0$ denoting the population size in normalized coordinates so that one unit represents many animals and we are justified in approximating the population size as a continuous variable. Bees fly from flower to flower gathering nectar for food, and the flowers also benefit from the bees for pollination. This is a typical example of mutualistic interaction. A model for their population is

$$\dot{b} = (3 - 3b + f)b, \qquad \dot{f} = (1 + b - f)f. \tag{3.7}$$

All the stationary points are $(0,0), (0,1), (1,0), (2,3)$. By evaluating the Jacobian

$$Df(b,f) = \begin{pmatrix} 3 - 6b + f & b \\ f & 1 + b - 2f \end{pmatrix}$$

we have the following classification:

(0,0): unstable node

(1,0): saddle node

(0,1): saddle node

(2,3): stable node

Basically, the stability/instability of any stationary point can be implied from the linearised system, when no eigenvalue has zero real part. These points are called *hyperbolic fixed points*. Otherwise, fixed points with zero real part in their eigenvalues (like centres) are called *non-hyperbolic fixed point*. The behaviours near these stationary points are more difficult to study: while all orbits around centres are periodic, there could be no periodic solutions when nonlinear higher order terms are added.

3.4 Stability and Lyapunov functions

We have seen that if $\mathrm{Re}\lambda_i < 0$ for all eigenvalues λ_i of A, solutions of linear system $\dot{x} = Ax$ converge to the origin. In fact, if A can be diagonised with the eigenpairs (λ_i, e_i), then the solution can be written as

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} e_i,$$

while the prefactor $e^{\lambda_i t}$ goes to zero as t goes to infinity. Is this true of the corresponding nonlinear systems near the stationary point, when the solution can not be obtained in explicit form? This is part of a much more general question about stability of stationary points, so first let us introduce some definitions. The main questions to be answered in this subsection are:

(1) How do stability results for linear systems carry over to nonlinear systems locally?

(2) How about the boundedness or stability of solutions for $\dot{x} = f(x), x \in \mathbb{R}^n$.

Definition 3.1 (Asymptotic stability)**.** A stationary point x^* of an autonomous system $\dot{x} = f(x)$ is asymptotically stable iff there exists an open neighbourhood U of x^* such that $\varphi_t(x_0) \to x^*$ as $t \to \infty$ for all $x_0 \in U$.

If all eigenvalues λ_i are negative, then the solution starting at any x_0 converges to the origin. If $A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is diagonal, this asymptotical stability can also be shown alternatively by considering the (squared) distance $V(x) = |x|^2$ between the solution $x(t)$ and the origin. Since

$$\frac{d}{dt}V(x) = 2x \cdot \dot{x} = 2\sum_{i=1}^{n} \lambda_i x_i^2 \leq 0,$$

the square distance $|x(t)|^2$ is strictly decreasing, until $x(t)$ reaches the origin. This means that $V(x(t)) = |x(t)|^2$ converges to zero .

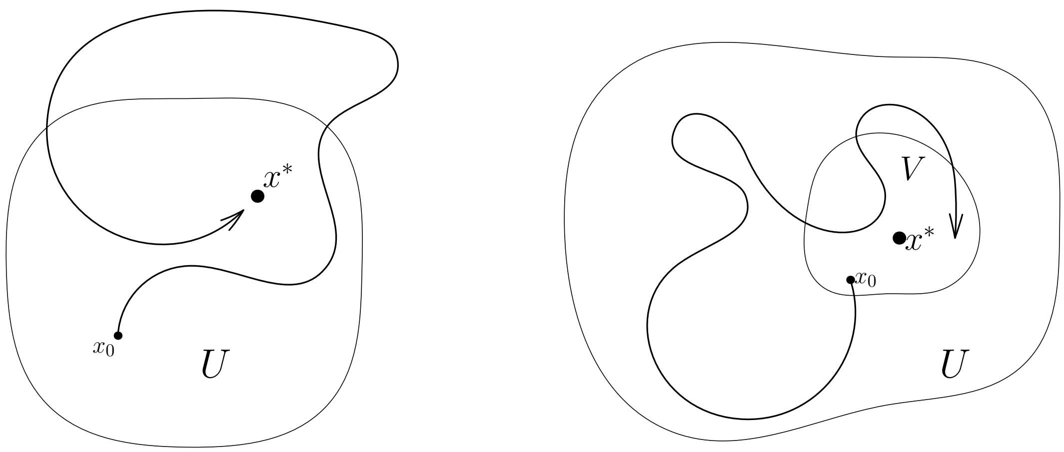


Figure 3.13: (a) Asymptotic stability; (b) Lyapunov stability.

This means that if a solution starts sufficiently close to x^*, its solution eventually becomes arbitrary close to x^*. But it does not imply the solution stays within U for all $t > 0$.

Definition 3.2 (Lyapunov stability)**.** A stationary point x^* of an autonomous ODE is Lyapunov stable iff for every open neighbourhood U of x^* there exists an open neighbourhood $W \subset U$ such that $x_0 \in W$ implies $\varphi_t(x_0) \in U$ for all $t \geq 0$.

In other words solutions stay close to x^* if they start close enough to x^*. Note that Lyapunov stability does not imply asymptotic stability (think of a linear centre). We have been thinking about linearisation, which motivates our final definition.

Definition 3.3 (Linear stability)**.** A stationary point x^* of an autonomous ODE is linearly stable iff the real parts of every eigenvalue of $Df(x^*)$ is negative.

For linear systems $\dot{x} = Ax$, if the eigenvalues of A have negative real parts then $|x(t)| \to 0$ as $t \to \infty$, i.e. solutions are asymptotically stable (we have shown this in the special case of distinct eigenvalues). We will take a geometric view of stability (see the textbook by Meiss for a more analytic treatment). The geometric approach is via motivating example for showing solutions are bounded (often an important first step in physics).

The proof of the lemma uses some results from calculus. First recall the chain rule for the derivative of a function of a function. In one dimension

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\mathrm{d}V(x(t))}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

and in higher dimensions ($x(t) \in \mathbb{R}^n$)

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \nabla V(x(t)) \cdot \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \nabla V(x) \cdot \dot{x}.$$

So if x satisfies the differential equation $\dot{x} = f(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \nabla V(x) \cdot f(x).$$

Lemma 3.1 (Bounding Lemma)**.** *Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, f smooth. Suppose there exists a compact set $U \subset \mathbb{R}^n$, $\epsilon > 0$ and a continuously differentiable function $V : U \to \mathbb{R}$ such that*

(a) $V(x) \geq 0$ in $\mathbb{R}^n \backslash U$

(b) $U \subset S_c = \{V(x) \leq c\}$ for all $c \geq c_0 > 0$.

(c) S_c is compact and $S_c \subset S_{c'}$ if $c < c'$

(d) $\dot{V}(x) < -\varepsilon$ for all $x \in \mathbb{R}^n \backslash U$.

then for all $x_0 \in \mathbb{R}^n$ there exists $t_0 > 0$ such that $\varphi_t(x_0) \in S_{c_0}$ for all $t \geq t_0$.

Proof. We first show that $x(t)$ enters S_{c_0} at some time, even initially x_0 is not in S_{c_0}. If $x_0 \notin S_{c_0}$ then $\dot{V} < -\varepsilon$ and $V(x_0) > c_0$, so

$$V(\varphi_t(x_0)) < V(x_0) - \varepsilon\, t$$

and so there exists $t_0 < \frac{V(x_0) - c_0}{\varepsilon}$ such that

$$V(\varphi_{t_0}(x_0)) = c_0, \qquad \text{i.e. } \varphi_{t_0}(x_0) \in S_{c_0}.$$

Once $x(t)$ is in S_{c_0}, we show that it stays in S_{c_0}. To make this argument formally, suppose

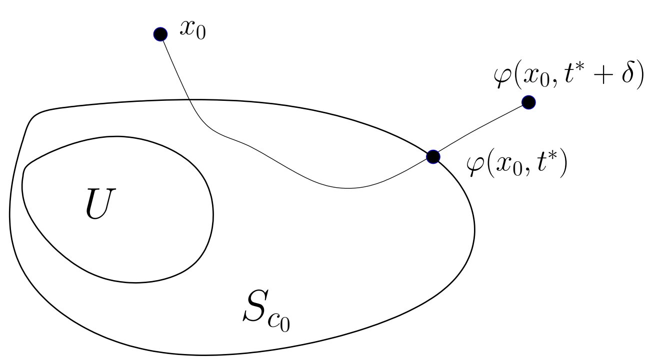


Figure 3.14: What if $\varphi_t(x_0)$ leaves S_{c_0}?

$\varphi_{t^*}(x_0) \in \partial S_{c_0}$ and for $\delta > 0$ sufficiently small $\varphi_t(x_0) \notin S_{c_0}$, for any $t \in (t^*, t^* + \delta)$ (see

Figure 3.14). Then $V(\varphi_{t^*+\delta}(x_0)) > c_0 = V(\varphi_{t^*}(x_0))$ and $\dot V(\varphi_t(x_0)) < -\epsilon$ for any $t \in (t^*, t^*+\delta)$. Then

$$0 < V(\varphi_{t^*+\delta}(x_0)) - V(\varphi_{t^*}(x_0)) = \int_{t^*}^{t^*+\delta} \frac{\mathrm{d}}{\mathrm{d}t} V(\varphi_t(x_0))\mathrm{d}t < \int_{t^*}^{t^*+\delta} (-\epsilon)\mathrm{d}t = -\delta\epsilon.$$

So we get a contradiction. □

Example 3.6 (Lorenz Equations)**.** The Lorenz equations are three coupled ODEs introduced as a simple model of the weather. They are one of the earliest examples which appeared to behave chaotically. The equations are

$$\dot x = \sigma(y-x), \qquad \dot y = rx - y - xz, \qquad \dot z = -bz + xy$$

with $\sigma, r, b \geq 0$. We wish to show that all solutions are bounded.

Need to 'create' a positive function $V(x,y,z)$ with the properties of the lemma. The obvious thing to do is to choose quadratic function, slightly less obvious is which quadratic function. The following approach is not optimal but works!

Set

$$V(x,y,z) = Ax^2 + By^2 + C(z-2r)^2$$

and we want to choose the positive constants A,B,C to satisfy the lemma. So

$$\begin{aligned}\frac{1}{2}\dot V &= Ax\dot x + By\dot y + C\dot z(z-2r)\\ &= A\sigma x(y-x) + By(rx-y-xz) + C(-bz+xy)(z-2r)\end{aligned}$$

where we have used the ODE to replace $\dot x, \dot y$, and $\dot z$. We want $\dot V < 0$ for large enough x, y and z, so the 'difficult' terms are cross multiplications like xy and xyz (good terms are quadratic terms, with negative coefficients). Choose the constants to remove them i.e.

$$\begin{aligned} B - C &= 0 \qquad && (xyz \text{ terms vanish }) \\ A\sigma + Br - 2Cr &= 0 && (xy \text{ terms vanish}) \end{aligned}$$

So set $B = C$ and $A\sigma = Br$, e.g. $A = r, B = C = \sigma$ (we have one degree of freedom to choose these constants: $A = r/\sigma, B = C = 1$ works as well), then

$$V(x,y,z) = \sigma r x^2 + \sigma y^2 + \sigma(z-2r)^2$$

and

$$\frac{1}{2}\dot V(x,y,z) = -\sigma r x^2 - \sigma y^2 - \sigma(bz^2 - 2rbz) = -\sigma r x^2 - \sigma y^2 - \sigma b(z-r)^2 + \sigma b r^2.$$

Notice it is $(z-r)^2$ on the right hand side, instead of $(z-2r)^2$ as in V.

The specific form of $\dot V/2$ also tells us which set to look at. So provided that (x,y,z) lies outside the set

$$\tilde U = \left\{(x,y,z) | \sigma r x^2 + y^2 + \sigma b(z-r)^2 \leq br^2\right\} \qquad \text{(an ellipsoid)}$$

then $\dot{V} < 0$. It is possible to modify the Bounding Lemma to deal with this, but the actual conditions of the bounding lemma require $\dot{V} < -\epsilon < 0$. Thus we choose U a little larger: pick $\epsilon > 0$ and set

$$U = \{(x, y, z) | \sigma r x^2 + \sigma y^2 + \sigma b(z - r)^2 \leq \sigma b r^2 + \epsilon\}$$

and so in $\mathbb{R}^3 \backslash U$

$$\dot{V} = -r\sigma x^2 - \sigma y^2 - \sigma b(z - r) + \sigma b r^2 \leq -2\epsilon < -\epsilon.$$

Now just choose c_0 sufficiently large that

$$S_{c_0} = \{(x, y, z) | r x^2 + \sigma y^2 + \sigma b(z - 2r)^2 \leq c_0\}$$

(another ellipsoid) contains U.

Remark. We can relate $\dot{V}$ to V in a differential inequality, to induce some information about the behaviour of V. More precisely, the above expression for $\dot{V}$ implies that

$$\dot{V}(x, y, z) = -2r\sigma x^2 - 2\sigma y^2 - \sigma b(z - 2r)^2 - \sigma b z^2 + 4\sigma b r^2 \leq -\mu V(x, y, z) + 4\sigma b r^2.$$

with $\mu = \min(2\sigma, 2, b)$. The advantage here is that we can "solve" this differential inequality. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\mu t} V(x, y, z)\right) = e^{\mu t}\left(\dot{V} + \mu V\right) \leq 4\sigma b r^2 e^{\mu t}.$$

Integrating both sides for time from zero to t, then

$$e^{\mu t} V(x, y, z) - V(x_0, y_0, z_0) \leq 4\sigma b r^2 \int_0^t e^{\mu \tau} \mathrm{d}\tau = \frac{4\sigma b r^2}{\mu}(e^{\mu t} - 1).$$

That is

$$V(x, y, t) \leq e^{-\mu t} V(x_0, y_0, z_0) + \frac{4\sigma b r^2}{\mu}\left(1 - e^{-\mu t}\right) \leq e^{-\mu t} V(x_0, y_0, z_0) + \frac{4\sigma b r^2}{\mu},$$

which is bounded for any $t \geq 0$. We can estimate the time when the trajectories enter the set (different from the above one)

$$\bar{U} = \left\{(x, y, z) | V(x, y, z) \leq \frac{4\sigma b r^2}{\mu} + \epsilon\right\},$$

that is $t \geq \frac{1}{\mu} \log \frac{\epsilon}{V(x_0, y_0, z_0)}$.

Remark. The function V above is not unique, and we can choose alternative ones (along with many others) like

$$V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2.$$

Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -2\sigma x^2 - 2y^2 - b(z - r - \sigma)^2 - b z^2 + b(r + \sigma)^2 \leq -\alpha V + b(r + \sigma)^2,$$

where $\alpha = \min(2\sigma, 2, b)$. Then we can choose the set to be

$$\tilde{U} = \left\{(x, y, z) \mid x^2 + y^2 + (z - \sigma - r)^2 \leq \frac{b(r + \sigma)}{\alpha}\right\},$$

or relax the right hand side by changing it to $b(r + \sigma)^2/\alpha + \epsilon$.

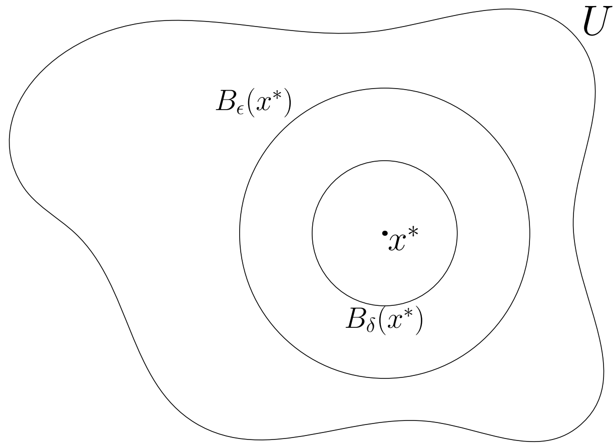


Figure 3.15: Regions defined in proof.

The same basic idea works for stationary points.

Definition 3.4 (Lyapunov functions)**.** A function $V : U \to \mathbb{R}$ is called a Lyapunov function on $U \subseteq \mathbb{R}^n$ iff it is continuously differentiable, $V(x) \geq 0$ on U and $\dot{V} \leq 0$ on U.

Theorem 3.2 (Lyapunov's Stability Theorem)**.** *Suppose $x^* \in \mathbb{R}^n$ is a stationary point of $\dot{x} = f(x)$ with f smooth. Let U be an open neighbourhood of x^* and suppose there exists a Lyapunov function $V : U \to \mathbb{R}$ such that $V(x) > 0$ on $U \setminus \{x^*\}$ and $V(x^*) = 0$. Then x^* is a Lyapunov stable. If in addition $\dot{V} < 0$ in $U \setminus \{x^*\}$ then x^* is asymptotically stable.*

Proof. Choose $\varepsilon > 0$ small enough so that $\{x \mid |x - x^*| \leq \varepsilon\}$ lies entirely in U, and let $c_0 = \min_{|x-x^*|=\varepsilon} V(x)$ which exists as $|x - x^*| = \varepsilon$ is compact (closed and bounded), and $c_0 > 0$ as $x^* \notin \{x \mid |x - x^*| = \varepsilon\}$. Let $\mathcal{B}_\varepsilon(x^*) = \{x \mid |x - x^*| < \varepsilon\}$. Now V is continuous, and $V(x^*) = 0$, so there exists $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(x^*), V(x) < \frac{1}{2}c_0$.

Consider $x_0 \in \mathcal{B}_\delta(x^*)$. Since $\dot{V} \leq 0$ in U, $V(\varphi_t(x_0)) \leq V(x_0) < \frac{1}{2}c_0$ for all t such that $\varphi_t(x_0) \in U$, and hence $V(\varphi_t(x_0)) < \frac{1}{2}c_0 < c_0$, the minimum on $|x - x^*| = \varepsilon$. Hence $\varphi_t(x_0) \in \mathcal{B}_\varepsilon(x^*)$ for all $t > 0$.

Suppose in addition that $\dot{V} < 0$ if $x \in U\setminus\{x^*\}$, (note that $\dot{V}(x^*) = 0$ since x^* is stationary). Then if $x_0 \in \mathcal{B}_\delta(x^*)$ as before, $V(\varphi_t(x_0))$ is strictly decreasing and hence tends to a limit $\bar{V} = \lim_{t\to\infty} V(\varphi_t(x_0))$. At the limit $\dot{V} = 0$ hence the limit must be x^*. □

Example 3.7. Consider the Lorenz equations with $0 < r < 1$.

$$\dot{x} = \sigma(y - x); \qquad \dot{y} = rx - y - xz; \qquad \dot{z} = -bz + xy$$

with $\sigma, b \geq 0$ and $0 < r < 1$. Try a Lyapunov function of the form

$$V(x, y, z) = Ax^2 + By^2 + Cz^2,$$

with $\frac{1}{2}\dot{V} = A\sigma x(y - x) + By(rx - y - xz) + Cz(-bz + xy)$. Choose $B = C$ to remove the xyz terms:

$$\frac{1}{2}\dot{V} = A\sigma(-x^2 + xy) + B(rxy - y^2 - bz^2)$$

We deal with the xy terms to ensure $\dot{V}$ is negative, by choosing A and B appropriately (matching the square!). Set $B = \sigma$, $A = r$ so all terms together

$$\begin{aligned}\frac{1}{2}\dot{V} &= r\sigma(-x^2 + 2xy) - \sigma y^2 - \sigma b z^2 \\ &= r\sigma(-(x-y)^2) - (\sigma - r\sigma)y^2 - \sigma b z^2 \\ &= -r\sigma(x-y)^2 - \sigma(1-r)y^2 - \sigma b z^2 \\ &< 0 \qquad\qquad \text{if } (x,y,z) \neq (0,0,0)\end{aligned}$$

if $0 < r < 1$. Hence the origin is asymptotically stable when $0 < r < 1$.

Example 3.8. Let us consider a degenerate node with repeated eigenvalues $\lambda = -2$:

$$\dot{x} = -2x + y, \quad \dot{y} = -2y.$$

Set $V(x,y) = x^2 + By^2$ and B is chosen later. Then

$$\frac{1}{2}\dot{V} = x(-2x+y) + By(-2y) = -2x^2 + xy - 2By^2 = -2(x - \frac{1}{4}y)^2 + \frac{1}{8}y^2 - 2By^2.$$

So for any $B > \frac{1}{16}$ the function is a Lyapunov function and the origin is asymptotically stable.

In many practical examples, the stationary point x^* can still be asymptotic stable, when the condition $\dot{V} < 0$ in $U \subset \{x^*\}$ in **Theorem 3.2** can be relaxed to $\dot{V} \leq 0$, provided that x^* is the only fixed point.

Theorem 3.3 (LaSalle's Invariance Principle)**.** *Suppose that $V : U \subset \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $\dot{x} = f(x)$. If the set $\{x \in U \mid \dot{V}(x) = 0\}$ contains only one fixed point x^*, then x^* is asymptotically stable.*

Example 3.9. Consider the equation $\ddot{x} + \mu\dot{x} + \omega^2 x = 0$ descibing the motion of a harmonic oscillator with friction ($\mu > 0$), or equivalently $\dot{x} = y, \quad \dot{y} = -\mu y - \omega^2 x$. If we choose $V(x,y) = \omega^2 x^2 + y^2$, then $\dot{V} = -2\mu y^2 \leq 0$. Since $(0,0)$ is the only fixed point in the set $\{(x,y) \mod \dot{V}(x,y) = 0\} = \{(x,y) \mid y = 0\}$, by Lasalle's invariance principle, the origin is asymptotically stable. In other words, the harmonic oscillator will eventially stop moving.

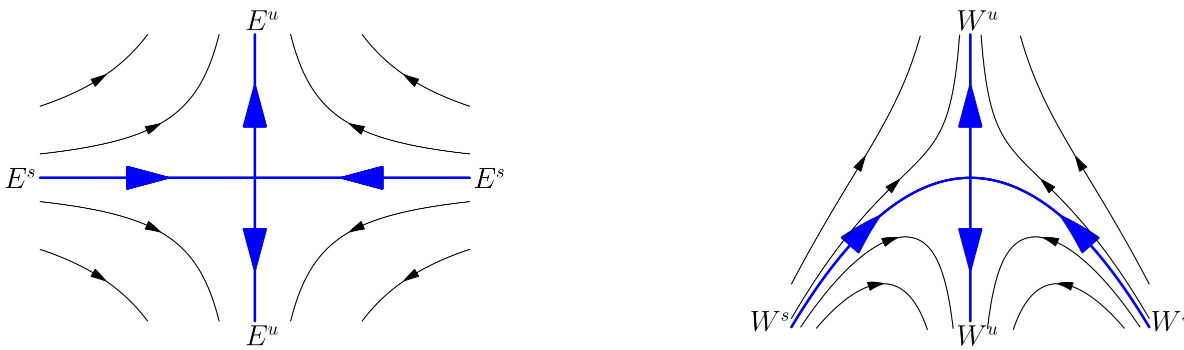
3.5 Linearisation and nonlinear terms

The previous result shows that if $\mathrm{Re}(\lambda) < 0$ and the eigenvalues are distinct, then a linearly stable fixed point is locally stable when nonlinear terms are added back in. This is an example of a range of persistence results for behaviour.

Example 3.10. The linearised system of the full nonlinear system

$$\dot{x} = -x, \quad \dot{y} = y + x^2$$

is $\dot{x} = -x, \dot{y} = y$, with two eigenvalues ± 1. The phase portrait for this saddle system should be well known now, but there are two special straight lines deserving more attention: the x-axis and and the y-axis. If the initial condition (x_0, y_0) is on the x-axis (that is $y_0 = 0$), then the solution $(x(t), y(t))$ converges to the origin, as time t goes to infinity. This linear space is called the *stable manifold*, denoted as E^s. Although the solution with initial condition away from the x-axis goes not infinity, as time t goes to positive infinity, initial condition (x_0, y_0) resides on the y-axis has the special property that its solution converges to the origin as time t goes to *negative infinity*. The y-axis is called *unstable manifold* (which is the stable manifold when time is reversed), denoted as E^u.



$\dot{x} = -x,\ \dot{y} = y$ $\qquad$ $\dot{x} = -x,\ \dot{y} = y + x^2$

Figure 3.16: Invariant manifolds W^s and W^u for the linearised system and full nonlinear system in Example 3.10.

Back to the nonlinear system, the correponding phase portrait is *deformed* from that of the linearised system. There are also special curves, the stable manifold W^s and the unstable manifodl W^u, whose solution converges to the origin, as time t goes to positive infinity and negative infinity, respectively. In general, W^s (or W^u) is no longer straight line, but it is tangent to E^s (or E^u) at the origin.

In fact, in this example, we can find the equations for W^s and W^u. Obviously, W^u is the y-axis: if $x_0 = 0$, then the ODE $\dot{x} = -x$ implies $x(t) = x_0e^{-t} \equiv 0$, and $y(t) = y_0e^t \to 0$ as $t \to -\infty$. The stable manifold is $W^s = \{(x, y) \mid y = -x^2/3\}$. If (x_0, y_0) is on W^s, that is $y_0 = -x_0^2/3$, then $x(t) = x_0e^{-t}$ and y satisfies the linear ODE

$$\dot{y} = y + x_0^2 e^{-2t}.$$

From the fact that $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(e^{-t}y\big) = e^{-t}(\dot{y} - y) = x_0^2e^{-3t}$, we get

$$e^{-t}y(t) = y_0 + \int_0^t x_0^2 e^{-3\tau}\mathrm{d}\tau = y_0 + \frac{x_0^2}{3}\big(1 - e^{-3t}\big) = \left(y_0 + \frac{x_0^2}{3}\right) - \frac{x_0^2}{3}e^{-3t} = -\frac{x_0^2}{3}e^{-3t}.$$

Since the solution $(x, y) = \left(x_0e^{-t}, -\frac{x_0^2}{3}e^{-2t}\right)$ stays on W^s and converges to the origin (as t goes to positive infinity), W^s is a stable manifold.

Remark. Here the word *manifold* is used for a smooth geometric curve or surface: if the manifold is one dimension, it is a smooth curve (inluding straight lines); if the manifold is two dimension, it is a smooth surface (including planes). But normally we do not know the dimension of the curve or the surface in advance, so it is better to use the generic name *manifold* instead of the more common curve or surface.

This above relationship between full system and its linearised system is summarised in the following theorem, provided that none of the eigenvalue has zero real part.

Theorem 3.4 (Stable Manifold Theorem)**.** *Suppose $\dot{x} = Ax + O(x^2)$ and A has no eigenvalues with* $\mathrm{Re}(\lambda) = 0$ *($x = 0$ is called a* hyperbolic *stationary point in this case). Then after a change of coordinates the system is*

$$\dot{x}_1 = A^+ x_1 + O(|x|^2), \qquad \dot{x}_2 = A^- x_2 + O(|x|^2)$$

where A^+ has $\mathrm{Re}(\lambda) > 0, A^-$ *has* $\mathrm{Re}(\lambda) < 0$*. Moreover there are invariant manifolds W^u and W^s with*

$$W^u = \{x \in U \mid \varphi_t(x) \to 0 \text{ as } t \to -\infty\}$$

and

$$W^s = \{x \in U \mid \varphi_t(x) \to 0 \text{ as } t \to \infty\}.$$

which are of of the same dimension as x_1 (resp. x_2) and tangential to $x_2 = 0$ (resp. $x_1 = 0$) at the origin.

This implies the *persistence of saddle structure* near a stationary point when the nonlinear terms are added back into the linearisation. The correponding changes in the phase portraits or the deformation of the stable/unstable manifolds are best discribed using language from *topology* (think about the deformation of a coffee mug into a donut). The persistence between structure are called *topologically conjugate* or *topologically equivalent*, but we will omit this complicated topological language and keep a mental picture instead, as in the following theorem.

Theorem 3.5 (Hartman–Grobmann)**.** *If $\dot{x} = Ax + O(x^2)$ and A has no eigenvalue with zero real part, then the behaviour near the neighbourhood of the origin is topologically equivalent to the linear system $\dot{x} = Ax$.*

Example 3.11. Now consider the system $\dot{x} = y, \dot{y} = x^2 + x$ and its linearised system $\dot{x} = y, \dot{y} = x$. The trajectories, governed by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + x}{y}$(which is separable), are given by $\dfrac{x^3}{3} + \dfrac{x^2 - y^2}{2} = C$. The stable manifold W^s concides with the unstable manifold W^u, and is $\dfrac{x^3}{3} + \dfrac{x^2 - y^2}{2} = 0$.

If A has eigenvalue with zero real part, then the situation is much more complicated, as we can see from the following two examples.

Example 3.12. Consider the system $\dot{x} = x, \dot{y} = y^2$, which you can solve the individual equations separately (they are de-coupled from each other). The lienarized system (near the origin) $\dot{x} = x, \dot{y} = 0$ has only horizontal phase curves. For a related system $\dot{x} = y^2, \dot{y} = x$ (you can also solve this explicitly).

If there is zero eigenvalue in the linearised system, their local behaviours of the full nonlinear system near the fixed point are different, as we can see from Figure 3.18 and 3.19.

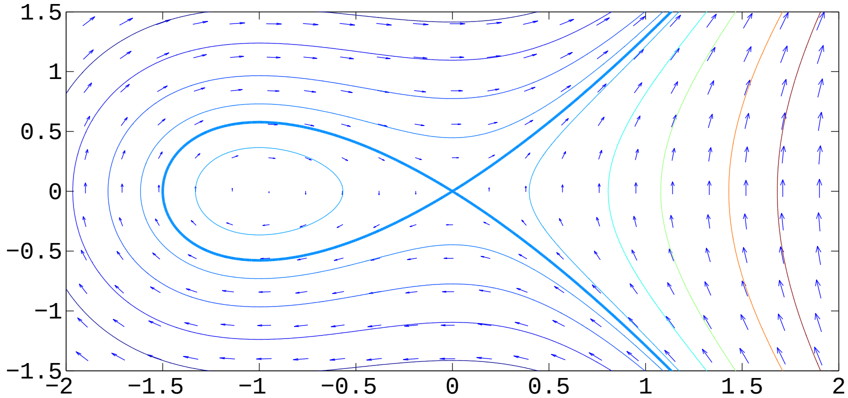


Figure 3.17: The phase portrait for $\dot{x} = y, \dot{y} = x^2 + x$ in Example 3.11.

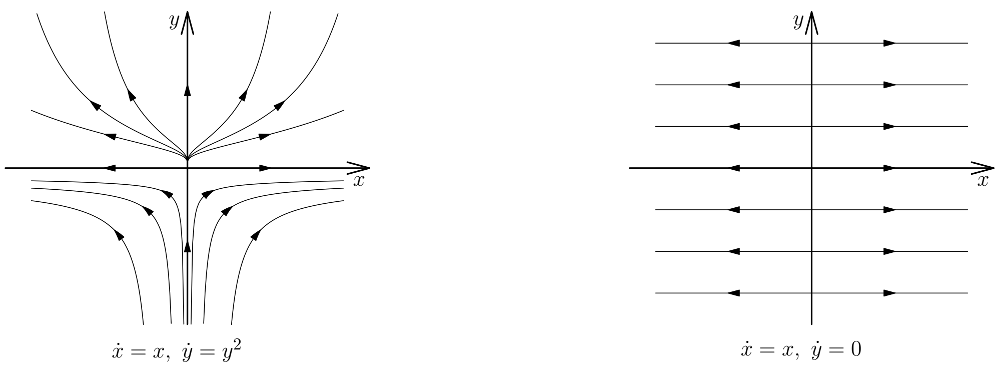


Figure 3.18: Phase portraits for the system $\dot{x} = x, \dot{y} = y^2$ with zero eigenvalues at the origin.

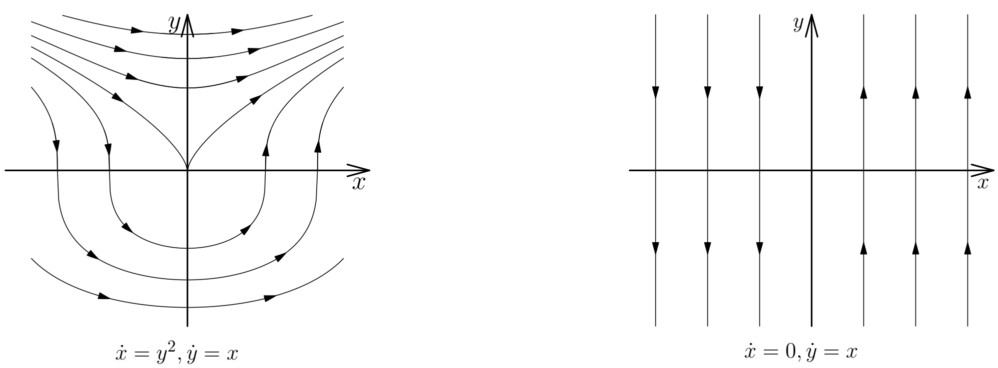


Figure 3.19: Phase portraits for the system $\dot{x} = y^2, \dot{y} = x$ with zero eigenvalues at the origin.

Exercise. Find the solution to the system

$$\dot{x} = x, \qquad \dot{y} = y^2$$

and

$$\dot{x} = y^2, \quad \dot{y} = x$$

whose phase portraits are given in Figure 3.18 and 3.19.

The situation is even worse if $f(x) = O(|x|^2)$, because the linearised system $\dot{x} = 0$ does not tell anything about the behaviour of the system near the origin. You can see two examples in Figure 3.20. Information can still be obtained in certain cases, by looking at the trajectories, or by transforming into polar coordinates.

Example 3.13. Consider the system $\dot{x} = -xy, \dot{y} = x^2 + y^2$, whose phase portrait is given in Figure 3.20(left figure). The trajectory, governed by the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2+y^2}{xy}$ is homogeneous. By the change of variable $y = zx$, the ODE becomes $\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{2z^2+1}{z}x$, which is separable. The solution is given by $x^4(2z^2 + 1) = C$ and the trajectories are given by $2x^2y^2 + x^4 = C$.

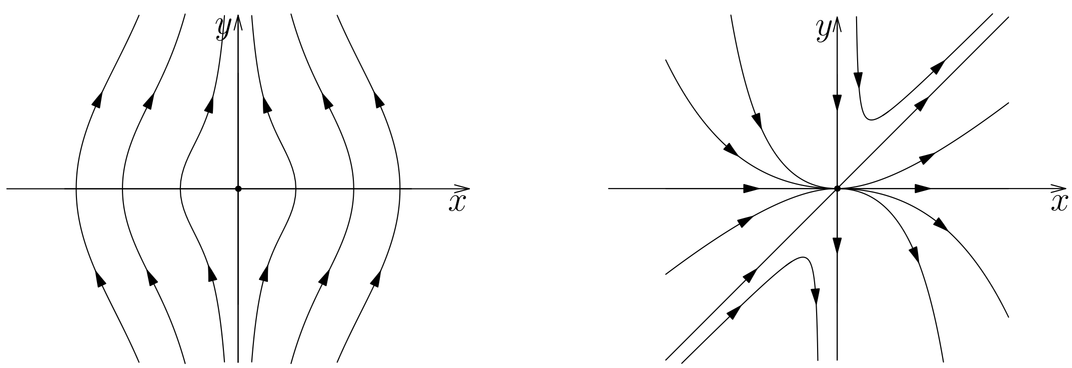


Figure 3.20: Phase portraits for systems with zero linear parts at the origin: Left: $\dot{x} = -xy, \dot{y} = x^2 + y^2$; Right: $\dot{x} = x^2, \dot{y} = y(2x - y)$.

3.6 Maps

Besides continuous dynamical systems using differential equations, discrete dynamical systems defined by maps are also popular in modelling (Fibonacci number for the population of animals), taking the form $x_{n+1} = f(x_n)$ with $x \in \mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}^k$. These systems are easier to deal with numerically, but more difficult analytically (precisely because of the discrete phase space). If there are parameters in the map, we write

$$x_{n+1} = f(x_n, \mu),$$

with $\mu \in \mathbb{R}^m$, $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$. Given an initial condition x_0, the trajectory is the sequence

$$(x_0, x_1, x_2, \dots),$$

i.e, a discrete set of points in phase space $\mathbb{R}^n$.

We often write

$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = \underbrace{f(f(\cdots(f(}_{n \text{ times}} x_0))\cdots)) = \underbrace{f \circ f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0),$$

where f^n means the n^{th} iteration of f, which is **NOT** $[f(x_0)]^n$, the n-th power of $f(x_0)$.

Similar to their continuous counterparts, we are mainly interested in the *fixed points* and *periodic orbits* of maps to understand their local behaviours, as a starting point for more complicated situations. A ***fixed point*** of the discrete dynamical system $x_{n+1} = f(x_n)$ is a solution of

$$x = f(x).$$

Periodic orbits are defined similarly: they satisfy

$$x_{n+p} = x_n \quad \text{for all } n \geq 0,$$

and p is called the period. Any point in the periodic orbit with period p is just a fixed point of f^p, that is,

$$x = f^p(x) = \underbrace{f \circ f \circ \cdots f}_{p \text{ times}}(x)$$

another algebraic equation! A period-p orbit is usually listed as a sequence of p points $(x_1^*, x_2^*, \cdots, x_p^*)$ such that

$$x_2^* = f(x_1^*), \quad x_3^* = f(x_2^*), \quad \cdots, \quad x_1^* = f(x_p^*),$$

while each of the point x_k^* satisfies $x = f^p(x)$. Because of the periodicity, the period-p orbit $(x_2^*, x_3^*, \cdots, x_p^*, x_1^*)$ is the same as $(x_1^*, x_2^*, \cdots, x_p^*)$, and we only need to choose one orbit out of the p equivalent ones.

As with ODEs, we are interested in qualitative properies like special solutions and their stabilities, invariant sets, long term behaviours and the dependence of these properties on parameters.

Example 3.14. Consider the simplest linear map $x_{n+1} = ax_n + b$. If $a = 1$, then $x_n = x_{n-1} + b = \cdots = x_0 + nb$ and there is no fixed point, unless $b = 0$. Otherwise if $a \neq 1$, the only fixed point is $x^* = b/(1-a)$. From the fact that

$$x_n - x^* = ax_{n-1} + b - \frac{b}{1-a} = ax_{n-1} - \frac{ab}{1-a} = a\left(x_{n-1} - x^*\right),$$

we get $x_n - x^* = a^n(x_0 - x^*)$ and

$$x_n = x^* + a^n(x_0 - x^*) = a^n x_0 + \frac{1-a^n}{1-a} b.$$

It is also easy to check that, if $a \neq 1$, there is no non-trivial period-2 orbits (check it!)—any period-2 orbit (x_1, x_2) satisfies $x_1 = x_2 = x^*$.

Example 3.15 (Compound interest)**.** Let P_n be the principal at n-th month with initial principal P_0, monthly interest rate r and monthly payment M, then P_n satisfies the relation

$$P_{n+1} = (1+r)P_n - M.$$

From the previous example, we get (with $a = 1+r, b = -M$) $P_n = (1+r)^n P_0 - \frac{M}{r}\big((1+r)^n - 1\big)$.

Remark. For a given continuous dynamical system (i.e., the ODE $\dot{x} = f(x)$), we can define an discrete dynamical system in the following ways shown in Figure 3.21. For any given time interval $T > 0$, we can take $x_n = x(nT)$, the solution of the ODE at $t = nT$. Then the sequence $(x_0, x_1, x_2, \cdots)$ is a dynamical system. Alternatively, we can define the discrete points at the intersection of $x(t) \in \mathbb{R}^n$ with a $n-1$ dimensional surface, called *return maps* or *Poincaré maps.*

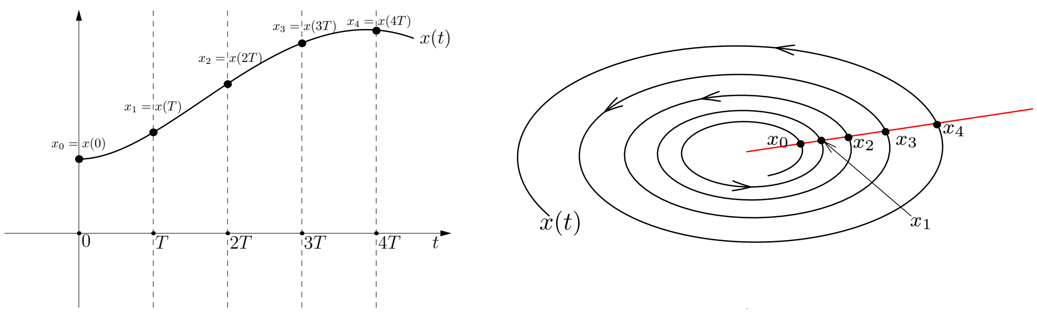


Figure 3.21: Two ways to get discrete dynamical systems from continuous ones, either by $x_n = x(nT)$ or the return map.

Maps also appear in the numerical approximations of ODEs. For example, if we want to consider the solution of $\dot{x} = x(1-x)$ at time $t = 0, h, 2h, \cdots$ (h is called the *time step*, which is usually small) and denote $x_n \approx x(nh)$, then by Taylor expansion,

$$x_{n+1} = x(nh+h) = x(nh) + hx'(nh) + \frac{h^2}{2!}x''(nh) + \cdots = x_n + h(1-x_n)x_n + O(h^2).$$

Therefore, to the leading order, we get the discrete map $x_{n+1} = x_n + hx_n(1-x_n)$.

Similarly, for discrete dynamical system governed by $x_{n+1} = f(x_n)$, a set $\Lambda \subseteq \mathbb{R}^n$ is *invariant* iff $x_0 \in \Lambda$ implies $x_n \in \Lambda$ for all $n \geq 0$. In fact, to show Λ is invariant, we only need to show that, if $x_n \in \Lambda$, then $x_{n+1} \in \Lambda$.

Example 3.16 (Logistic map)**.** The logistic map $x_{n+1} = \mu x_n(1-x_n)$ is the simplest discrete dynamical system exhibiting chaotic behaviours (for some parameters of μ). We will study in detail how these behaviours and the associated bifurcations depend on the parameter μ. We can show that the interval $\Lambda = [0,1]$ is invariant when $\mu \in [0,4]$. In fact, for $x_n \in [0,1]$ and $\mu \in [0,4]$, then $x_{n+1} = \mu x_n(1-x_n) \geq 0$ and

$$x_{n+1} = \mu(x_n - x_n^2) = \mu\left[\frac{1}{4} - \left(x_n - \frac{1}{2}\right)^2\right] \leq \frac{\mu}{4} \leq 1.$$

Example 3.17 (2D system)**.** Consider the system

$$x_{n+1} = x_n\, f(y_n), \qquad y_{n+1} = g(x_n, y_n).$$

The line $x = 0$ is invariant ($x_n = 0 \implies x_{n+1} = 0$), and on $x = 0$, $y_{n+1} = g(0, y_n)$.

Linear Maps: The local behaviours of discrete maps can also be inferred from linearisation near fixed points. Let x^* be a fixed point of $x_{n+1} = f(x_n)$ with $x^* = f(x^*)$. If x is close to x^* such that $y_n = x_n - x^*$ is small, then

$$y_{n+1} = x_{n+1} - x^* = f(x_n) - f(x^*) \approx Df(x^*)(x - x^*) = Df(x^*)y_n.$$

Therefore, the linearised equation is $y_{n+1} = Ay_n$, with the constant matrix $A = Df(x^*)$. The solution can be written as $y_n = A^n y_0$. If the matrix can be diagonalised as $A = S\Lambda S^{-1}$ (the columns of S are eigenvectors of A), then $A^n = S\Lambda^n S^{-1}$. The change of variable $z_n = S^{-1}y_n$ leads to the *normal form*

$$z_{n+1} = S^{-1}y_{n+1} = S^{-1}Ay_n = S^{-1}AS(S^{-1}z_n) = \Lambda z_n,$$

where the matrix power Λ^n in the solution $z_n = \Lambda^n z_0$ can be calculated easily. For example,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \mapsto \Lambda^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_m^n \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mapsto \Lambda^n = \begin{pmatrix} \lambda^n & (n+1)\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

The general solution of $x_{n+1} = Ax_n$ when A has m distinct eigenvalues λ_m with eigenvectors e_m is

$$x_n = \sum_{j=1}^{m} c_j \lambda_j^n e_j.$$

Here the coefficients c_j are determined from the initial condition ($n = 0$ in the previous equation)

$$x_0 = \sum_{j=1}^{m} c_j e_j.$$

We can start with the simplest case to motivate the criteria of stability. For linear ODES, the canonical example is the scalar ODE $\dot{x} = \lambda x$ with solution $x(t) = x_0 e^{\lambda t}$. Therefore, the stability of the ODE is determined by $\exp(\lambda t)$ as t goes to infinity, or equivalently the boundary $\mathrm{Re}\lambda = 0$. Similarly, if we look at the simplest map $x_{n+1} = \lambda x_n$, then $x_n = \lambda^n x_0$. The stability is determined by λ^n as n goes to infinity, or equivalently the boundary $|\lambda| = 1$ ($|\lambda| < 1$ implies stability in the corresponding eigenspace). Now we can proceed for general cases in general dimensions.

Example 3.18 (Saddle)**.** In Normal Form coordinates, the map is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad \text{with} \qquad |\lambda_1| < 1 < |\lambda_2|.$$

These two components $x_{n+1} = \lambda_1 x_n, y_{n+1} = \lambda_2 y_n$ can be solved explicitly, to give

$$x_n = \lambda_1^n x_0, \qquad\qquad y_n = \lambda_2^n y_0.$$

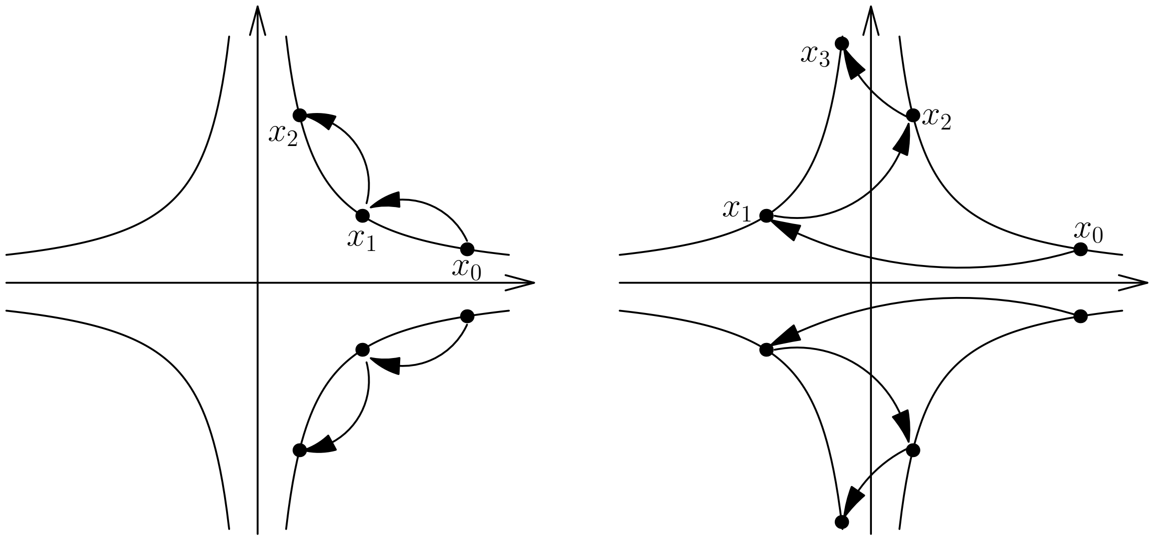


Figure 3.22: Saddles for discrete time equations (a) $0 < \lambda_1 < 1 < \lambda_2$; (b) $-1 < \lambda_1 < 0$, $\lambda_2 > 1$.

We can take modulus on the solutions, $\frac{|x_n|}{|x_0|} = |\lambda_1|^n$, $\frac{|x_n|}{|x_0|} = |\lambda_1|^n$. That is, the solution (x_n, y_n) lies on the generalised hyperbola

$$\left\{ (x, y) \;\middle|\; \left|\frac{x}{x_0}\right| = \left|\frac{y}{y_0}\right|^{\ln|\lambda_1|/\ln|\lambda_2|} \right\}.$$

The motion of these hyperbolas is discrete; an orbit hops along the relevant curve or curves as shown in Figure 3.22. If an eigenvalue is negative then the orbit of a point will oscillate between negative and positive values in that eigen-direction as indicated in Figure 3.22(b).

Example 3.19 (Focus)**.** The map in normal form is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

The geometric interpretation is clearer if we write the coefficient matrix as

$$\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} = \sqrt{\rho^2 + \omega^2} \begin{pmatrix} \frac{\rho}{\sqrt{\rho^2+\omega^2}} & -\frac{\omega}{\sqrt{\rho^2+\omega^2}} \\ \frac{\omega}{\sqrt{\rho^2+\omega^2}} & \frac{\rho}{\sqrt{\rho^2+\omega^2}} \end{pmatrix} = \underbrace{\lambda}_{\text{dilation}} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation by } \theta}$$

where $\lambda = \sqrt{\rho^2 + \omega^2}$ and $\theta = \tan^{-1}(\omega/\rho)$. If we define $z_n = x_n + iy_n$, then in complex notation

$$\begin{aligned} z_{n+1} = x_{n+1} + iy_{n+1} &= \lambda\big[(x_n \cos\theta - y_n \sin\theta) + i(x_n \sin\theta + y_n \cos\theta)\big] \\ &= \lambda\big(\cos\theta + i\sin\theta\big)(x_n + iy_n) = \lambda e^{i\theta} z_n. \end{aligned}$$

Therefore, the solution can be written as $z_n = \lambda^n e^{in\theta} z_0$, or equivalently

$$x_n = \lambda^n (x_0 \cos n\theta - y_0 \sin n\theta), \quad y_n = \lambda^n (x_0 \sin n\theta + y_0 \cos n\theta).$$

Therefore, the solution (x_n, y_n) converges to the origin if and only if $\lambda = \sqrt{\rho^2 + \omega^2} < 1$.

4 Periodic orbits

The behaviour of a dynamical system could be very complicated, even we just consider those limiting ones, as chaotic trajectories could appear. But in the plane, the limiting trajectories on a compact set have relative simple structures, consisting only fixed points and closed orbits (that connecting fixed points or periodic). We have talked about fixed points, while periodic orbits are much harder to work with than stationary points, as the explicit expressions are not always available. In this section, we will describe three aspects:

in the plane: Poincaré-Bendixson Theorem (existence of periodic solutions)

linearisation: Floquet Theory (solutions of linear ODEs with periodic coefficients)

and we will show how single periodic orbit could arise from bifurcation.

4.1 Poincaré-Bendixson Theorem

In the plane, there is a classic result called *Poincaré Bendixson Theorem* for proving the existence of periodic orbits. This theorem is stated without proof (can be found in Chapter 6.6 in the book by Meiss), but you need to be able to state the conditions correctly and apply it to examples.

Theorem 4.1 (Poincaré-Bendixson Theorem for the existence of periodic orbits)**.** *Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and f smooth. If there exists a compact (closed and bounded) subset $\mathcal{D} \subset \mathbb{R}^2$ containing no stationary points and $p \in \mathcal{D}$ such that $\varphi_t(p) \in D$ for all $t \geq 0$ (i.e, $\mathcal{D}$ is invariant), then there is at least one periodic orbit in $\mathcal{D}$ and the orbit of p tends to this periodic orbit.*

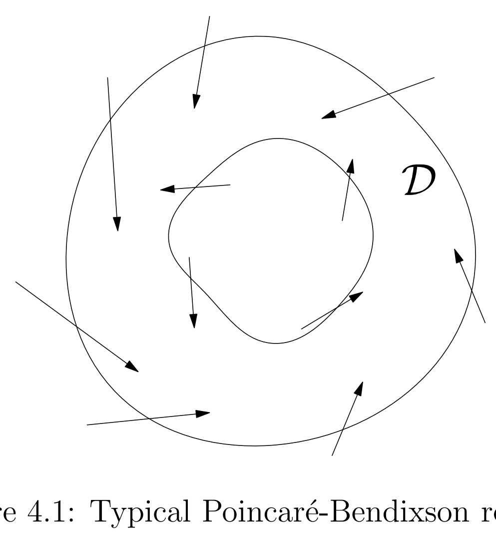


Figure 4.1: Typical Poincaré-Bendixson region.

How do we apply the theorem? In applications the region $\mathcal{D}$ is usually annular (as show in Figure 4.1). The strategy will be to show that solutions enter and do not leave an annular region containing no stationary points, hence we can apply the Poincaré-Bendixson

Theorem stated above. This is usually done by constructing a region lying between two closed curves $\partial\mathcal{D}_\pm$ defined by $V_\pm(x) = c_\pm$ with $\partial\mathcal{D}_-$ lying inside $\partial\mathcal{D}_+$ and $V_\pm$ increasing outwards (like Lyapunov functions). If in addition (for example) $\dot{V}_-(x) > 0$ on $V_-(x) = c_-$ and $\dot{V}_+(x) < 0$ on $V_+(x) = c_+$ then trajectories on both boundaries point into the region and trajectories inside the region cannot cross outwards.

Remark. The function V is not unique, and many functions could lead to the same conclusion (of course the annular region $\mathcal{D}$ could be different). However, a good choice of the function V can enormously simplify your calculation, motivated from the corresponding normal form.

Example 4.1. Consider the system

$$\begin{aligned}\dot{x} &= x + y - 4x(x^2+y^2)\\ \dot{y} &= -x + y - 4y(x^2+y^2)\end{aligned}$$

The stationary Point $(0,0)$ is obvious. Are there any others?

From the first equation

$$4x(x^2+y^2) = x + y \quad \text{i.e. } 4xy(x^2+y^2) = xy + y^2$$

and from the the second equation

$$4y(x^2+y^2) = -x + y \quad \text{i.e. } 4xy(x^2+y^2) = -x^2 + xy$$

Therefore,

$$4xy(x^2+y^2) = xy + y^2 = -x^2 + xy,$$

which implies that $x^2 + y^2 = 0$, or there is no solutions other than $(0,0)$.

Now consider $V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ (can your think of any motivation for this choice?). Then

$$\begin{aligned}\dot{V} &= x\dot{x} + y\dot{y}\\ &= x(x + y - 4x(x^2+y^2)) + y(-x + y - 4y(x^2+y^2)\\ &= x^2 + y^2 - 4(x^2+y^2)^2. \end{aligned} \tag{4.1}$$

Define

$$\mathcal{D} = \left\{(x,y) \mid \frac{1}{8} \le x^2 + y^2 \le 1\right\}.$$

Then $\dot{V} > 0$ on $x^2 + y^2 = \frac{1}{8}$ (which is the set $V(x,y) = \frac{1}{16}$) i.e. solutions enter $\mathcal{D}$ across this boundary; $\dot{V} < 0$ on $x^2 + y^2 = 1$ (which is the set $V(x,y) = \frac{1}{2}$) i.e. solutions enter $\mathcal{D}$ across this boundary.

So solutions enter and do not leave $\mathcal{D}$ which is closed, bounded and contains no stationary points, and hence there is at least one periodic orbit in $\mathcal{D}$. (Note that *any* outer boundary bigger than $\frac{1}{4}$ and lower boundary less than this will do.)

Remark. In polar coordinates this example is easy:

$$\dot{r} = r(1 - 4r^2), \qquad \dot{\theta} = -1.$$

Thus the $\dot{\theta}$ equation shows that $(0,0)$ is the only stationary point, whilst the $\dot{r}$ equation shows $r \to \frac{1}{2}$!

Equivalently we could have noted that the equation for $\dot{V}$ in (4.1) can be written as

$$\dot{V} = 2V(1-8V).$$

So solutions $V = \frac{1}{2}r^2 \to \frac{1}{8}$.

If the boundaries are given by $V(x) = c$, $x \in \mathbb{R}^2$, and V increases outwards then $\dot{V} = \nabla V \cdot f(x)$ (by the Chain Rule) and so if $\dot{V} > 0$ on the inner boundary then f points **into** $\mathcal{D}$ on the lower boundary and if $\dot{V} < 0$ on the outer boundary then f also points **into** $\mathcal{D}$ on the outer boundary. The region $\mathcal{D}$ is a trapping region (no solutions escape from $\mathcal{D}$). This can be proved rigorously using the Mean Value Theorem as in sections on Lyapunov functions, but this geometric description is self-evident and does not really need further explication.

What if $\dot{V} \geq 0$ on the outer boundary and $\dot{V} \leq 0$ on the inner boundary (i.e, with non-strict inequalities)? Here the geometric argument does not hold (trajectories can be tangent to the boundary at some places) and we need to work a little harder. If we choose

$$\mathcal{D} = \left\{ (x,y) \mid \frac{1}{4} \leq x^2 + y^2 \leq 1 \right\}$$

in the previous example, then $\dot{V} \geq 0$ (actually $\dot{V} \equiv 0$) on $\partial\mathcal{D}_- = \{(x,y) \mid x^2+y^2 = 1/4\}$. In this case $\partial\mathcal{D}_-$ is exactly the periodic orbit.

Example 4.2. Prove existence of a periodic orbit for

$$\dot{x} = y, \qquad \dot{y} = -x + y(1-3x^2-6y^2). \tag{4.2}$$

First consider stationary points: from $\dot{x} = 0$ we find $y = 0$ and then $\dot{y} = 0$ implies $x = 0$ so $(0,0)$ is a unique stationary point.

Try $V(x,y) = x^2 + y^2$, then

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2xy + 2y\left[-x + y(1-3x^2-6y^2)\right] = 2y^2(1-3x^2-6y^2).$$

So if $1-3x^2-6y^2 < 0$ (i.e. in the neighbourhood of the origin), $\dot{V} \leq 0$, and if $1-3x^2-6y^2 > 0$ (i.e. far away from the origin), $\dot{V} \geq 0$. To apply the Poincaré-Bendixson Theorem, we have to understand how the geometry of these ellipses $\{(x,y) \mid 1-3x^2-6y^2 = 0\}$ describing the sign of $\dot{V}$ interacts with the geometry of the circles of constant $V(x,y) = x^2+y^2$ that will be used to define $\mathcal{D}$. Moreover, every such circle contains some points on the x-axis (with $y = 0$), and $\dot{V} = 0$ on some parts of $V = c$. As a result, the simple geometric arguments used above will not hold (we need $\dot{V}$ strictly less or greater than zero.

How does the $\dot{V}$ equation relate to the circles $V = c$? First,

$$\dot{V} = 2y^2(1-3x^2-6y^2) \leq 2y^2(1-3x^2-3y^2)$$

So if $x^2+y^2 = V \geq \frac{1}{3}$, then $\dot{V} \leq 0$.

Let the outer boundary of $\mathcal{D}$ be $x^2+y^2=\frac{1}{3}$, call this $\partial\mathcal{D}_+$. Then if $p=(x_0,y_0)$ with $|p|=\sqrt{x_0^2+y_0^2}\le\frac{1}{3}$ and $V(p)\le\frac{1}{3}$, we claim $V(\varphi_t(p))=|\varphi_t(p)|^2\le\frac{1}{3}$ for all $t\ge 0$. Here $\varphi_t(p)$ is the solution to (4.2) starting from p at time $t=0$ (this form is used to emphasise the dependence of the initial condition on p). Suppose this claim is not true, then there exist two times t_0 and t_1, such that

$$V(\varphi_{t_0}(p))=\frac{1}{3}$$

and

$$V(\varphi_t(p))>\frac{1}{3}\quad\text{for all}\quad t\in(t_0,t_1].$$

In particular

$$V(\varphi_{t_1}(p))-V(\varphi_{t_0}(p))>0.$$

Hence by the Mean Value Theorem,

$$V(\varphi_{t_1}(p,t_1))-V(\varphi_{t_0}(p))=\dot{V}(\varphi_t(p))(t_1-t_0)\quad\text{for some}\quad t\in[t_0,t_1].$$

On the other hand,

$$\frac{dV(\varphi_t(p)}{dt}\le 0 \text{ for all } t\in[t_0,t_1]\,.$$

Therefore, $0<V(\varphi_{t_1}(p,t_1))-V(\varphi_{t_0}(p))=\dot{V}(\varphi_t(p))(t_1-t_0)\le 0$, which is a contradiction.

Now we consider the inner boundary. The fact

$$\dot{V}=2y^2(1-3x^2-6y^2)\ge 2y^2(1-6x^2-6y^2)$$

implies that $\dot{V}\ge 0$ on the set $\{(x,y)\mid x^2+y^2\le 1/6\}$, and motivates the choice of the inner boundary

$$\partial\mathcal{D}_-=\left\{(x,y)\mid x^2+y^2=\frac{1}{6}\right\}.$$

We claim that if $V(p)\ge\frac{1}{6}$ then $V(\varphi_t(p))\ge\frac{1}{6}$ for all $t\ge 0$. The proof is almost exactly the same as the above case (try it!).

So if we choose $\mathcal{D}=\left\{(x,y)\mid\frac{1}{6}\le V(x,y)\le\frac{1}{3}\right\}$, a solution that starts in $\mathcal{D}$ stays in $\mathcal{D}$. Moreover, $\mathcal{D}$ contains no stationary points and hence by the Poincaré-Bendixson Theorem there is at least one periodic orbit in $\mathcal{D}$.

Now we consider a much more complicated example, in which the Poincaré-Bendixson region is much more difficult to construct.

Example 4.3 (Glycolysis oscillation)**.** In this system of ODEs modelling the process how the human body converts glucose (sugar) into energy, x is the ADP concentration and y is the F6P (fructose-t-phosphate) concentration ($a>0$):

$$\dot{x}=-x+ay+x^2y,\qquad\dot{y}=\frac{1}{2}-ay-x^2y.$$

We want to show that there are oscillations (periodic orbits) if the positive parameter a is sufficiently small. Start by considering stationary points and their stability.

Stationary Points: Solving the system

$$x^2y = x - ay, \qquad x^2y = \frac{1}{2} - ay,$$

we get $x^* = \frac{1}{2}$ and $y^* = \frac{1}{2(a+x^2)} = \frac{2}{4a+1}$. So $\left(\frac{1}{2}, \frac{2}{4a+1}\right)$ is the only stationary point.

Stability: The Jacobian matrix is

$$J = \begin{pmatrix} -1+2xy & x^2+a \\ -2xy & -(x^2+a) \end{pmatrix}.$$

At the stationary point $\left(\frac{1}{2}, \frac{2}{4a+1}\right)$,

$$J(x^*, y^*) = \begin{pmatrix} -1+y^* & \frac{1}{2y^*} \\ -y^* & -\frac{1}{2y^*} \end{pmatrix}.$$

Therefore, we get the determinant and the trace

$$\det J = \frac{1}{2y^*} > 0, \quad \mathrm{tr} J = -1 + y^* - \frac{1}{2y^*}.$$

Since the product of the roots ($\det J$) is positive the stationary point is not a saddle, and so it is stable if $\mathrm{Tr}\ J < 0$ and unstable if $\mathrm{Tr}\ J > 0$. In other words, the stationary point is unstable if $1 + \frac{1}{2y^*} - y^* < 0$ or

$$1 + \frac{4a+1}{4} - \frac{2}{4a+1} < 0.$$

This equation can be written as

$$4(4a+1) + (4a+1)^2 - 8 < 0 \qquad \text{or} \qquad 16a + 4 + 16a^2 + 8a + 1 - 8 < 0$$

which is simplified as

$$3 - 24a - 16a^2 > 0.$$

Thus if a is sufficiently small, then the fixed point is unstable and there is a small closed curve containing the fixed point that solutions cross outwards.

Constructing a PB region(when $a > 0$ is sufficiently small): As noted earlier, if a is sufficiently small then the stationary point $(\frac{1}{2}, \frac{2}{1+4a})$ is unstable and so we can use the Lyapunov function constructed from adjoint eigenvectors in reverse time to construct an inner boundary that trajectories cross outwards. In what follows $\epsilon > 0$ is a small constant.

The remainder of the Poincaré-Bendixson region will be constructed using four straight lines (see Figure 4.2): the x-axis, a vertical line near the y-axis, a horizontal line at larger y and a part of $x + y = c$, with c chosen later. The first two are obvious. If $y = 0$ (the x-axis) then $\dot{y} = 1/2 > 0$ and so trajectories cross the x-axis upwards into $y > 0$. If $x = -\epsilon < 0$ (near the y-axis) then $\dot{x} = \epsilon + ay$ and so $\dot{x} > 0$ if $y > 0$. Thus solutions cross the x-axis and a horizontal line near the y-axis into the positive quadrant formed by the two lines. (The use of ϵ is so that we don't have to worry about the behaviour at the origin, where $\dot{x} = 0$ had we chosen the two coordinate axes.)

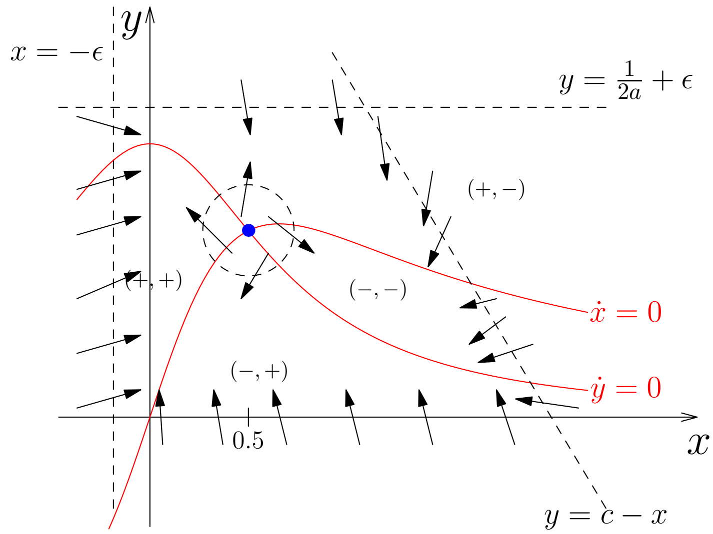


Figure 4.2: Poincaré-Bendixson region for the glycolysis model.

To think about the remaining straight lines of the boundary, draw the two nullclines:

$$y = \frac{x}{a + x^2} \qquad \text{on which } \dot{x} = 0,$$
$$y = \frac{1}{2(a + x^2)} \qquad \text{on which } \dot{y} = 0.$$

In other words, the direction field (or the vector field) is vertical on the line $y = x/(a + x^2)$ and horizontal on $y = 1/2(a + x^2)$.

Thus the nullcline $y = 1/2(a + x^2)$ for $\dot{y}$ has a maximum at $y = \frac{1}{2a}$; if $y > \frac{1}{2a}$, then $\dot{y} < 0$ and trajectories have a component downwards. This suggests using the line

$$y = \frac{1}{2a} + \epsilon$$

which all solutions cross downwards.

But we can not choose a vertical line $x = M$ for some large M to complete the closed region, because of the term x^2y ($\dot{x}$ may not be negative on this vertical line). However, we note that the sign for $\dot{x} + \dot{y} = -x + 1/2$ is much simpler (negative when $x > 1/2$). This motivate the choice of straight lines of the form $x + y = c$ for some c. If we choose $F(x, y) = x + y - c$ and define the region $D = \{(x, y) \mid F(x, y) < 0$, then $\dot{F} < 0$ if $x > 1/2$. In other words, trajectories on the line $x + y = c$ cross inwards to D.

The choice of c is determined by closing the outer boundary as simply as possible: choose it so that $x + y = c$ intersects $y = \frac{1}{2a} + \epsilon$ when $x = \frac{1}{2}$, i.e.

$$c = \frac{1}{2} + \frac{1}{2a} + \epsilon.$$

So consider region (see Figure 4.2) bounded by

(a) $x = 0, \qquad 0 \leq y \leq \dfrac{1}{2a} + \epsilon$

(b) $y = 0, \qquad 0 \le x \le \frac{1}{2} + \frac{1}{2a} + \epsilon$

(c) $y = \frac{1}{2a} + \epsilon, \qquad 0 \le x \le \frac{1}{2}$

(d) $x + y - (\frac{1}{2} + \frac{1}{2a} + \epsilon) = 0, \qquad \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{2a} + \epsilon$

Solutions enter but do not leave this region bounded by these curves. Together with the small closed curve about the unstable (a sufficiently small) stationary point which solutions cross outwards, this forms a Poincaré-Bendixson region. Hence there exists a periodic orbit.

4.2 Floquet theory

Suppose $u(t)$ is a periodic solution of $\dot{x} = f(x)$, so $u(t+T) = u(t)$ for all $t \in \mathbb{R}$. What is the linearisation of the equation about this solution? Note that since it *is* a solution u satisfies $\dot{u} = f(u)$. Set $x(t) = u(t) + v(t)$ where $|v| \ll 1$. Then

$$\dot{x} = \dot{u} + \dot{v} = f(u+v) = f(u) + \mathrm{D}f(u)v + O(|v|^2).$$

i.e. $\dot{v} = \mathrm{D}f(u)v$ is the linearisation of the equation about u, where $\mathrm{D}f(u)$ is the $n \times n$ Jacobian matrix evaluated at u. This system of ODEs is linear (in v), but its coefficient is periodic, different from the constant coefficient ODEs we considered earlier. The Floquet theory deals with the structure of such system of ODEs

$$\dot{v} = A(t)v, \quad A(t+T) = A(t).$$

Example 4.4 (One dimensional case)**.** The one-dimensional case is elementary and somewhat misleading if we are thinking about its generalization to higher dimensions. Nonetheless it is worth considering as an exercise. We have

$$\dot{v} = a(t)v, \qquad a(t+T) = a(t),$$

where both $a(t)$ and $v(t)$ are scalar. This ODE is separable, and the solution is given by

$$v(t) = v_0 \exp\left(\int_0^t a(s)\, ds\right).$$

To find the instruction of the solution, we look at $t = nT$ first, that is $v(nT) = v_0 \exp(\int_0^{nT} a(s)\, ds)$. Then because a is periodic,

$$\begin{aligned}\int_0^{nT} a(s)\, ds &= \int_0^T a(s)\, ds + \cdots + \int_{(n-1)T}^{nT} a(s)\, ds \\ &= n\int_0^T a(s)\, ds \qquad \text{as} \qquad a(s+T) = a(s)\end{aligned}$$

This implies that $v(nT) = v_0\Phi^n$ where the constant $\Phi = \exp\left(\int_0^T a(s)\,ds\right)$. Therefore, $|v(nT)|$ is increasing/decreasing (i.e. unstable or stable) according to whether $|\Phi|$ is greater than or less than one. If we write

$$v(t) = w(t)\Phi^{t/T} = w(t)\exp\left(\frac{t}{T}\int_0^T a(s)ds\right), \tag{4.3}$$

then $w(t)$ is a periodic. In fact, we have

$$v(t+T) = v(t)\exp\left(\int_t^{t+T} a(s)ds\right) = v(t)\exp\left(\int_0^T a(s)ds\right) = w(t)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right).$$

Comparing with the expression (4.3) (at $t+T$ instead of t), then

$$v(t+T) = w(t+T)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right) = w(t)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right),$$

that $w(t) = w(t+T)$ and w is periodic with period T. Therefore, in general the solution v is no periodic, unless $\Phi = 1$.

The nice solution in terms of exponentials does not work in $\mathbb{R}^n$: even though the solution to $\dot{x} = A(t)x$ with initial condition $v(0) = v_0$ can be written as $v(t) = \Phi(t)v_0$, the matrix $\Phi(t)$ depends on the coefficient $A(t)$ in a much more complicated way than that in one dimension. Nevertheless, such a matrix $\Phi(t)$ exists, and plays a similar role of as in one dimension.

Example 4.5 (A one-way coupled linear PDE with periodic coefficient)**.** Consider the following two systems

$$(1)\qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & \sin t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; \qquad (2)\qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \sin t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where the coefficient $A(t)$ is periodic with period 2π. In both cases, x is governed by the same equation $\dot{x} = -x$ and the general solution is given by $x(t) = x_0 e^{-t}$. For system (1), y is governed by $\dot{y} = x + y\sin t$, whose solution is given by

$$y(t) = x_0 e^{-\cos t}\int_0^t e^{-\tau+\cos\tau}d\tau + y_0 e^{1-\cos t}.$$

For system (2), y is governed by $\dot{y} = x\sin t + y$,

$$y(t) = -\frac{e^{-t}}{5}(\cos t + 2\sin t)x_0 + \left(y_0 + \frac{x_0}{5}\right)e^t.$$

Question: are there any periodic solution with period 2π for either system with particular initial condition?

The Structure of linear ODEs with periodic coefficients. Let v_k be the solution to the system

$$\dot{v} = A(t)v, \qquad A(t+T) = A(t)$$

with the initial condition $v_k(0) = e_k$, the k-th canonical basis in $\mathbb{R}^n$ (the vector with 1 at k-th entry, and 0 otherwise). Because of the linearity, any solution with initial condition $v(0) = (\alpha_1, \cdots, \alpha_n)^t = \alpha_1 e_1 + \cdots + \alpha_n e_n$ is

$$v(t) = \sum_{k=1}^{n} \alpha_n v_k(t) = \Phi(t)v(0),$$

where $\Phi(t)$ is the so called *fundamental matrix* with v_k at k-th column, i.e,

$$\Phi(t) = \begin{bmatrix} v_1(t) & v_2(t) & \cdots & v_n(t) \end{bmatrix}.$$

In other words, $\Phi(t)$ is the solution to the matrix ODEs $\dot{\Phi} = A(t)\Phi$ starting with the identity matrix as initial condition. We can get from the solutions in Example 4.4 that the fundamental matrices are

$$\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-\cos t}\int_0^t e^{-\tau+\cos\tau} d\tau & e^{1-\cos t} \end{pmatrix}$$

and

$$\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ \frac{e^{-t}}{5}\left(e^t - e^{-t}\cos t\right) - \frac{2}{5}e^{-t}\sin t & e^t \end{pmatrix}$$

respectively. There is no need to start with $v_k(0) = e_k$ as the initial conditions to construct $\Phi(t)$. We can start with any n-linearly independent solutions $\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n$ of the ODE $\dot{x} = A(t)x$. If we define $\widetilde{\Phi}(t) = [\tilde{v}_1 \ \tilde{v}_2 \ \cdots \tilde{v}_n]$, then $\Phi(t) = \widetilde{\Phi}(t)\widetilde{\Phi}(0)^{-1}$ is the desired fundamental matrix.

Remark. The fundamental matrix $\Phi(t,s)$ exists for general linear ODEs $\dot{x} = Ax$, where A is any general matrix (no need to be periodic).

Now we can see how the periodicity of the coefficient matrix A appears in the solution. In general, for non-autonomous ODEs (as the ones with periodic coefficients considered in this section), if $v(t)$ is a solution, $v(t+s)$ does not have to be a solution. But for periodic solution, if we differentiate both sides of $\tilde{v}(t) = v(t+T)$, then

$$\frac{d}{dt}\tilde{v}(t) = \dot{v}(t+T) = A(t+T)v(t+T) = A(t)\tilde{v}(t).$$

That is $\tilde{v}(t) = v(t+T)$ is also a solution. Then from the facts that

$$\tilde{v}(t) = \Phi(t)\tilde{v}(0) = \Phi(t)v(T) = \Phi(t)\Phi(T)v(0)$$

and $v(t+T) = \Phi(t+T)v(0)$ for any $v(0)$, we get $\Phi(t+T) = \Phi(t)B$ and $\Phi(nT) = B^n$ for any integer n, where $B \equiv \Phi(T)$ is called the *monodromy matrix* of the system. Similar to the one dimensional case, if there is a constant matrix H such $B = \exp(TH)$, then $\Phi(t)\exp(-tH)$ is periodic, or

$$\Phi(t) = P(t)\exp(tH),$$

the fundamental matrix $\Phi(t)$ is the product of a periodic matrix $P(t)$ and a matrix exponential $\exp(tH)$. The stability of periodic solutions is reduced to the eigenvalues of the monodromy matrix $B = \Phi(T) = \exp(TH)$: if all eigenvalues have modulus less than unit, the periodic solutions are locally stable; otherwise if there is one eigenvalue with modulus greater than unit, then the periodic solutions are unstable.

Example 4.6 (Another linear ODEs with periodic coefficients)**.** If

$$A(t) = \begin{pmatrix} -1 + \frac{1}{2}\cos^2 t & 1 - \frac{1}{2}\cos t \sin t \\ -1 - \frac{1}{2}\cos t \sin t & -1 + \frac{1}{2}\sin^2 t \end{pmatrix}$$

with period π or

$$\begin{aligned} \dot{u} &= (-1 + \tfrac{1}{2}\cos^2 t)u + (1 - \tfrac{1}{2}\cos t \sin t)v \\ \dot{v} &= (-1 - \tfrac{1}{2}\cos t \sin t)u + (-1 + \tfrac{1}{2}\sin^2 t)v \end{aligned}$$

Then

$$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t}, \qquad \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}$$

are solutions with initial conditions e_1 and e_2. Therefore

$$\Phi(t) = \begin{pmatrix} e^{-\frac{t}{2}}\cos t & e^{-t}\sin t \\ -e^{-\frac{t}{2}}\sin t & e^{-t}\cos t \end{pmatrix}$$

and

$$B = \Phi(\pi) = \begin{pmatrix} e^{-\frac{1}{2}\pi} & 0 \\ 0 & e^{-\pi} \end{pmatrix}$$

and hence the origin is stable.

For linear ODEs with periodic coefficients from linearisation, more information is available about the eigenvalues.

Theorem 4.2 (Special value w.r.t perturbed periodic solutions)**.** *Let $\phi(t)$ be a periodic solution of the autonomous system $\dot{x} = f(x)$, and $\dot{v} = A(t)v$ with $A(t) = Df(\phi(t))$ is the linearisation around $\phi(t)$. Then the monodromy matrix B corresponding to $A(t)$ always have eigenvalue 1.*

Proof. Let $v = \dot{\phi}$, then taking derivative of the equations for ϕ, $v = f(\phi(t))$, we have

$$\dot{v} = Df(\phi(t))\dot{\phi}(t) = A(t)v.$$

That is the derivative $v = \dot{\phi}$ satisfies the linearised ODEs and is periodic with period T (the same period as ϕ). Therefore,

$$v(0) = v(T) = \Phi(T)v(0) = Bv(0),$$

and 1 is an eigenvalue of B with eigenvector $v(0) = \dot{\phi}(0)$. $\square$

Another result is to generalise the solution $x(t) = x_0 \exp(\int_0^T a(s)ds)$ of the scalar ODE $\dot{x} = a(t)x$ into higher dimension.

Theorem 4.3 (Evolution of the determinant)**.** *If $\Phi(t)$ is a non-singular matrix that satisfies the system of ODEs $\dot{x} = A(t)x$ (the matrix A does not have to be periodic and $\Phi(0)$ does not have to be the identity matrix), then*

$$\det\Phi(t) = \exp\left(\int_s^t \mathrm{tr}A(s)ds\right)\det\Phi(s).$$

Sketched proof. We can actually show the equivalent versions

$$\frac{d}{dt}\det\Phi(t) = \mathrm{tr}A(t)\det\Phi(t).$$

Without loss of generality, we can assume $s = 0$ and $\Phi(0) = I$. Then $\Phi(t) = I + tA(0) + O(t^2)$ when t is small. Then

$$\det\Phi(t) = \det\big(I + tA(0) + O(t^2)\big) = 1 + t\mathrm{tr}A(0) + O(t^2).$$

Therefore, $\frac{d}{dt}\det\Phi(t)\big|_{t=0} = \mathrm{tr}A(0)$. □

Remark. This theorem is a more general fact: if the points evolve under the ODE $\dot{x} = f(x,t)$, then the rate of change of the local volume element is $\mathrm{div}f(x,t)$, the divergence of the vector field $f(x,t)$. The volume (or area in two dimension) does not change if and only if the divergence of the vector field is zero. In the special case of linear ODEs with $f(x,t) = A(t)x$, we recover the above result since $\mathrm{div}f(x,t) = \mathrm{tr}A(t)$.

Now we can look at the stability. The eigenvalues of the monodromy matrix are denoted as $\rho_1, \rho_2, \cdots, \rho_n$, also called *characteristic multipliers*. Their logarithms divided by T are called *characteristic exponents*, i.e. $\rho_k = \exp(\mu_k T)$. Therefore

$$\rho_1\rho_2\cdots\rho_n = \exp(\mu_1 T)\exp(\mu_2 T)\cdots\exp(\mu_n T) = \det(B) = \exp\left(\int_0^T \mathrm{tr}A(s)ds\right).$$

If the ODEs $\dot{v} = A(t)v$ is two dimensional, and are derived from periodic solutions of

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2).$$

Then we have one characteristic multiplier $\rho_1 = 1$ (from Theorem 4.2) and the other one

$$\rho_2 = \exp\left(\int_0^T \mathrm{tr}A(s)ds\right) = \exp\left(\int_0^T \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) ds\right).$$

Therefore, the stability is determined by the *sign* of the integral of the divergence of the vector field (f_1, f_2) along the periodic solution. If the periodic solution is known, the above integral could be evaluated in some special cases, leading to conclusions about the stability of the periodic solution.

Example 4.7. Consider the ODEs

$$\dot{x} = x - y - x(x^2 + y^2), \qquad \dot{y} = x + y - y(x^2 + y^2).$$

In polar coordinates $x(t) = r(t)\cos\theta(t), y(t) = r(t)\sin\theta(t)$, the equations become

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

There is a periodic solution $r(t) = 1$ with period $T = 2\pi$. The stability of this periodic solution is easy in the polar coordinates, or using the criteria above. We have

$$A = \frac{\partial}{\partial x}\big(x - y - x(x^2 + y^2)\big) + \frac{\partial}{\partial y}\big(x + y - y(x^2 + y^2)\big) = 2 - 4(x^2 + y^2).$$

Therefore, on the periodic orbits $r = 1$ (or $x^2 + y^2 = 1$), $A(t)$ is the constant $-2 < 0$ or $\rho_2 = \exp(-2T) = \exp(-4\pi) < 1$. The periodic solution is stable.

5 Bifurcation and centre manifold

For the general ODE $\dot{x} = f(x)$ near its stationary point x^*, we learned early that if none of the eigenvalues of the Jacobian $Df(x^*)$ has a real part zero, then the behaviour of $\dot{x} = f(x)$ is determined by its linearised system $\dot{y} = Df(x^*)y$ with $y \approx x - x^*$. What happens if the Jacobian matrix $Df(x^*)$ has eigenvalues with zero real part?

If some eigenvalues have zero real part, nonlinear terms are expected to play a role, and the behaviour could change accordingly. The study of these qualitative changes in the behaviours (mainly stability/instability of stationary points and periodic orbits), subject to changes in certain parameters, is call *bifurcation theory*. Since the stability/instability of fixed points is indicated precisely by the real part of the eigenvalues, we are going to see how these eigenvalues pass the imaginary axis, as the parameter changes.

Example 5.1. Consider the following two systems

$$(a) \quad \dot{x} = \mu x, \qquad (b) \quad \begin{cases} \dot{x} &= \mu x + \omega y, \\ \dot{y} &= -\omega x + \mu y. \end{cases}$$

It is easy to see that, the eigenvalue $\lambda(\mu) = \mu$ in (a), and $\lambda(\mu) = \mu \pm i\omega$ in (b). The stability is changed when μ crosses zero. More general scenario is shown in Figure 5.1.

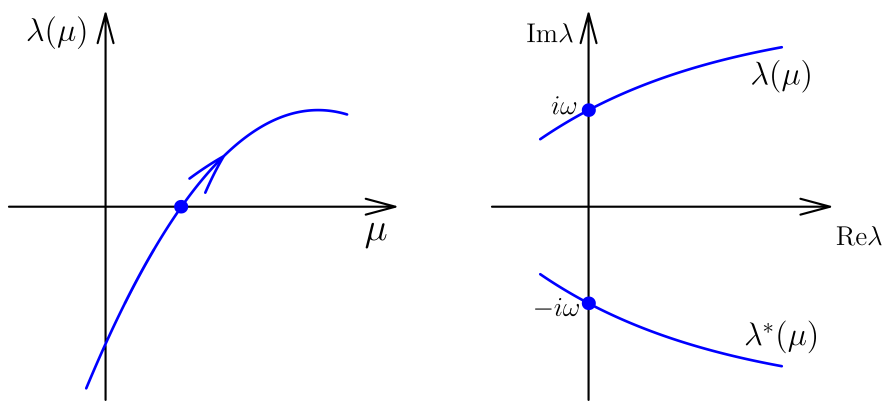


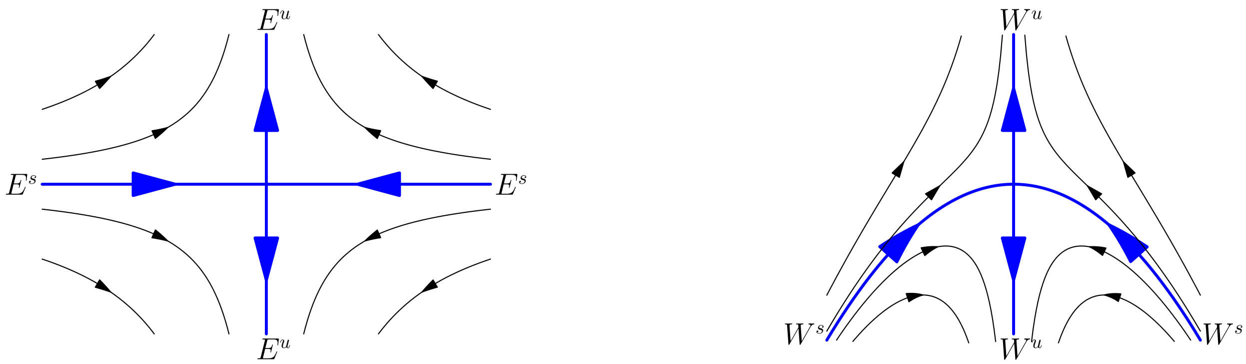
Figure 5.1: Left figure: real eigenvalue passing through zero as a function of μ; Right figure: complex eigenvalues passing through the imaginary axis (think of the eigenvalues as parametrised curves in the complex plane).

5.1 Centre manifold theorem

We learned *Stable Manifold Theorem* earlier, which states that the structure of the system near a hyperbolic fixed point does not change when nonlinear terms are added. Consider the system

$$\dot{x} = -x, \quad \dot{y} = y + x^2$$

and its linearised system as shown in Figure 5.2.



$\dot{x} = -x, \ \dot{y} = y$ $\qquad$ $\dot{x} = -x, \ \dot{y} = y + x^2$

Figure 5.2: The nonlinear system and its linearised system.

The stable manifold E^s and the unstable manifold E^u for the linearised system (in normal form) are easy to calculate, which is just the horizontal and vertical axis. Therefore, the notation E^s and E^u (instead of E^s and E^u) is used for the linearised system, to emphasize that they are linear vector spaces. The corresponding stable and unstable manifold for the nonlinear system are usually curved. For the system $\dot{x} = -x, \dot{y} = y + x^2$, the unstable manifold is still the vertical axis (show this!), but the stable manifold is different, and can be approximated as a local series expansion

$$W^s = \big\{(x, y) \mid y = M(x) = a_2 x^2 + a_3 x^3 + \cdots \big\}. \tag{5.1}$$

The constant term in $M(x)$ vanishes because W^s passes the origin, and the linear term vanishes because W^s should be tangent to E^s (the horozontal axis), the stable manifold of the linearised system. Now the coefficients $a_2, a_3, \cdots$ representing W^s can be obtained by taking the derivative of both sides of $y = M(x)$. On one hand

$$\dot{y} = y + x^2 = (a_2 + 1)x^2 + a_3 x^3 + \cdots .$$

On the other hand,

$$\frac{d}{dt} M(x) = \left(2a_2 x + 3a_3 x^2 + \cdots\right) \dot{x} = (-x) \left(2a_2 x + 3a_3 x^2 + \cdots\right).$$

Matching the two expressions for $\dot{y} = \frac{d}{dt} M(x)$, we get $a_2 = -1/3, a_3 = a_4 = \cdots = 0$. In other words, the stable manifold is exactly $y = -x^2/3$.

Because the real parts of the eigenvalues are away from zero, the nonlinear system is stable under changes in the parameters or nonlinear terms. However, if there is any eigenvalue with zero real part, we expect some qualitative changes in the property when certain parameter changes, which precisely why bifurcation theory and *Centre Manifold Theorem* are studied together.

Theorem 5.1 (Centre Manifold Theorem)**.** *Given $\dot{x} = f(x), x \in \mathbb{R}^n$, f smooth and suppose $x = 0$ is a stationary point. Suppose the Jacobian matrix $Df(0)$ has eigenvalues in sets σ_u with $\mathrm{Re}(\lambda) > 0$, σ_s with $\mathrm{Re}(\lambda) < 0$ and σ_c with $\mathrm{Re}(\lambda) = 0$ and corresponding generalized linear eigenspaces E^u, E^s and E^c respectively. Then there exist unstable and stable manifolds*

W^u, W^s of the same dimension as E^u, E^s and tangential to E^s and E^u at $x = 0$; and an invariant centre manifold W^c tangential to E^c at $x = 0$.

So in general, locally $\mathbb{R}^n = W^c \oplus W^u \oplus W^s$ with the approximate governing equations on each manifold

$$\begin{aligned} \dot{x} &= g(x) && \text{on } W^c \\ \dot{y} &= By && \text{on } W^s && \text{(stable directions)} \\ \dot{z} &= Cz && \text{on } W^u && \text{(unstable directions)}\ , \end{aligned}$$

where $g(x)$ is quadratic (or higher order) in x, all eigenvalues of B have negative real parts, and all eigenvalues of C have positive real parts.

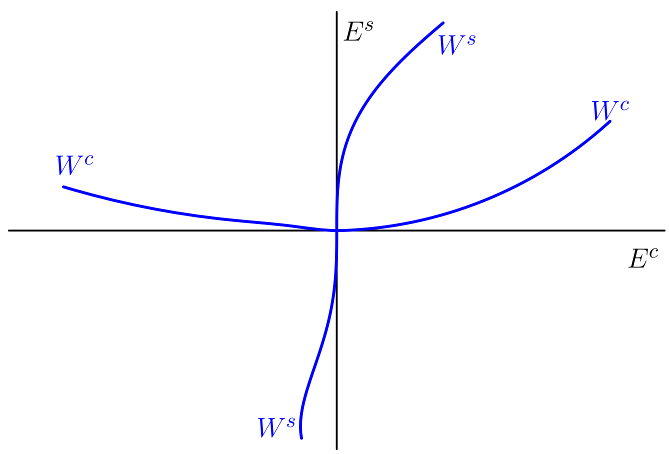


Figure 5.3: Behaviour on W^c depends on nonlinear terms, behaviour off W^c is dominated by exponential contraction in the E^s direction.

In Figure 5.3, there is no unstable direction and in the stable direction the dynamics is attracting, so solutions tend to the centre manifold very quickly. The dynamics on W^c depends on nonlinear terms, is usually much slower and characterise the dynamics of the whole system in the long time. So the question is how this decomposition can be useful in general, and how the centre manifold can be approximated or computed.

5.2 Calculating the centre manifold W^c

Suppose that after a change of coordinate transformation, the hyperplane (or line, if x is one dimension) $(x, 0)$ is spanned by E^c and $(0, y)$ by E^s, then the centre manifold is tangential to $y = 0$ at $(0, 0)$ and we may assume that

$$W^c = \{(x, y) \mid y = h(x), h(0) = 0, Dh(0) = 0\}.$$

In this coordinate, the system can be written as

$$\dot{x} = Ax + f_1(x, y), \qquad \dot{y} = Cy + f_2(x, y),$$

where all eigenvalues of A have real parts zero and those of C have real parts non-zero, f_i contain only nonlinear terms. So on the centre manifold W^c,

$$\dot{x} = Ax + f_1(x, h(x))$$

and $\dot{y}$ can be calculated on the centre manifold in two ways: directly from the $\dot{y}$ equation above, or by differentiating $y = h(x)$, i.e.

$$\dot{y} = Ch(x) + f_2(x, h(x)) \qquad \text{and} \qquad \dot{y} = \frac{d}{dt}h(x) = Dh(x)\dot{x} = Dh(x)\left[Ax + f_1(x, h(x))\right].$$

where $Dh(x)$ is the (matrix) partial derivatives of $h(x)$, in one dimension simply $h'(x)$.

Expanding h as a Taylor series (noting that the constant and linear terms vanish), the two equations for $\dot{y}$ provide two different polynomials and the coefficients of different monomials can be equated to determine the coefficients of the Taylor expansion.

For a specific problem, here is the general procedure to calculate the centre manifold (which is very similar if you want to find the stable/unstable manifold):

(a) Change the system into normal form (if needed), such that the linearised system is a diagonal matrix

(b) Identify the centre manifold E^c of the linearised system, which is the linear space spanned by the eigenvectors associated with the zero eigenvalues.

(c) Parameterise the centre manifold. You can parameterised E^c first, and then for W^c. For instance, if E^c is the y-axis, then E^c is parameterised as $x = 0$ (and $z = 0$ if in three dimension), and W^c (also a line!) is parameterised by $x = a_2y^2 + a_3y^3 + \cdots$ and $z = b_2y^2 + b_3y^3 + \cdots$. If E^c is the xy-plane in three dimension, then E^c is parameterised by $z = 0$ and W^c is parameterised by

$$z = ax^2 + bxy + cy^2 + \cdots.$$

(d) Finally determine the coefficients in the parmaterisation by differentiation on both sides.

Example 5.2. Consider the system

$$\dot{x} = xy, \quad \dot{y} = -y - x^2.$$

The linear normal form (based on the linearisation at the origin) has the constant matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \text{diag}\,(0, -1).$$

Then the eigenpairs are

$$\lambda_1 = 0, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = -1, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since the matrix A is already in normal form, no coordinate transformation is needed.

Now the centre manifold takes the form

$$y = h(x) = ax^2 + bx^3 + cx^4 + O(x^5), \tag{5.2}$$

There is no constant term, because the centre manifold passes through the origin; there is no linear term, because this manifold should be tangent to e_1 (or equivalently E^c, the centre manifold of the linearised system).

We can determine the coefficients by comparing two ways for calculating $\dot{y}$. Directly from the $\dot{y}$ equation of the system

$$\dot{y} = -y - x^2 = -(ax^2 + bx^3 + cx^4) - x^2 + O(x^5). \tag{5.3}$$

On the other hand, differentiating (5.2) w.r.t t gives $\dot{y} = dh(x)/dt = \dot{x}h'(x)$, i.e.,

$$x(ax^2 + bx^3 + cx^4 + \cdots)(2ax + 3bx^2 + 4cx^3 + \cdots) = 2ax^4 + \cdots . \tag{5.4}$$

Equating coefficients of x^2, x^3 and x^4 in (5.3) and (5.4) gives

$$-a - 1 = 0, \quad -b = 0, \quad -c = 2a^2,$$

i.e. $a = -1$, $b = 0$ and $c = -2$.

Thus the centre manifold is $y = -x^2 - 2x^4 + O(x^5)$ and the dynamics on the centre manifold is

$$\dot{x} = xh(x) = -x^3 - 2x^5 + O(x^7).$$

Thus $\dot{x} < 0$ if $x > 0$ and $\dot{x} > 0$ if $x < 0$. So the origin is stable and the solutions look like a stable node, but the motion onto the centre manifold in the y-direction is much faster than the motion on the centre manifold, leading to a phase portrait as shown in Figure 5.4.

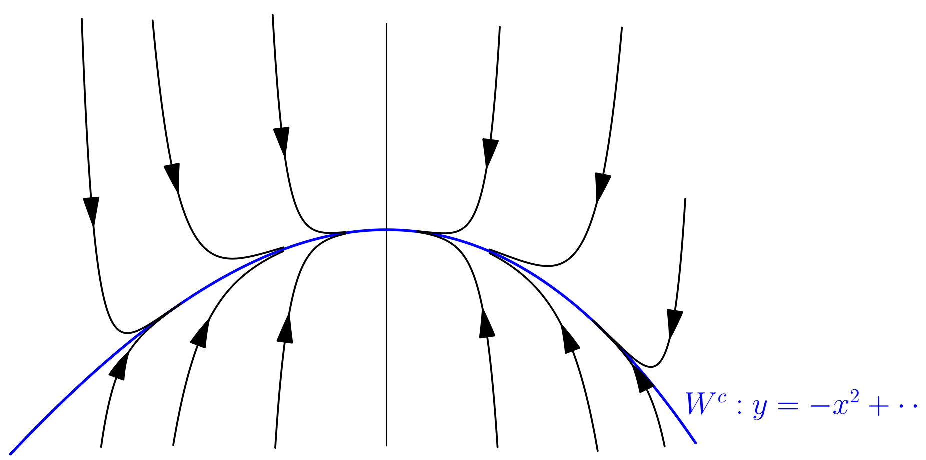


Figure 5.4: Phase portrait showing exponential collapse onto the centre manifold and then slow motion towards $(0, 0)$ on the centre manifold.

Remark. If you try higher order terms, you get

$$y = -x^2 - 2x^4 - 12x^6 - 112x^8 - 1360x^{10} - 19872x^{12} + \cdots .$$

The fast increasing of the coefficients implies that this approximation is valid only in a small neighbourhood of the origin.

Example 5.3. Consider the system

$$\dot{x} = y - x + xy, \qquad \dot{y} = x - y - x^2.$$

First we have to convert the linearised system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

into normal form. It is easy to calculate the eigenpairs of A,

$$\lambda_1 = 0, \quad e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -2, \quad e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let

$$U = [e_1, e_2]^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

the change of variable from (x, y) to (u, v) (in normal form) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+y)/2 \\ (y-x)/2 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = U^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = [e_1, e_2] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ u + v \end{pmatrix}.$$

The new system in (u, v) is

$$\dot{u} = uv - v^2, \dot{v} = -2v + uv - u^2.$$

The centre manifold is parametrised by $v = h(u) = au^2 + bu^3 + cu^4 + \cdots$, then

$$\begin{aligned} \dot{v} &= (2au + 3bu^2 + 4cu^3 + \cdots)\dot{u} \\ &= (2au + 3bu^2 + 4cu^3 + \cdots)\big(u(au^2 + bu^3 + cu^4 + \cdots) - (au^2 + bu^3 + cu^4 + \cdots)^2\big) \\ &= 2a^2u^4 + \cdots \end{aligned}$$

and on the other hand

$$\begin{aligned} \dot{v} &= -2v + uv - u^2 \\ &= -2(au^2 + bu^3 + cu^4 + \cdots) + u(au^2 + bu^3 + cu^4 + \cdots) - u^2 \\ &= (2a + 1)u^2 + (2b - a)u^3 + (3c - b)u^4 + \cdots. \end{aligned}$$

Comparing the coefficients of u^2, u^3 and u^4 of the two expressions of $\dot{v}$, we get

$$a = -\frac{1}{2}, \quad b = -\frac{1}{4}, \quad c = -\frac{3}{8}, \qquad \text{or} \qquad v = -\frac{1}{2}u^2 - \frac{1}{4}u^3 - \frac{3}{8}u^4 + \cdots.$$

The dynamics on the centre manifold is

$$\dot{u} = uv - v^2 = -\frac{1}{2}u^3 - \frac{1}{4}u^4 - \frac{3}{8}u^5 + \cdots,$$

which is stable if u is small. Going back to the original coordinates, the centre manifold is approximately

$$y - x = -\frac{1}{4}(x+y)^2 - \frac{1}{16}(x+y)^3 - \frac{3}{64}(x+y)^4 + \cdots$$

Remark. The calculation could be quite involved if you are calculating unnecessary higher order terms than needed in the end. In general, the lowest power appears in equations with stable linear part. For instance, the second expression above starts with u^2, while the first expression starts with u^4.

Remark. Strictly speaking, the change of variables from (x, y) to (u, v) is not necessary, but we need to know that the centre manifold is represented as

$$y - x = a_2(x + y)^2 + a_3(x + y)^3 + \cdots .$$

Take the derivative of both sides (w.r.t t),

$$\dot{y} - \dot{x} = \big(2a_2(x + y) + 3a_3(x + y)^2 + \cdots\big)(\dot{x} + \dot{y}).$$

After substituting $\dot{x}$ and $\dot{y}$, we compare the coefficients of powers of $(x + y)$ on both sides, and we should get the same answer. (Probably it is worth the effect to perform the change of variable at the very beginning).

5.3 Extended centre manifold

As it stands, the CMT does not allow us to deal with parameters. To include the effect of parameters and hence to treat bifurcations, we work on the extended centre manifolds by augmenting the equation with the apparently trivial equation $\dot{\mu} = 0$:

$$\begin{aligned}
\dot{x} &= Ax + f_1(x, y, \mu),\\
\dot{y} &= Cy + f_2(x, y, \mu),\\
\dot{\mu} &= 0.
\end{aligned}$$

The additional equation allows us to parametrise the centre manifold as $y = h(x, \mu)$ instead of the form $y = h(x)$ considered in the last section (hence the *extended* centre manifold).

The trivial equation $\dot{\mu} = 0$ adds one more dimension to the centre manifold and allows us to work in a neighbourhood of both $(x, y) = (0, 0)$ in phase space *and* $\mu = 0$ in parameter space, where $\mu = 0$ is the value at which the bifurcation occurs. So A has the zero real part eigenvalues and C has stable and unstable manifolds, and f_1, f_2 contain only nonlinear terms. The CMT gives the motion on the stable and unstable manifolds,W^s and W^u in y, and there is a $n_c + 1$ dimensional centre manifold (where n_c is the dimension of x), valid for $|x|$ and $|\mu|$ small.

This time, if coordinates are chosen so that the central motion is in normal form, the extended centre manifold can be parametrised by $y = h(x, \mu)$, with

$$h(0, 0) = 0, \qquad Dh(0, 0) = 0.$$

Notice that $Dh = [D_x h, D_\mu h]$, which is the partial derivative w.r.t both variables. Then $\dot{x} = Ax + f_1(x, h(x, \mu), \mu)$ is the equation on the (extended) centre manifold.

There are three typical equations (to leading order) on the extended centre manifold if $A = 0$ and x is a scalar:

$$\begin{aligned}
&\dot{x} = \mu - x^2 && \text{(saddlenode bifurcation)}\\
&\dot{x} = \mu x - x^2 && \text{(transcritical bifurcation)}\\
&\dot{x} = \mu x - x^3 && \text{(pitchfork bifurcation)}
\end{aligned} \tag{5.5}$$

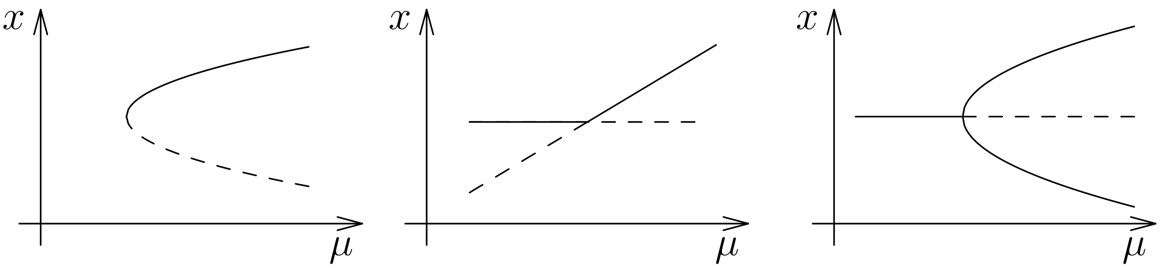


Figure 5.5: (μ, x) plane for local bifurcations of stationary points: saddlenode, transcritical and pitchfork (supercritical).

Typical behaviour is sketched in the (x, μ) plane: these are called *bifurcation diagrams*. By convention, dotted lines are used to show unstable solutions and continuous lines for stable solutions. The pitchfork illustrated her is *supercritical*, meaning that the non-trivial stationary points are stable; if they were unstable then it would be a *subcritical* pitchfork bifurcation (a *subcritical* pitchfork bifurcation when the solid line is dashed, and the dashed line is solid). More details will be given in the next subsection.

Example 5.4. Consider the second order ODE

$$\ddot{u} + \dot{u} - \mu u + u^2 = 0$$

with a parameter μ. Setting $v = \dot{u}$, we get the equivalent system of ODEs

$$\dot{u} = v, \quad \dot{v} = -v + \mu u - u^2. \tag{5.6}$$

At the origin, the matrix for the linear part is $\begin{pmatrix} 0 & 1 \\ \mu & -1 \end{pmatrix}$. The eigenvalues satisfies $s(s+1) - \mu = 0$, so there is a eigenvalue with real part zero, if and only if $s = 0$, or $\mu = 0$. Therefore we expect a bifurcation at the origin if $\mu = 0$.

Rough Explanation of what happens for μ small: The stationary points are governed by $v = 0$, $\mu u - u^2 = 0$. That is $u = 0$, or $u = \mu$, and we expect a transcritical bifurcation (exchange of stability).

The general procedure: (a) Transform to normal form (including in the $\dot{\mu} = 0$ equation); (b) Expand extended CM; (c) Calculate dynamics on CM.

(a). Transformation: The linear part (5.6) at the origin if $\mu = 0$ is not in normal (diagonal) form. We have the eigenpairs,

$$\lambda_1 = 0,\ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -1,\ e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence the change of coordinate uses the matrix of eigenvectors and the NEW coordinates x, y are defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Hence the coordinate transform is

$$x = u + v, \quad y = -v \qquad \text{or} \qquad u = x + y, \quad v = -y.$$

In terms of these new coordinates, the system (5.6) becomes

$$\begin{aligned}\dot{x} &= \dot{u} + \dot{v} = v - v + \mu u - u^2 = \mu(x+y) - (x+y)^2, \\ \dot{y} &= -\dot{v} = v - \mu u + u^2 = -y - \mu(x+y) + (x+y)^2.\end{aligned}$$

(b). Extended Centre Manifold Now the extended system in the new coordinates is

$$\dot{x} = \mu(x+y) - (x+y)^2, \quad \dot{y} = -y - \mu(x+y) + (x+y)^2, \quad \dot{\mu} = 0.$$

The extended centre manifold should be tangential to the (x, μ) plane (or $y = 0$) at $(x, y, \mu) = (0, 0, 0)$, and is parametrised by

$$y = h(x, \mu) = ax^2 + bx\mu + c\mu^2 + h.o.t.$$

From the $\dot{y}$ equation

$$\dot{y} = -(ax^2 + bx\mu + c\mu^2 + \dots) - \mu(x + \dots) + x^2 + \cdots = (1-a)x^2 - (b+1)x\mu + \cdots$$

From the definition of the extended centre manifold

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \mu}\dot{\mu} = (2ax + b\mu)\dot{x} = \cdots,$$

where all the terms are at least cubic. So equating coefficients of the quadratic terms (of which there are none in the second equation!) gives $a = 1$, $b = -1$, and the extended centre manifold is

$$y = x^2 - x\mu + \dots.$$

(c). Dynamics on the centre manifold. Locally on the extended Centre Manifold $\dot{\mu} = 0$ is trivial so it is the $\dot{x}$ equation that is interesting:

$$\begin{aligned}\dot{x} &= \mu(x + x^2 - \mu x + \dots) - (x^2 + 2x(x^2 - \mu x) + \dots) \\ &= \mu x - x^2 + O(x^3)\end{aligned}$$

Substituting back into the equation for $\dot{x}$ we get (to leading order)

$$\underbrace{\dot{x} = \mu x - x^2}_{\text{Standard Form for transcritical}} \qquad + O(x^3).$$

The phase portrait for the reduced dynamics for x is shown in Figure 5.6 and the phase portrait for the original system is in Figure 5.7.

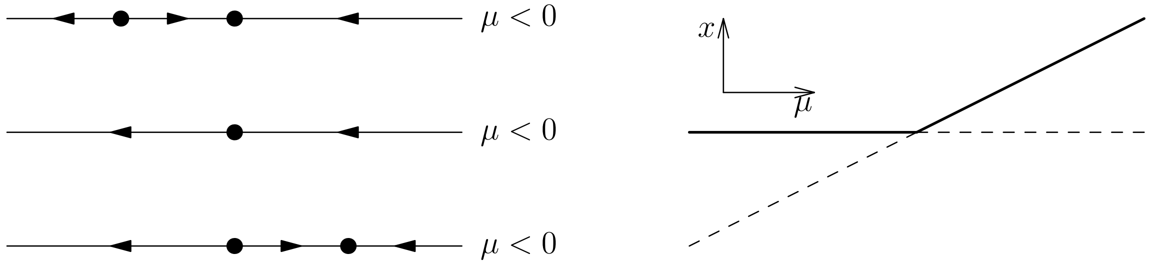


Figure 5.6: Phase portraits on the (one-dimensional) centre manifold and the bifurcation diagram.

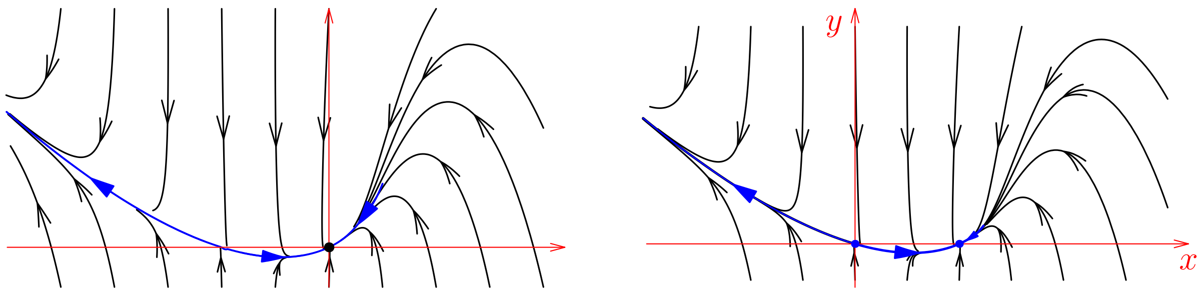


Figure 5.7: Full phase portraits of the dynamics in $\mu < 0$ and $\mu > 0$.

Remark. If the stable manifold is of higher dimension, then $y_1 = h_1(x, \mu), y_2 = h_2(x, \mu)$ and we need to find h_1, h_2 using the same method. For example, for the system

$$\dot{x} = \mu x - yz, \quad \dot{y} = -y + x^2, \quad \dot{z} = -z + x^3.$$

Add $\dot{\mu} = 0$ to this system, then the stable manifold expanded by y and z is parameterised by x and μ, that is

$$y = h_1(x, \mu) = a_1 x^2 + a_2 x\mu + a_3 \mu^2 + \cdots, \quad z = h_2(x, \mu) = b_1 x^2 + b_2 x\mu + b_3 \mu^2 + \cdots.$$

Then $a_1 = 1, a_2 = a_3 = b_1 = b_2 = b_3 = 0$. That is $y = x^2 + \cdots$, but we have to go to cubic polynomials to find the stable manifold for z, which gives $z = x^3 + \cdots$. Therefore, the reduced dynamics on the stable manifold is

$$\dot{x} = f_\mu(x) = \mu x - x^5.$$

If $\mu < 0$, $x = 0$ is the only stable fixed point. If $\mu > 0$, there are three fixed point $0, \mu^{1/4}$ and $-\mu^{1/4}$. Since

$$f'_\mu(0) = \mu > 0, \quad f'_\mu(\pm\mu^{1/4}) = \mu - 5(\pm\mu^{1/4})^4 = -4\mu < 0,$$

the fixed point 0 is unstable, and the fixed points $\pm\mu^{1/4}$ are stable.

5.4 Classifications of bifurcations

Suppose $x = 0 \in \mathbb{R}^n$ is a stationary point of the system of ODEs $\dot{x} = f(x, \mu)$ if $\mu = 0$, and $D_x f(0, 0)$ has a single zero eigenvalue. (If the stationary point is x^* at μ^*, then we simply work in shifted coordinates $x - x^*$ and $\mu - \mu^*$). Now we consider the extended centre manifold for the system governed by ($x \in \mathbb{R}$)

$$\dot{x} = f(x, \mu), \qquad \dot{\mu} = 0$$

where f satisfies $f(0,0) = 0$ and $f_x(0,0) = 0$. Consider the Taylor series expansion of f for $|x|, |\mu|$ small:

$$\dot{x} = f(0,0) + f_x(0,0)x + f_\mu(0,0)\mu + \frac{1}{2!}(f_{xx}x^2 + f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu) + O(|x|^3, |\mu|^3)$$

where all partial derivatives are evaluated at $(0,0)$.

By the assumption that $f(0,0) = 0$ ($x = 0$ is the stationary point on the centre manifold for $\mu = 0$) and $f_x(0,0) = 0$ (there is a zero eigenvalue), the above Taylor series is simply

$$\dot{x} = f_\mu(0,0)\mu + \frac{1}{2}\left(f_{xx}(0,0)x^2 + f_{\mu\mu}(0,0)\mu^2 + 2f_{x\mu}(0,0)x\mu\right) + \cdots .$$

Different bifurcations could occur, depending on whether the partial derivatives vanish or not.

Saddle-node Bifurcation $\dot{x} = \mu - x^2$

If both $f_\mu(0,0)$ and $f_{xx}(0,0)$ are non-zero, then

$$\dot{x} = f_\mu(0,0)\mu + \frac{1}{2}f_{xx}(0,0)x^2 + O(|x\mu|, |\mu|^2, \cdots) \approx \mu f_\mu(0,0) + \frac{x^2}{2}f_{xx}(0,0).$$

The stationary points are

$$x^*_\pm \approx \pm\sqrt{-\frac{2f_\mu(0,0)}{f_{xx}(0,0)}\mu} \tag{5.7}$$

if $\mu f_\mu(0,0)/f_{xx}(0,0) \le 0$. So the stability is determined for sufficiently small $|x|$ and $|\mu|$ by the sign of f_{xx} and f_μ: there is no solution, if $\mu f_\mu/f_{xx} > 0$, and there are two solutions given by (5.7) if $\mu f_\mu/f_{xx} \le 0$. Since

$$\frac{\partial}{\partial x}f(x,\mu)\bigg|_{x=x^*_\pm} \approx x^*_\pm f_{xx}(0,0) = \pm\sqrt{-\frac{2f_\mu(0,0)}{f_{xx}(0,0)}\mu}\, f_{xx}(0,0).$$

Therefore, if $\mu f_\mu/f_{xx} > 0$, x_+ is stable, x_- unstable if $f_{xx} < 0$ and x_- is stable, x_+ unstable if $f_{xx} > 0$ This is a *saddle-node* bifurcation, also called tangential bifurcation or fold bifurcation.

Transcritical Bifurcation $\dot{x} = \mu x - x^2$

If in addition to $f(0,0) = f_x(0,0) = 0$, $f_\mu(0,0)$ is zero, but $f_{xx}(0,0) \neq 0$, the ODE equation becomes

$$\dot{x} \approx \frac{1}{2}\Big(f_{xx}(0,0)x^2 + 2f_{x\mu}(0,0)x\mu + f_{\mu\mu}(0,0)\mu^2\Big)$$

Then the possible stationary points are $x_\pm^* = k_\pm \mu$, where

$$k_\pm = \frac{-f_{x\mu} \pm \sqrt{f_{x\mu}^2 - f_{xx} f_{\mu\mu}}}{f_{xx}}.$$

So if $f_{x\mu}^2 - f_{xx} f_{\mu\mu} > 0$, there are two branches of solutions which intersect at the bifurcation point $(0,0)$. This is a *transcritical* bifurcation. Stability is determined by looking at the leading order terms of the derivative $f_x(x,\mu)$ and a relatively simple manipulation shows that one branch is stable and the other is unstable, with stability being *exchanged* as μ passes through zero. To show the stability,

$$\frac{\partial}{\partial x} f(x,\mu)\bigg|_{x=x_\pm^*} = f_{xx}(0,0) x_\pm^* + f_{x\mu}(0,0)\mu = \pm\mu\sqrt{f_{x\mu}^2 - f_{xx} f_{\mu\mu}}.$$

So the fixed point x_+ is stable if $\mu < 0$ and unstable if $\mu > 0$; x_- has the opposite stability property.

Pitchfork Bifurcation $\dot{x} = \mu x - x^3$

If $f_\mu(0,0) = f_{xx}(0,0) = 0$ then

$$\dot{x} \approx \frac{1}{2}\big(f_{\mu\mu}\mu^2 + 2 f_{x\mu} x\mu\big) + \frac{1}{6}\big(f_{xxx} x^3 + f_{\mu\mu\mu}\mu^3 + \dots\big). \tag{5.8}$$

If $f_{x\mu} \neq 0$, there is one branch of solutions with $x \approx -\frac{f_{\mu\mu}}{2 f_{x\mu}}\mu$. However there is a second set of solutions by balancing the second term $f_{x\mu} x\mu$ and the third terms $f_{xxx} x^3$:

$$f_{x\mu} x\mu + \frac{1}{6} f_{xxx} x^3 = 0$$

from which, provided $f_{xxx} \neq 0$,

$$x^2 = -\frac{6 f_{x\mu}}{f_{xxx}}\mu \tag{5.9}$$

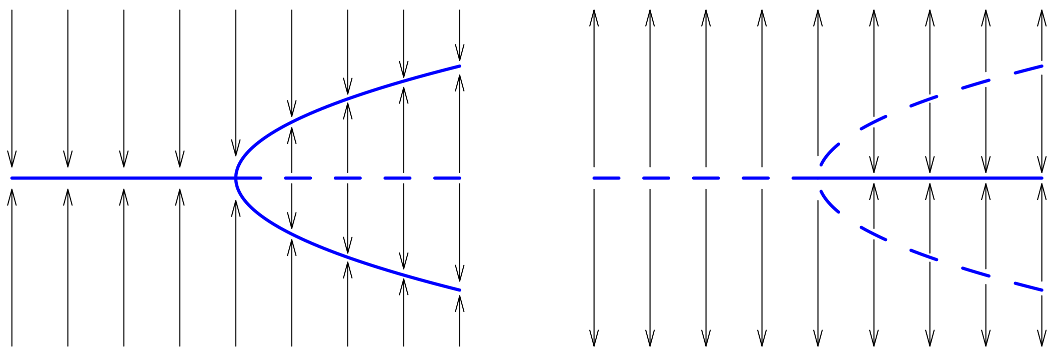
giving two new solutions in whichever sign of μ makes the right hand side positive. There are no other ways of balancing leading order terms (by posing $x \sim \mu^\alpha$) so these are the only bifurcating solutions. Since

$$\frac{\partial}{\partial x} f(x,\mu) = f_{x\mu}\mu + \frac{1}{2} f_{xxx} x^2 + \cdots, \tag{5.10}$$

we see that the solution $x \approx -\frac{f_{\mu\mu}}{2 f_{x\mu}}\mu$ is stable (locally) if $f_{x\mu}\mu < 0$ and unstable if $f_{x\mu}\mu > 0$. So the sign of $f_{x\mu}$ determines on which side of $\mu = 0$ this branch is stable.

The stability of second set of solutions is determined by substituting (5.9) into (5.10) giving $-2 f_{x\mu}\mu$ and so the stability is the opposite of the simple branch described above.

This is called a *pitchfork* bifurcation: if the non-trivial branch is stable it is called a *supercritical* pitchfork bifurcation and if the non-trivial branch is unstable it is called a *subcritical* pitchfork bifurcation, as shown in Figure 5.8.



Supercritical Pitchfork Bifurcation $\dot{x} = \mu x - x^3$ Subcritical Pitchfork Bifurcation $\dot{x} = -\mu x + x^3$

Figure 5.8: Two types of Pitchfork Bifurcation

Remark. It should be noted that the classification of bifurcation is based on the behaviour near the bifurcation point: in saddle-node bifurcation, the number of fixed points is from zero to two, one stable and one unstable; in transcritical bifurcation, two fixed points always exist and exchange stability; in pitchfork bifurcation, the number of fixed points changes from one to three, and the stability is exchanged.

5.5 Hopf bifurcations

The only 'typical' case not treated in the previous section is the appearance of purely imaginary (simple) eigenvalues $\pm i\omega$, instead of zero eigenvalues. Usually, there is only one zero eigenvalue (as seen in all examples in the previous section), and the centre manifold is only one dimension. But in the simplest case with bifurcation with two purely imaginary eigenvalues, the centre manifold at $\mu = 0$ is two dimensional and the extended centre manifold is three dimensional. The equations governing the bifurcations could very complicated, but a standard example will be enough for our purposes.

Consider the canonical example

$$\dot{x} = \mu x - \omega y - x(x^2 + y^2), \quad \dot{y} = \omega x + \mu y - y(x^2 + y^2)$$

or in polar coordinates $r = \sqrt{x^2 + y^2}, \theta = \arctan y/x$

$$\dot{r} = \mu r - r^3, \quad \dot{\theta} = \omega. \tag{5.11}$$

The linearisation about the origin is

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

with eigenvalues $\mu \pm i\omega$. So if $\mu < 0$, the origin is stable and if $\mu > 0$ it is unstable. Observe that the $\dot{r}$ equation implies that if $\mu > 0$, $r \to \sqrt{\mu}$, i.e. there is a periodic orbit of radius $\sqrt{\mu}$, which is stable (sketch the right hand side of the $\dot{r}$ equation if this is not obvious).

This is an example of a Hopf bifurcation, also known as Poincaré-Andronov-Hopf bifurcation. As the parameter is varied, a stationary point changes its stability and a periodic orbit

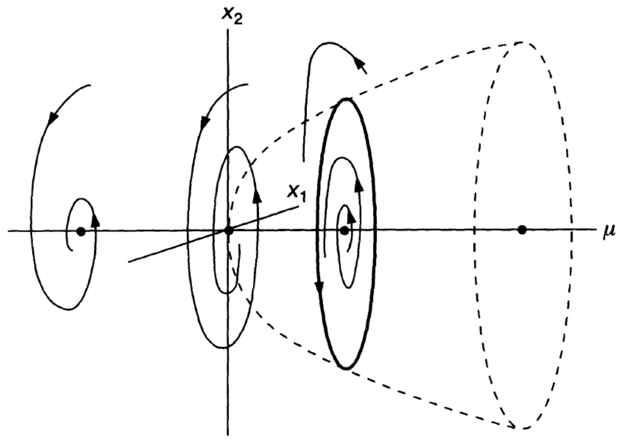


Figure 5.9: Hopf bifurcation: when μ increases, the stable focus becomes unstable, and a periodic solution called *limit cycle* appears.

is created with the opposite stability (like a pitchfork bifurcation in r). If the bifurcating periodic orbit is stable, then this is a *supercritical Hopf bifurcation* and if it is unstable this is a *subcritical Hopf bifurcation*. The above canonical example can also be written with complex numbers. That is, if we define $z(t) = x(t) + iy(t)$, then the above system is equivalent to

$$\frac{d}{dt}z = (\mu + i\omega)z - z|z|^2.$$

Remark. If only the first equation in (5.11) in the radial variable r is taken, a pitchfork bifurcation happens at $\mu = 0$. This connection (at least qualitatitively) between Hopf bifurcation (for two variables) and pitchfork bifurcation (in radial like variable) is true in general.

Example 5.5 (Brusselator model for autocatalytic reaction)**.** Consider the system of ODEs

$$\dot{x} = a - (b+1)x + x^2y, \qquad \dot{y} = bx - x^2y,$$

where a and b are two positive parameters. The unique steady state is $(a, b/a)$ with the Jacobian

$$J = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}.$$

Since the determinant of J is a^2, the only possible bifurcation Hopf bifurcation, which occurs only when the trace is zero. That is when $b^* = 1 + a^2$. For $b < b^*$, the steady state $(a, b/a)$ is stable; for $b > b^*$, it is unstable and a new periodic solution (a limit cycle) appears.

Example 5.6. We considered in Example 4.3 the following model

$$\dot{x} = -x + ay + x^2y, \qquad \dot{y} = b - ay - x^2y$$

for glycolysis oscillation with $b = 1/2$ and $a > 0$. For general b, the only stationary point is

$$x^* = b, \qquad y^* = \frac{b}{a+b^2}.$$

We can find the conditions for possible bifurcations. From the Jacobian matrix

$$J(x,y) = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2, \end{pmatrix}$$

we have

$$J(x^*,y^*) = \begin{pmatrix} \frac{b^2-a}{b^2+a} & a+b^2 \\ -\frac{2b^2}{a+b^2} & -a-b^2 \end{pmatrix}$$

Since $\det J(x^*,y^*) = a+b^2 > 0$, the only possible bifurcation is the real parts of the complex eigenvalues pass zero (real eigenvalues can not pass zero, which leads to zero determinant), or Hopf bifurcation. This happens when the trace of $J(x^*,y^*)$ is zero, that is,

$$\mathrm{tr}J(x^*,y^*) = \frac{b^2-a-a^2-2ab^2-b^4}{a+b^2}.$$

If $\mathrm{tr}J(x^*,y^*) < 0$, the fixed point (x^*,y^*) is stable; otherwise it is not stable.

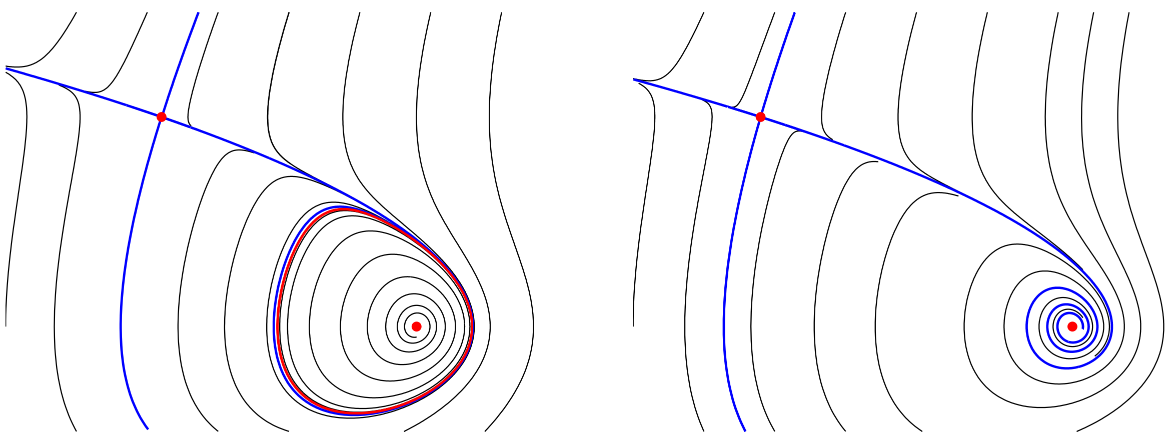


Figure 5.10: The phase portrait of the system (5.12) for $\mu = 1.8$ (left figure) and $\mu = 2.2$ (right right). As μ passes 2, the periodic solution (or the limit cycle) disappear, and the unstable focus at $(\mu-1,-1)$ becomes a stable focus. (Can you add arrows to indicate the direction of the trajectories, based on the local behaviours of the stationary points?)

Example 5.7. Consider the following system

$$\dot{x} = 1-y^2, \qquad \dot{y} = -x-\mu y-y^2, \tag{5.12}$$

for $\mu \geq 0$.

The fixed points are $(\mu-1,-1)$ and $(-\mu-1,1)$ with the Jacobian matrix is $J = \begin{pmatrix} 0 & -2y \\ -1 & -\mu-2y \end{pmatrix}$.

At the fixed point $(-\mu-1,1)$, $J = \begin{pmatrix} 0 & -2 \\ -1 & -\mu-2 \end{pmatrix}$. Since $\det J = -2 < 0$, the two eigenvalues have opposite signs, and this is always a saddle point.

At the fixed point $(\mu-1,-1)$, $J = \begin{pmatrix} 0 & 2 \\ -1 & -\mu+2 \end{pmatrix}$ and the eigenvalues are the roots of

$$\lambda^2 + (\mu-2)\lambda + 2 = 0,$$

or $\lambda_{\pm} = \frac{2-\mu\pm\sqrt{\mu^2-4\mu-4}}{2}$. Since $\lambda_+\lambda_- = 2 > 0$, the real parts of the eigenvalues pass zero if and only if μ passes 2. At $\mu = 2$, the two eigenvalues are $\pm\sqrt{2}i$, this is a unstable focus becomes a stable focus. When μ continues to increase beyond $1+2\sqrt{2}$, the discriminant $\Delta = \mu^2 - 4\mu - 4$ becomes positive, and the two eigenvalues are both real and negative. Therefore, the stable focus becomes a stable node, as shown in Figure 5.10.

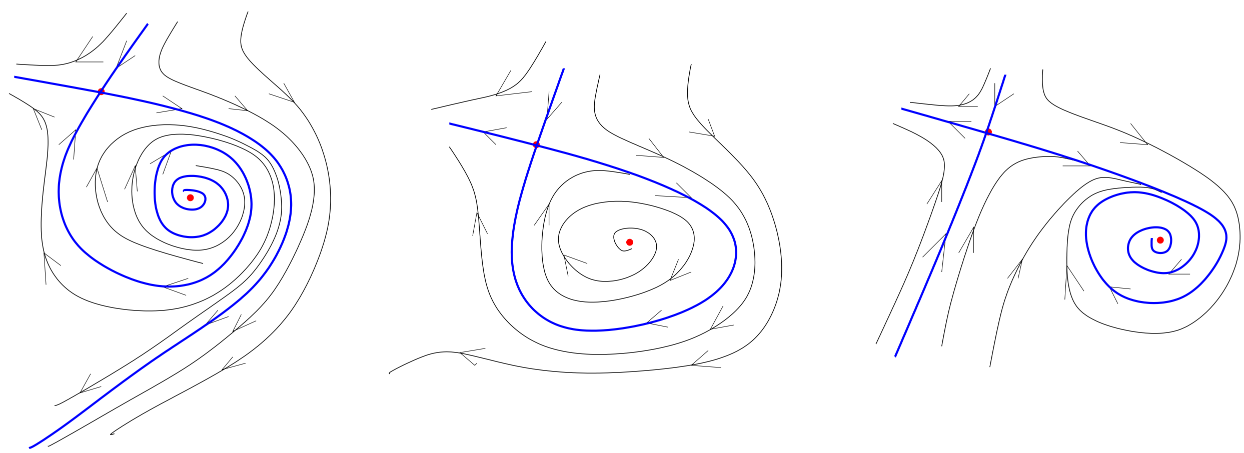


Figure 5.11: The global bifurcation at about $\mu^* = 1.63$, where there is a homoclinic orbit at $(-\mu-1, 1)$.

Remark. There is actually another bifurcation at about $\mu^* = 1.63$, with a homoclinic orbit at the fixed point $(-\mu-1,1)$: the unstable manifold W^u coincides with the stable manifold W^s through this fixed point. This kind of *global bifurcation* is much more difficult to study, where the critical parameter μ^* can not be determined as the local bifurcation in the previous few examples.

6 Maps and their bifurcation

6.1 Fixed points and periodic orbits of maps

Recall that a discrete time system (or map) is defined by a difference equation

$$x_{n+1} = f_\mu(x_n), \qquad x_n \in \mathbb{R}^n,$$

with parameter(s) μ (the subscript μ may be omitted if no confusion arises). Similar to continuous dynamical system, simple solutions include:

i) **Fixed Points:** $x_{n+1} = x_n$, that is solutions of $x^* = f(x^*)$.

ii) **Periodic orbits:** $(x_0, \ldots, x_{p-1})$ with $x_k = f(x_{k-1}), k = 1, \ldots, p-1$ and $x_0 = f(x_{p-1})$. Therefore,

$$x_k = f^p(x_k) = \underbrace{f(\cdots(f(x_k))\cdots)}_{p \text{ iterations}}, \qquad k = 0, 1, 2, \cdots, p-1.$$

That is, periodic points are *fixed points* of an iterate f^p of the map. We usually work with smallest period.

The *stability* of fixed points or periodic orbits can also be studied via linearisation. If x_n is close to the fixed point, let $y_n = x_n - x^*$, then

$$y_n = x_n - x^* = f(x_{n-1}) - f(x^*) \approx (x_{n-1} - x^*)f'(x^*) = f'(x^*)y_{n-1},$$

and $y_n \approx [f'(x^*)]^n y_0$. Therefore, a fixed point x^* is *linearly stable* if $|f'(x^*)| < 1$.

For a periodic orbit with period p, the condition for the fixed points is

$$|(f^p)'(x_k)| < 1 \qquad k = 0, 1, 2, \cdots p-1.$$

If fact, we only need to check one k, since $(f^p)'(x_0) = (f^p)'(x_1) = \cdots = (f^p)'(x_{p-1})$. Using the chain rule (try the case of $p = 2$ and $p = 3$ to see how it works),

$$\begin{aligned}
\frac{d}{dx} f^p(x_k) &= f'(\underbrace{f(\cdots f(x_k)\cdots)}_{p-1 \text{ iterations}})f'(\underbrace{f(\cdots f(x_k)\cdots)}_{p-2 \text{ iterations}})\cdots f'(x_k) \\
&= f'(x_{k+p-1})f'(x_{k+p-2})\cdots f'(x_k) \\
&= f'(x_{k-1})f'(x_{k-2})\cdots f'(x_k) \\
&= f'(x_0)f'(x_1)\cdots f'(x_{p-1}).
\end{aligned} \tag{6.1}$$

So there is only one condition for the linear stability of a periodic orbit: $\prod_{k=0}^{p-1} |f'(x_k)| < 1$.

The concept of *invariant set* can also be defined for maps, but is less used than that for the continuous dynamical systems.

Example 6.1. Let λ be any non-zero constant. Then the parabola $P = \{(x, y) \mid y = x^2\}$ is an invariant set for the map $x_{n+1} = \lambda x_n, \quad y_{n+1} = \lambda^2 y_n$. In fact, if $(x_n, y_n) \in P$, then $y_n = x_n^2$ and

$$y_{n+1} = \lambda^2 y_n = \lambda^2 x_n^2 = (\lambda x_n)^2 = x_{n+1}^2.$$

That is $(x_{n+1}, y_{n+1}) \in P$ as well. Therefore, the parabola P is invariant.

Because of the points x_n are discrete in space, special graphic tools (other than the phase portrait) are used. First, fixed points for the one dimensional map $x_{n+1} = f(x_n)$ can be viewed as the intersection of the straight line $y = x$ and the curve $y = f(x)$. If $f'(x^*)$ is positive at the fixed point x^*, the stability can also be determined graphically, by comparing $f'(x^*)$ with the slope of $y = x$ (see Figure 6.1).

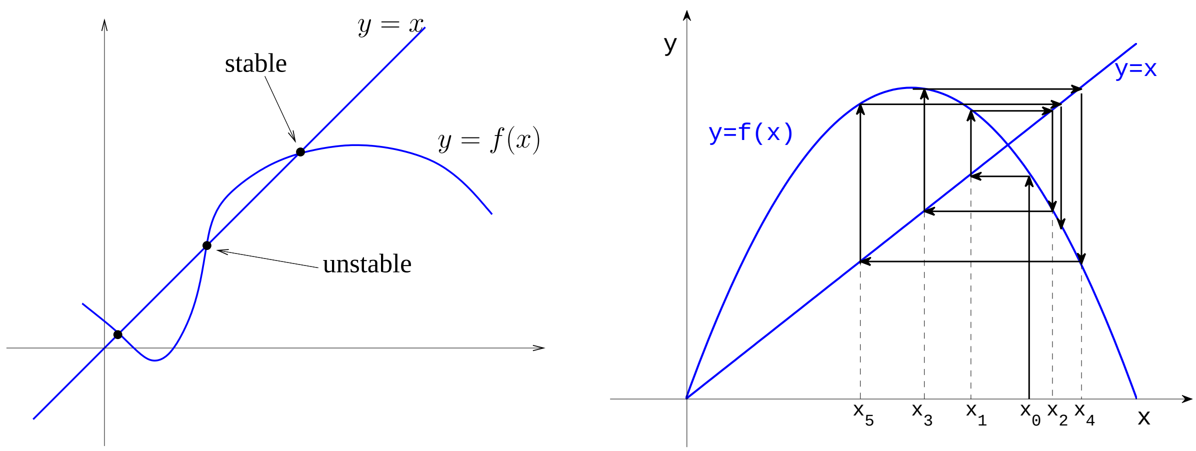


Figure 6.1: Left figure: graphic representation of the fixed points at the intersection between the straight line $y = x$ and the curve $y = f(x)$. Right figure: cobweb diagram showing the iteration of the map $x_{n+1} = f(x_n)$.

The iteration of the trajectory $x_0, x_1, \cdots$ can be viewed from the cobweb diagram (right figure in Figure 6.1): (1) the vertical line $x = x_n$ intersect the curve $y = f(x)$ at $(x_n, f(x_n)) = (x_n, x_{n+1})$; (2) the horizontal line through (x_n, x_{n+1}) intersect the line $y = x$ at $(x_{n+1}, f(x_{n+1})$; (3) then the vertical line through (x_{n+1}, x_{n+1}) becomes $x = x_{n+1}$ and the whole process can be continued again.

The behaviour of the map $x_{n+1} = f_\mu(x_n)$ near a fixed point can be understood using the cobweb diagram as shown in Figure 6.2. While the stability (inward towards the fixed point x^* or not) is determined by whether $|f'_\mu(x^*)|$ is greater than unit, the sign of $f'_\mu(x^*)$ determines whether the diagram looks like stairs ($f'_\mu(x^*) > 0$) or spirals ($f'_\mu(x^*) < 0$).

6.2 Bifurcation of maps

Similarly, as the parameter μ in the map $x_{n+1} = f_\mu(x_n)$ varies, bifurcation could occur at the fixed point $x^* = f_\mu(x^*)$ if $|f'_\mu(x^*)|$ passes one. To compare with the continuous dynamical systems $\dot{x} = f_\mu(x)$, we can consider the analogous discrete maps $x_{n+1} = x_n + f_\mu(x_n)$ (instead of $x_{n+1} = f_\mu(x_n)$, such that the fixed points in both cases coincide, that is $f_\mu(x^*) = 0$ and the bifurcation diagrams are exactly the same.

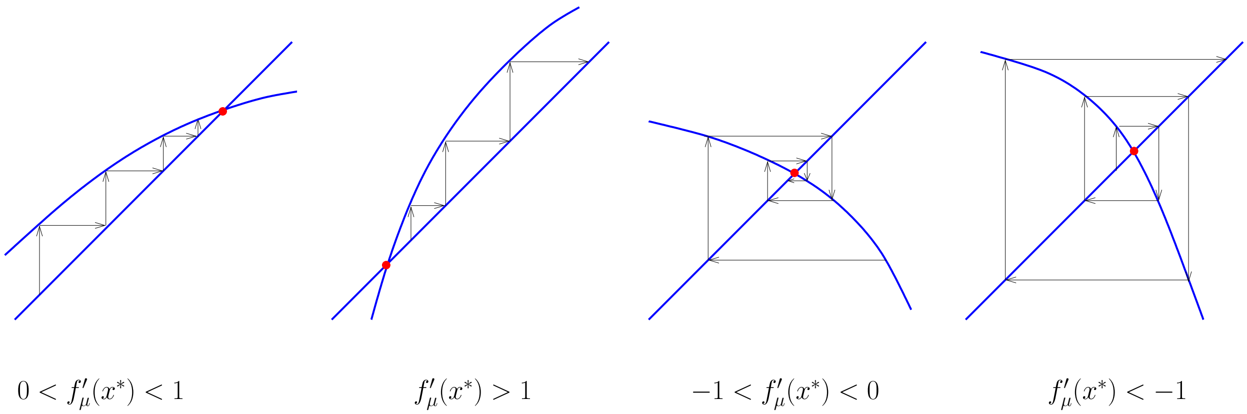


Figure 6.2: The behaviour near a fixed point in terms of the cobweb diagram

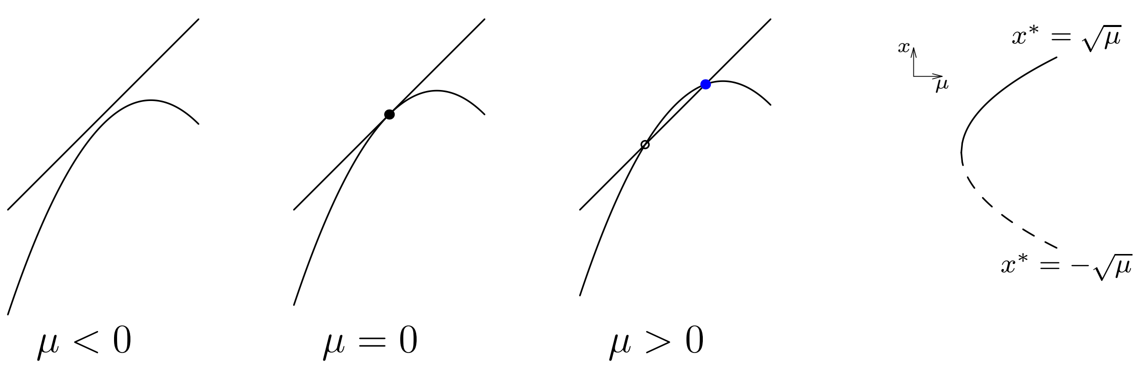


Figure 6.3: Saddle-node (tangential) bifurcation.

Saddle-node (tangential) bifurcation for $x_{n+1} = \mu + x_n - x_n^2$: If $\mu > 0$, there are two fixed points $x_{\pm}^* = \pm\mu^{1/2}$ (two intersection points between the curve $y = x$ and $y = \mu + x - x^2$); the fixed point $x_+^* = \mu^{1/2}$ is stable but $x_-^* = -\mu^{1/2}$ is not stable. If $\mu < 0$, there is no fixed point. Because bifurcation occurs when the straight line $y = x$ touches the parabola $y = \mu + x - x^2$ tangentially at $\mu = 0$, this saddle-node bifurcation is also called tangential bifurcation (see Figure 6.3).

Transcritical bifurcation for $x_{n+1} = (1 + \mu)x_n - x_n^2$: There are always two fixed points $x^* = 0$ and $x^* = \mu$. The fixed point $x^* = 0$ is stable for $\mu < 0$, but becomes unstable for $\mu > 0$, while the other fixed point $x^* = \mu$ is stable.

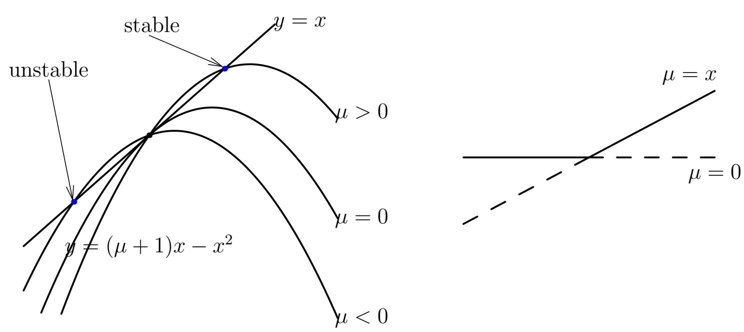


Figure 6.4: Transcritical bifurcation.

Supercritical pitchfork bifurcation for $x_{n+1} = (1 + \mu)x_n - x_n^3$: When $\mu < 0$, there is only one fixed point $x^* = 0$, which is stable. When $\mu > 0$, there are three fixed points; $x^* = \pm\mu^{1/2}$ are stable, but $x^* = 0$ unstable.

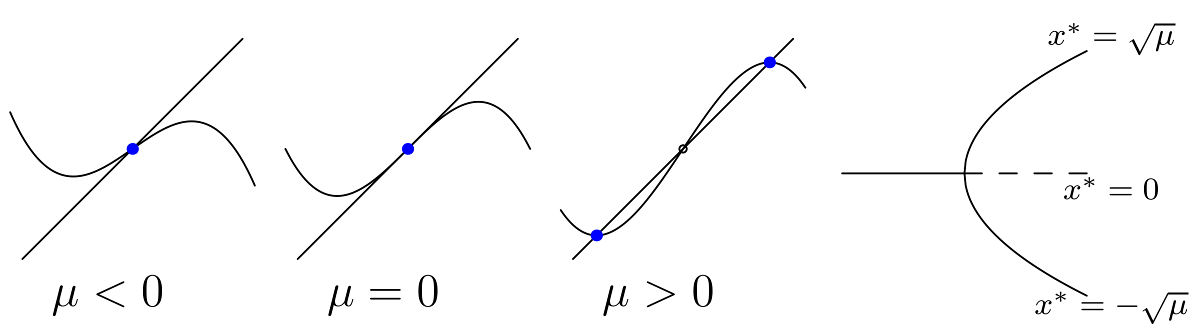


Figure 6.5: Pitchfork bifurcation.

Remark. Although the bifurcation diagrams of the three bifurcations look the same as those for the continuous differential equations, the situation for discrete maps is more complicated:

(a) The bifurcation are valid only *locally* for discrete maps. Take the map $x_{n+1} = 1 + x_n - x_n^2$ ($\mu = 1$) and the initial condition $x_0 = 10$ (or any initial such that x_0 is large), then $x_n \to -\infty$ as n goes to infinity, different from the stable fixed point $x^* = \mu^{1/2} = 1$. But for the continuous dynamical system $\dot{x} = 1 - x^2$, $x(t)$ always converges to $x^* = \mu^{1/2} = 1$, if x_0 is large.

(b) Another bifurcation could happen along the stable fixed point, as μ further increases or decreases. The Jacobian of the map $x_{n+1} = \mu + x_n - x^2$ at the stable fixed point $x^* = \mu^{1/2}$ is
$$f'_\mu(x^*) = 1 - 2x^* = 1 - 2\mu^{1/2}.$$
Then as $\mu > 1$, $f'_\mu(x^*) < -1$, which becomes unstable (a period-doubling bifurcation happens as we show below).

For continuous dynamical systems, Hopf bifurcation is common when stable spiral becomes unstable spiral (real parts of the eigenvalue becomes positive), and periodic solution appears. For maps, the analogous situation is period-two bifurcation: the original fixed point becomes unstable, and a period-two orbit appear. This will be examined in the next subsection, for the special logistic map.

6.3 Logistic map

The *logistic map* is the simplest quadratic family of maps
$$f_\mu(x) = \mu x(1-x), \qquad \mu \geq 0,$$
in which chaotic behaviours can arise. In the context of population dynamics, the two terms μx and $-\mu x^2$ in this map can be interpreted as reproduction and starvation (density-dependent mortality) respectively.

If this map is invariant on the interval $[0,1]$, then $\mu \in [0,4]$, since we only have to make sure $\max_{x\in[0,1]} f_\mu(x) = f_\mu(1/2) = \mu/4 \leq 1$. The behaviour of the map for small and moderately large μ can be explained by examining the stability of the fixed points and the periodic orbits.

Fixed Points: $x^* = \mu x^*(1-x^*)$. So $x^* = 0$ or $x^* = (\mu-1)/\mu$ provided $\mu \geq 1$.

Linear Stability: First $f'_\mu(x) = \mu - 2x\mu$. If $0 \leq \mu < 1$, the fixed point $x^* = 0$ is stable since $|f'_\mu(0)| = \mu < 1$, and the fixed point $x^* = (\mu-1)/\mu$ is not in the range $[0,1]$. As $\mu \geq 1$, the fixed point $x^* = 0$ becomes unstable. But $x^* = (\mu-1)/\mu \in (0,1)$ become stable, as long as
$$|f'_\mu\big((\mu-1)/\mu\big)| = |2-\mu| < 1,$$
or $1 < \mu < 3$. Because the fixed points $x^* = 0$ and $x^* = (\mu-1)/\mu$ exchange stability at $\mu = 1$, this is a transcritical bifurcation.

Period-doubling bifurcation: As μ passes 3, $f'_\mu\big((\mu-1)/\mu\big)$ passes -1 and $x^* = (\mu-1)/\mu$ becomes unstable (see Figure 6.6 for sample iterations at $\mu = 3.35$). A period-two orbit (x^*_+, x^*_-) appears, such that
$$x^*_+ = f_\mu(x^*_-), \quad x^*_- = f_\mu(x^*_+).$$
In other words, both x^*_+ and x^*_- are fixed points of $x = f_\mu(f_\mu(x))$, but not fixed points of $x = f_\mu(x)$. This is called *period-doubling bifurcation*, signified by $f_{\mu^*}(x^*) = -1$ at $\mu^* = 3$.

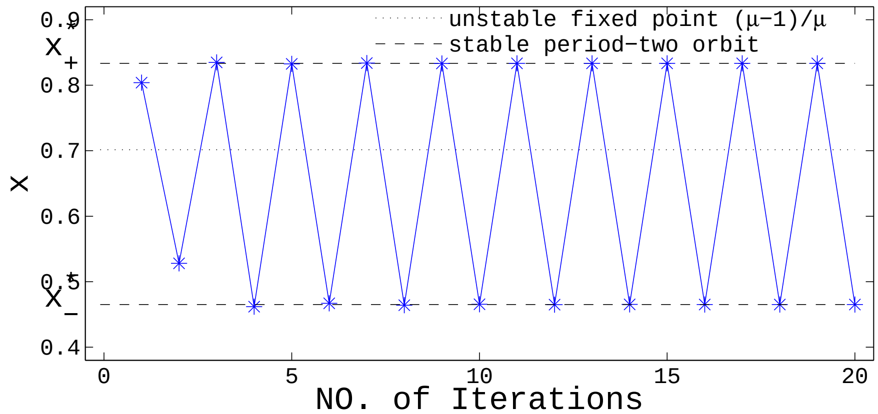


Figure 6.6: The fixed point $x^* = (\mu - 1)/\mu$ becomes unstable as $\mu > 3$, and a period-two orbit emerges (the iteration for $\mu = 3.35$ is plotted here).

Since

$$x - f_\mu\big(f_\mu(x)\big) = x(\mu x - \mu + 1)(\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1),$$

all fixed points of $x = f_\mu\big(f_\mu(x)\big)$ are

$$x^* = 0, \quad x^* = \frac{\mu - 1}{\mu}, \quad x^*_\pm = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}.$$

The first two are inherited from $x^* = f_\mu(x^*)$, and the last two form the period two orbits, solving the quadratic equation $\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1 = 0$. A more involved calculation shows that the this period-two orbit loses its stability, when the modulus

$$\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\Big|_{x^*_\pm} = f'_\mu\big(f_\mu(x)\big) f'_\mu(x)\Big|_{x^*_\pm} = f'_\mu(x^*_+) f'_\mu(x^*_-)$$

is greater than unit. First from

$$x^*_+ + x^*_- = \frac{\mu + 1}{\mu}, \quad x^*_+ x^*_- = \frac{\mu + 1}{\mu^2},$$

the Jacobian $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x^*_\pm}$ can be simplified as

$$f'_\mu(x^*_+) f'_\mu(x^*_-) = \mu^2(1 - 2x^*_-)(1 - 2x^*_+) = \mu^2\big(1 - 2(x^*_+ + x^*_-) + 4x^*_- x^*_+\big) = 4 + 2\mu - \mu^2.$$

By solving $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x^*_\pm} = \pm 1$, we get $\mu = -1$ or $\mu = 3$ for $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x^*_\pm} = 1$ and $\mu = 1 \pm \sqrt{6}$ for $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x^*_\pm} = -1$. We do not need to check the negative values of $\mu = -1$ or $\mu = 1 - \sqrt{6}$; in fact the fixed points $x^*_\pm$ exists only for $\mu \geq 3$. Therefore, the only possible bifurcation is at $\mu^* = 1 + \sqrt{6} \approx 3.449$, with $\frac{d}{dx} f_{\mu^*}\big(f_{\mu^*}(x)\big)\big|_{x^*_\pm} = -1$. The value -1 suggests another period-doubling bifurcation for $x = f_\mu(f_\mu(x))$, leading to period-four orbits, which are fixed points of $x = f_\mu(f_\mu(f_\mu(f_\mu(x))))$.

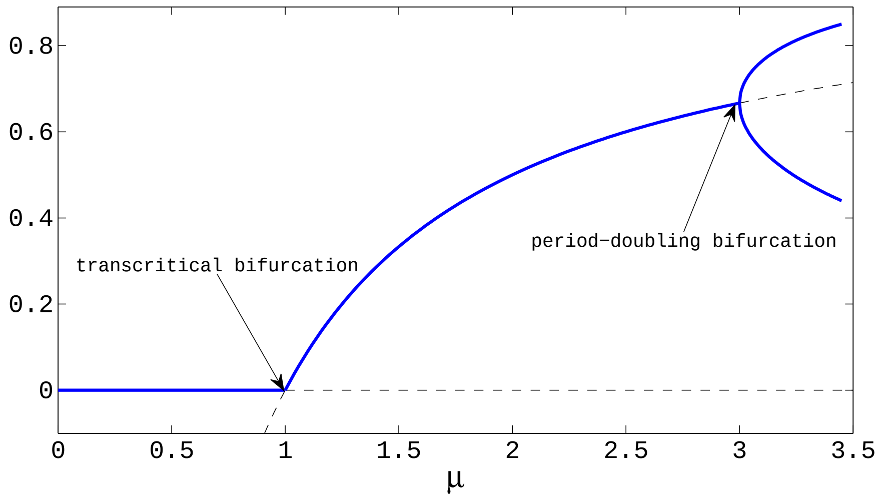


Figure 6.7: Bifurcation diagram for the logistic map, when μ is not too close to 4.

In fact there is an infinite cascade of 'period-doubling' bifurcations:

$$\begin{aligned} &\mu_1 = 3 && \text{period } 1 \to 2 \\ &\mu_2 = 1+\sqrt{6} && \text{period } 2 \to 4 \\ &\quad\vdots && \\ &\mu_n && \text{period } 2^{n-1} \to 2^n. \end{aligned}$$

Moreover, μ_n has a finite limit (about 3.56995), when the period-doubling cascade ends and chaotic behaviours start. There is also a universal *Feigenbaum constant* defined as the limit of ratio between the lengths of two successive bifurcation intervals, i.e,

$$\lim_{n\to\infty} \frac{\mu_{n-1}-\mu_n}{\mu-\mu_{n+1}} \approx 4.669$$

Many other maps exhibit similar (period-doubling) bifurcations, and the above limiting ratio is independent of the details of the map.

Remark. Although the bifurcation diagrams for both pitchfork and period-doubling bifurcation look similar, the behaviours of the fixed points are totally different: new "fixed points" for period-doubling bifurcation are actually fixed points of $x = f_\mu(f_\mu(x))$ instead of $x = f_\mu(x)$.

6.4 Bifurcation of two-dimensional maps

The same approach can be used to study the bifurcation of two-dimensional maps, by looking at when the change in the parameter leads to eigenvalues of the *Jacobian matrix* with unit modulus.

Example 6.2. Consider the map

$$x_{n+1} = \mu y_n + x_n - x_n^2, \qquad y_{n+1} = x_n,$$

where $|\mu|$ is small. The fixed point (x^*, y^*) satisfies the equations $x = \mu x + x - x^2, y = x$. That is, there are two fixed points

$$(x_1^*, y_1^*) = (0, 0), \qquad (x_2^*, y_2^*) = (\mu, \mu).$$

From the Jacobian matrix

$$J(x, y) = \begin{pmatrix} \frac{\partial}{\partial x}(\mu y + x - x^2) & \frac{\partial}{\partial y}(\mu y + x - x^2) \\ \frac{\partial}{\partial x} x & \frac{\partial}{\partial y} x \end{pmatrix} = \begin{pmatrix} 1 - 2x & \mu \\ 1 & 0 \end{pmatrix},$$

we get

$$J(x_1^*, y_1^*) = \begin{pmatrix} 1 & \mu \\ 1 & 0 \end{pmatrix}, \qquad J(x_2^*, y_2^*) = \begin{pmatrix} 1 - 2\mu & \mu \\ 1 & 0 \end{pmatrix}.$$

At the fixed point (x_1^*, x_2^*), the two eigenvalues are governed by

$$\det\left(\lambda I - J(x_1^*, y_1^*)\right) = \det\begin{pmatrix} \lambda - 1 & -\mu \\ -1 & \lambda \end{pmatrix} = \lambda^2 - \lambda - \mu = 0.$$

That is

$$\lambda_1^{\pm} = \frac{1 \pm \sqrt{1 + 4\mu}}{2}.$$

As μ passes zero, λ_1^+ pass 1 and this fixed point (x_1^*, y_1^*) becomes unstable.

At the fixed point (x_2^*, y_2^*), the two eigenvalues are governed by

$$\det\left(\lambda I - J(x_2^*, y_2^*)\right) = \det\begin{pmatrix} \lambda - 1 + 2\mu & -\mu \\ -1 & \lambda \end{pmatrix} = \lambda^2 - (1 - 2\mu)\lambda - \mu = 0.$$

That is,

$$\lambda_2^{\pm} = \frac{1 - 2\mu \pm \sqrt{(1 - 2\mu)^2 + 4\mu}}{2} = \frac{1 - 2\mu \pm \sqrt{1 + 4\mu^2}}{2}.$$

In this case, λ_2^- is close to zero ($|\lambda_2^+|$ is far away from unit) and can not trigger any instability. If μ is small and negative,

$$\sqrt{1 + 4\mu^2} > \sqrt{1 + 4\mu + 4\mu^2} = 1 + 2\mu$$

and

$$\lambda_2^+ = \frac{1 - 2\mu + \sqrt{1 + 4\mu^2}}{2} > \frac{1 - 2\mu + (1 + 2\mu)}{2} = 1.$$

As a result, the fixed point (x_2^*, y_2^*) is unstable. On the other hand, if μ becomes positive (and small), $\sqrt{1 + 4\mu^2} < \sqrt{1 + 4\mu + 4\mu^2} = 1 + 2\mu$ and

$$\lambda_2^+ = \frac{1 - 2\mu + \sqrt{1 + 4\mu^2}}{2} < \frac{1 - 2\mu + (1 + 2\mu)}{2} = 1.$$

Therefore, the stability of the two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) are exchanged, indicating the *transcritical bifurcation* at $\mu = 0$.

The bifurcation is also clear from Figure 6.8. For $\mu \in (-1/4, 0)$, $|\lambda_1^{\pm}| < 1$ and the fixed point $(x_1^*, y_1^*) = (0, 0)$ is stable. The other fixed point $(x_2^*, y_2^*) = (\mu, \mu)$ is stable for $\mu > 0$, but becomes unstable again when $\lambda_2^- = -1$, or $\mu = 2/3$. A period-doubling bifurcation occurs her (associated with eigenvalue -1).

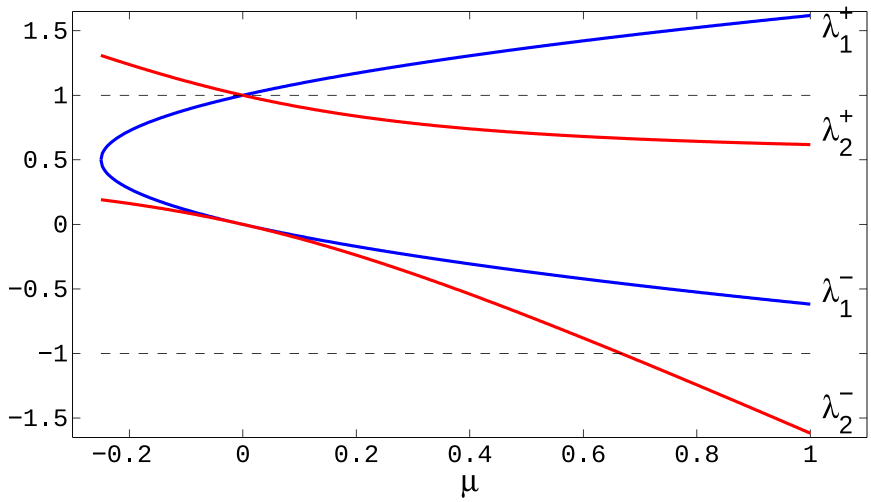


Figure 6.8: The eigenvalues of the Jacobian matrix near the two fixed points $(0, 0)$ and (μ, μ).

6.5 Other concepts: intermittancy, Lyapunov exponent and the route to chaos

There are other important concepts motivated from maps, like

- Complex iterations from fractals (Julia sets)
- Chaos and its characterisation (sensitive dependence on initial data, existence of "strange attractor", . . .)
- Intermittancy (jumping between nearly periodic and chaotic motions) in chaotic regime
- Lypunov exponents (rate of separation of close trajectories)

These concepts will be briefly mentioned (you will see them more in books for popular audience), but will not appear in the final exam.