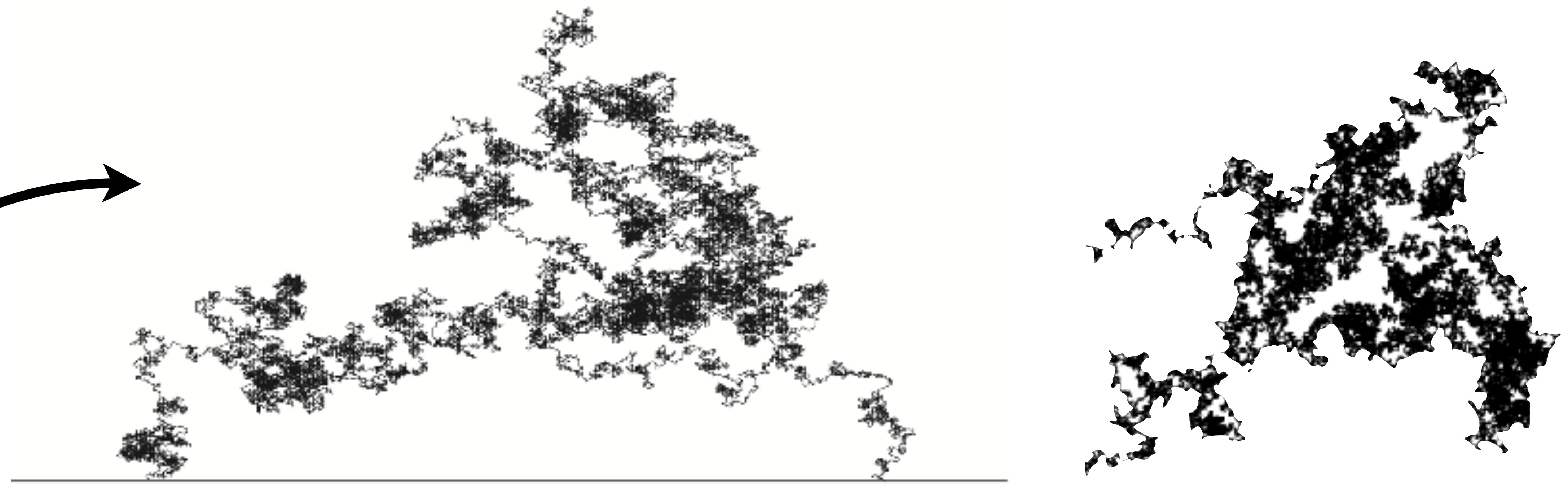
Growth Fragmentations, Brownian Motion and Random Geometry

UK Easter Probability Meeting, Manchester, March 2023

Ellen Powell, Durham University.
Based on joint work with Juhan Aru, Nina Holden, Xin Sun

Aim

Planar Brownian Excursions



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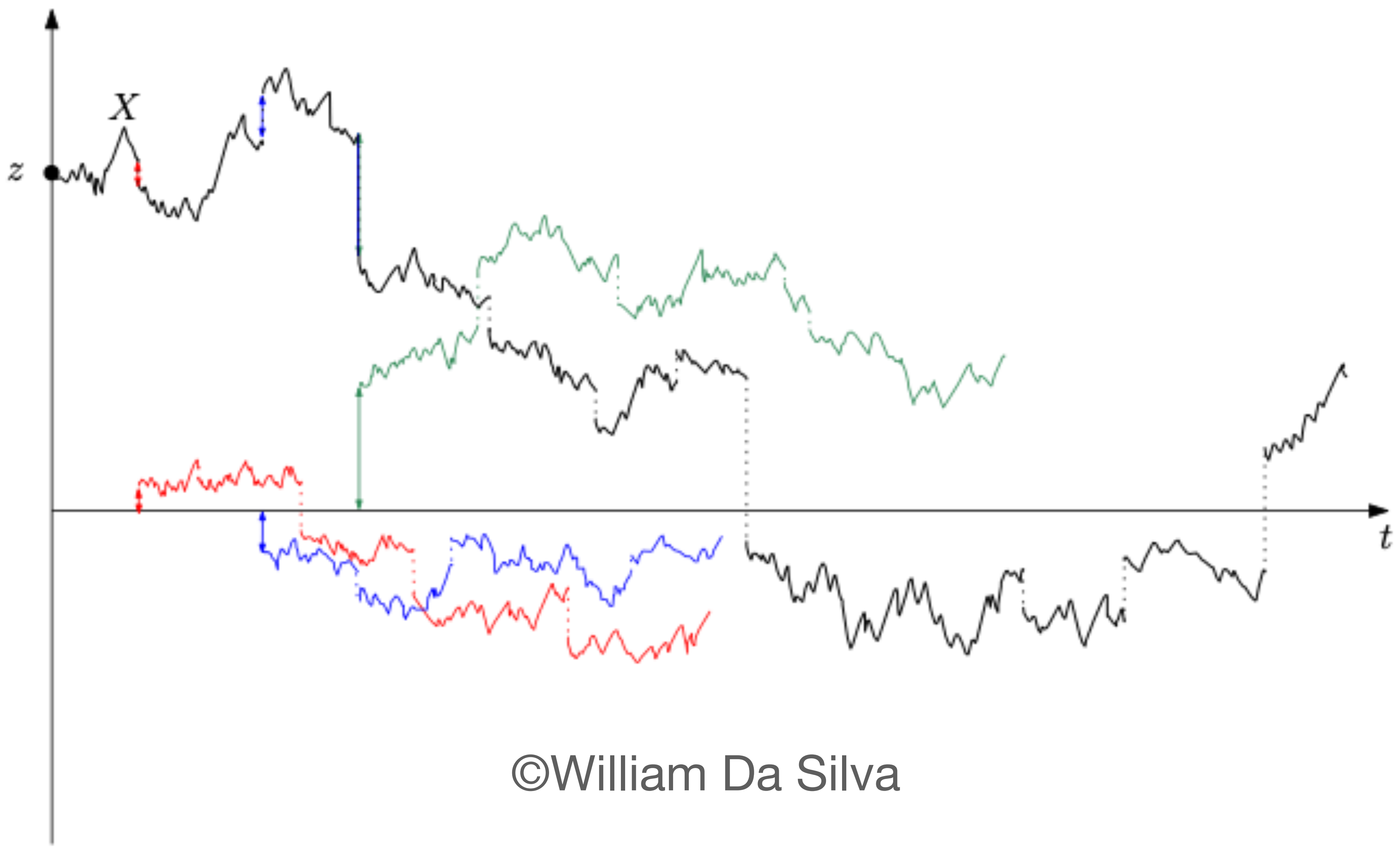
Growth Fragmentations



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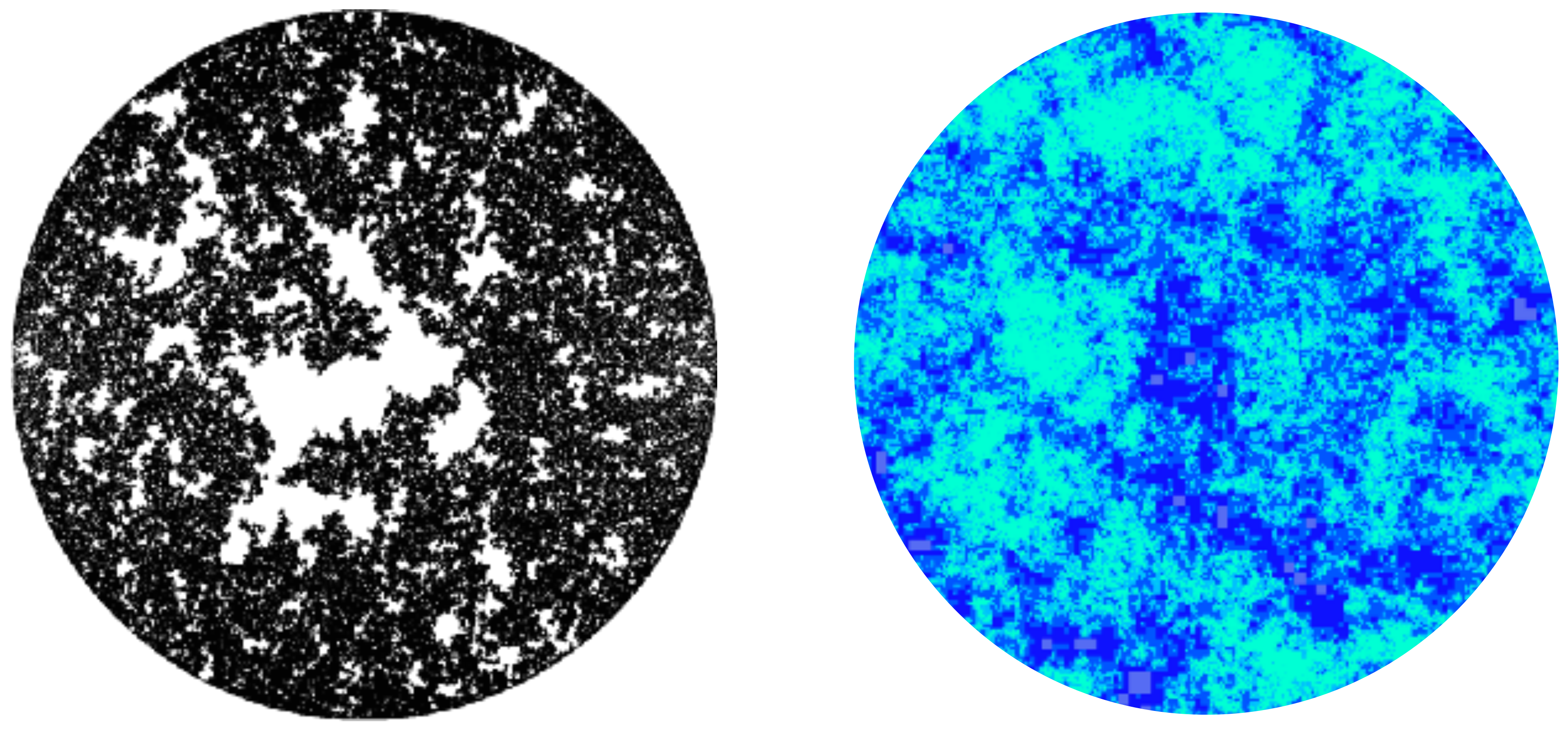
Gaussian Multiplicative Chaos & Conformal Loop Ensembles



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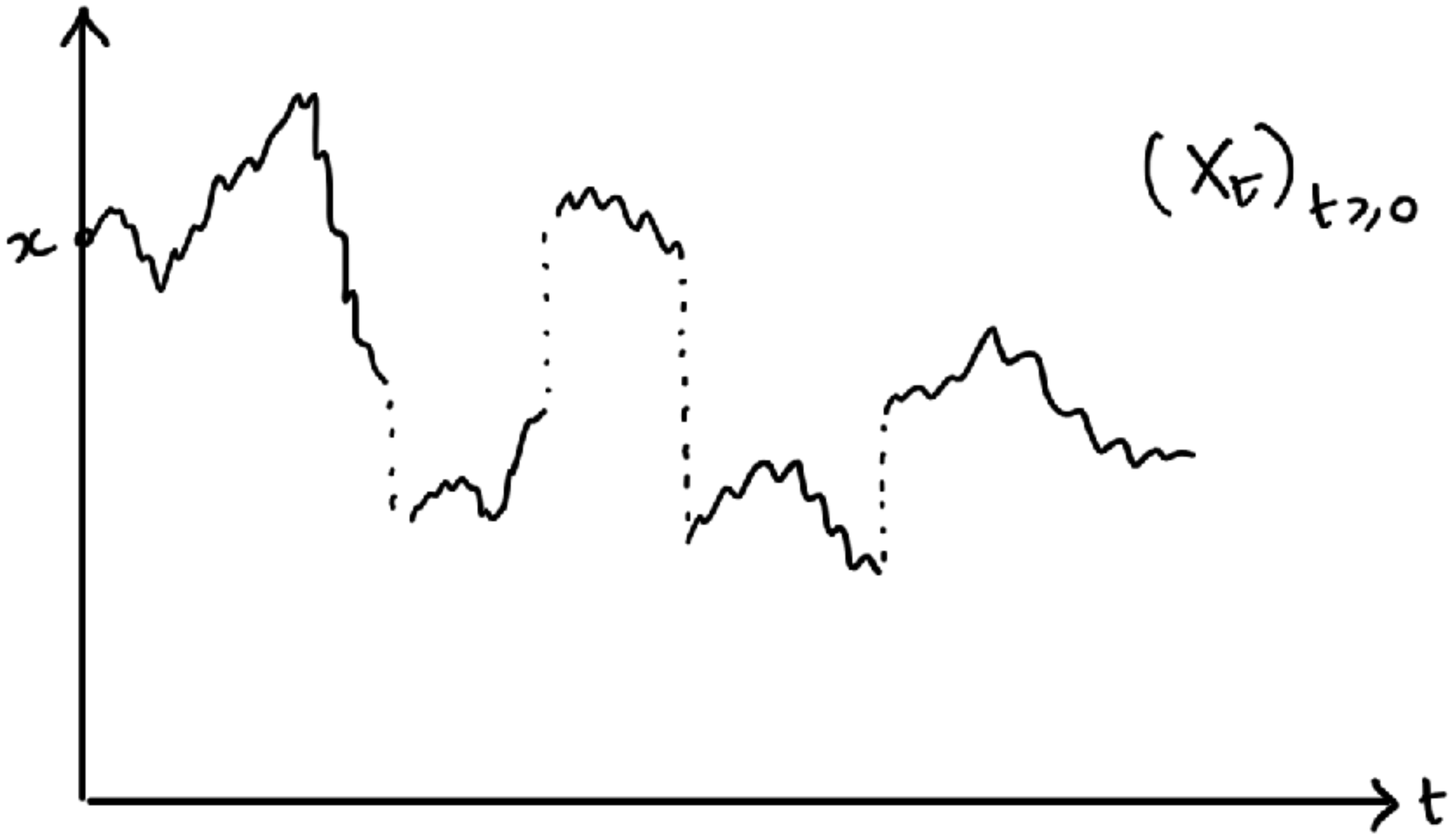
Growth fragmentations

Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- *E.g. Stable Lévy process conditioned to be die continuously at 0*



Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X

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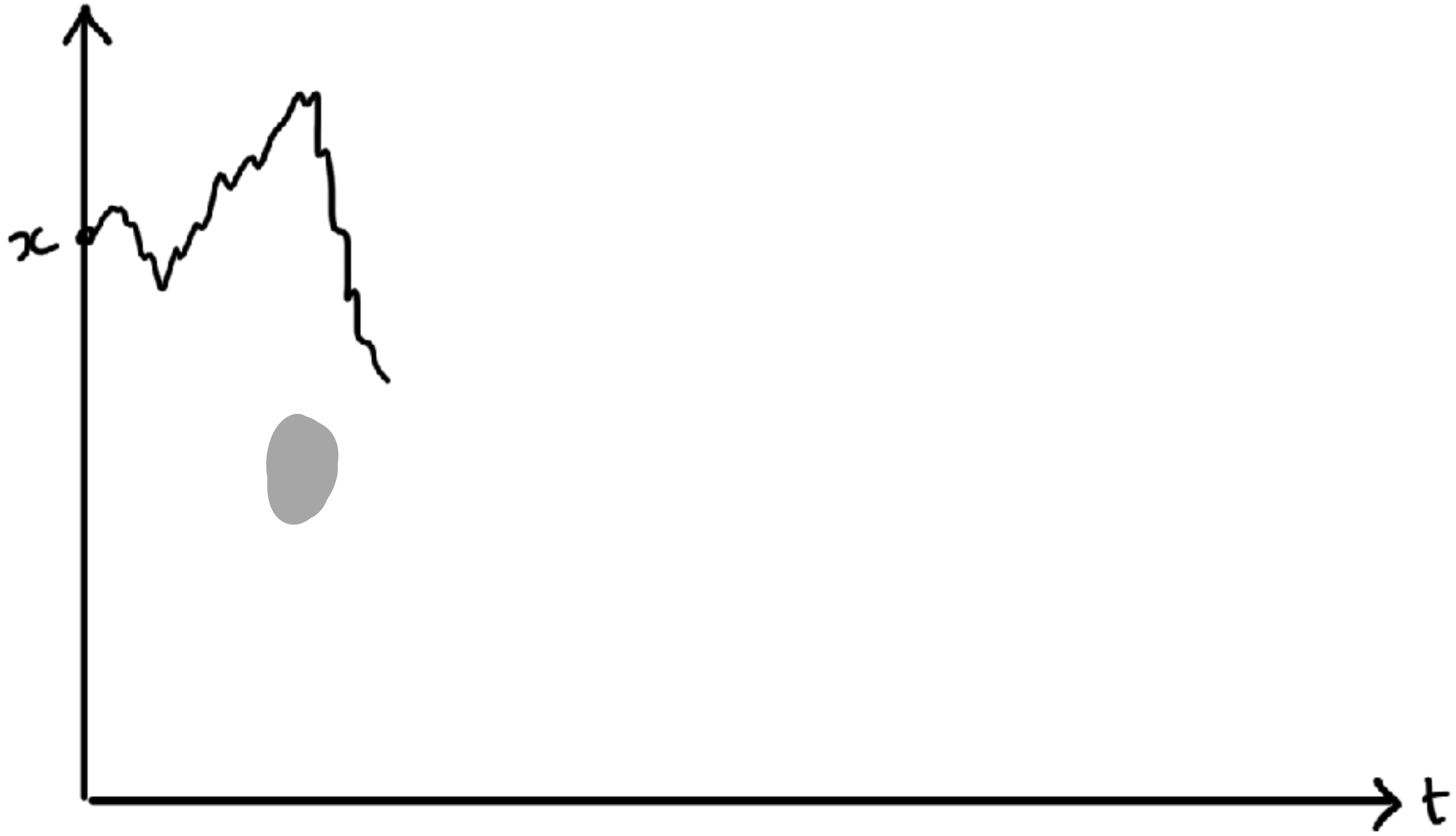

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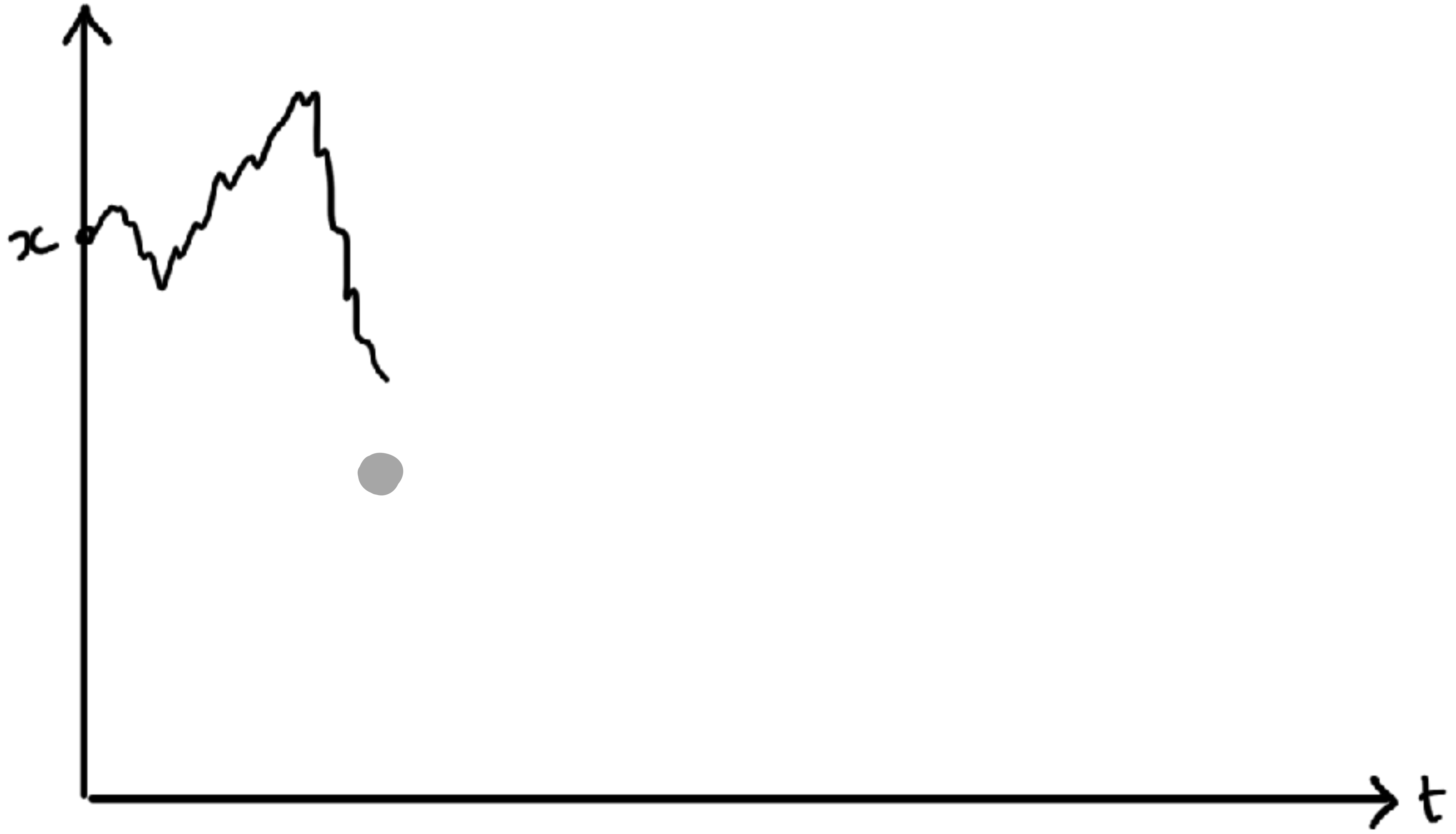
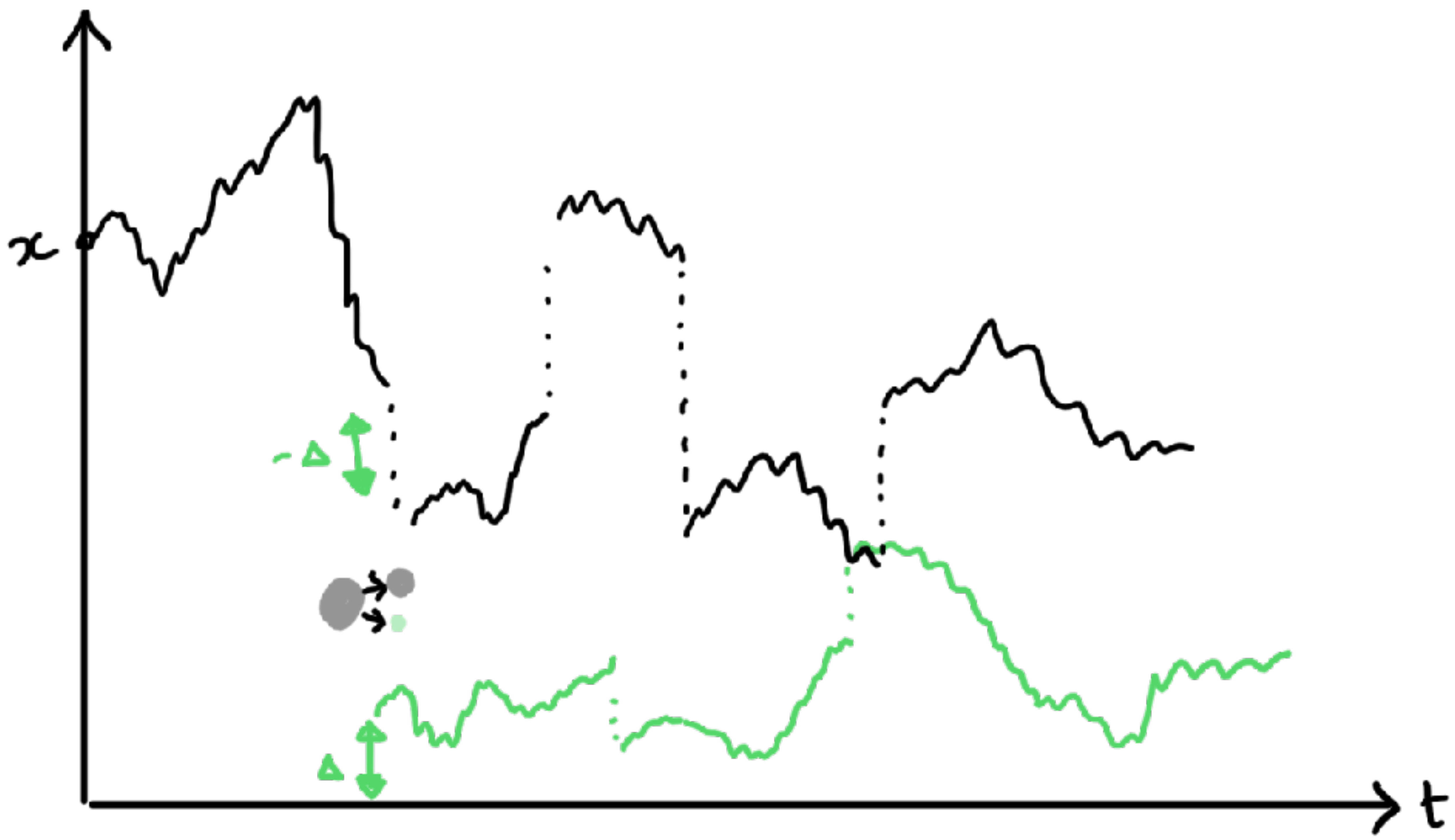


Growth Fragmentations

- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X
- **Fragmentation:** negative jump $-\Delta$ of $X \rightsquigarrow$ new particle with initial size Δ, then evolves independently under same law as X **(mass is conserved)**



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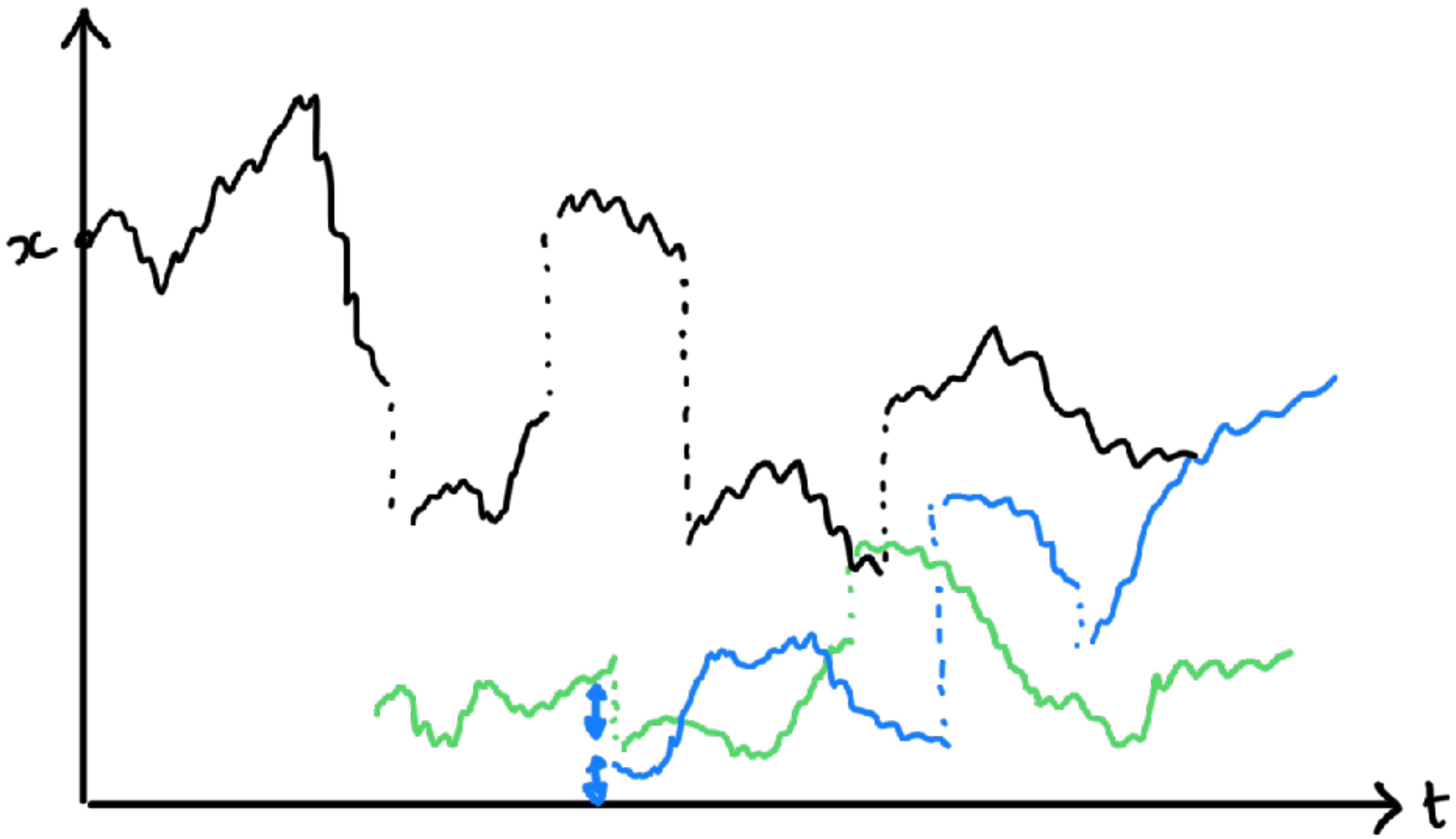
- **Iterates**



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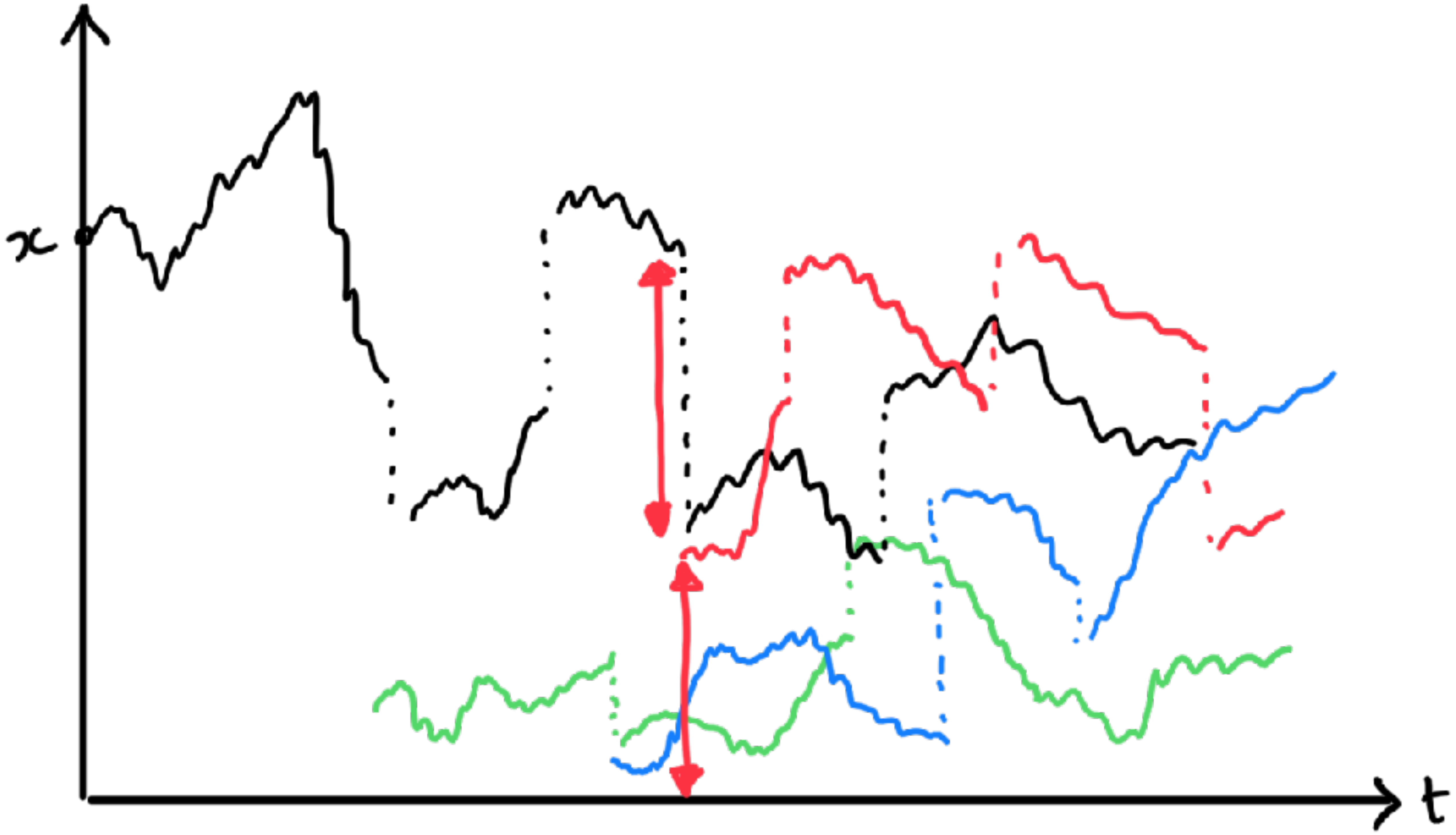

Growth Fragmentations

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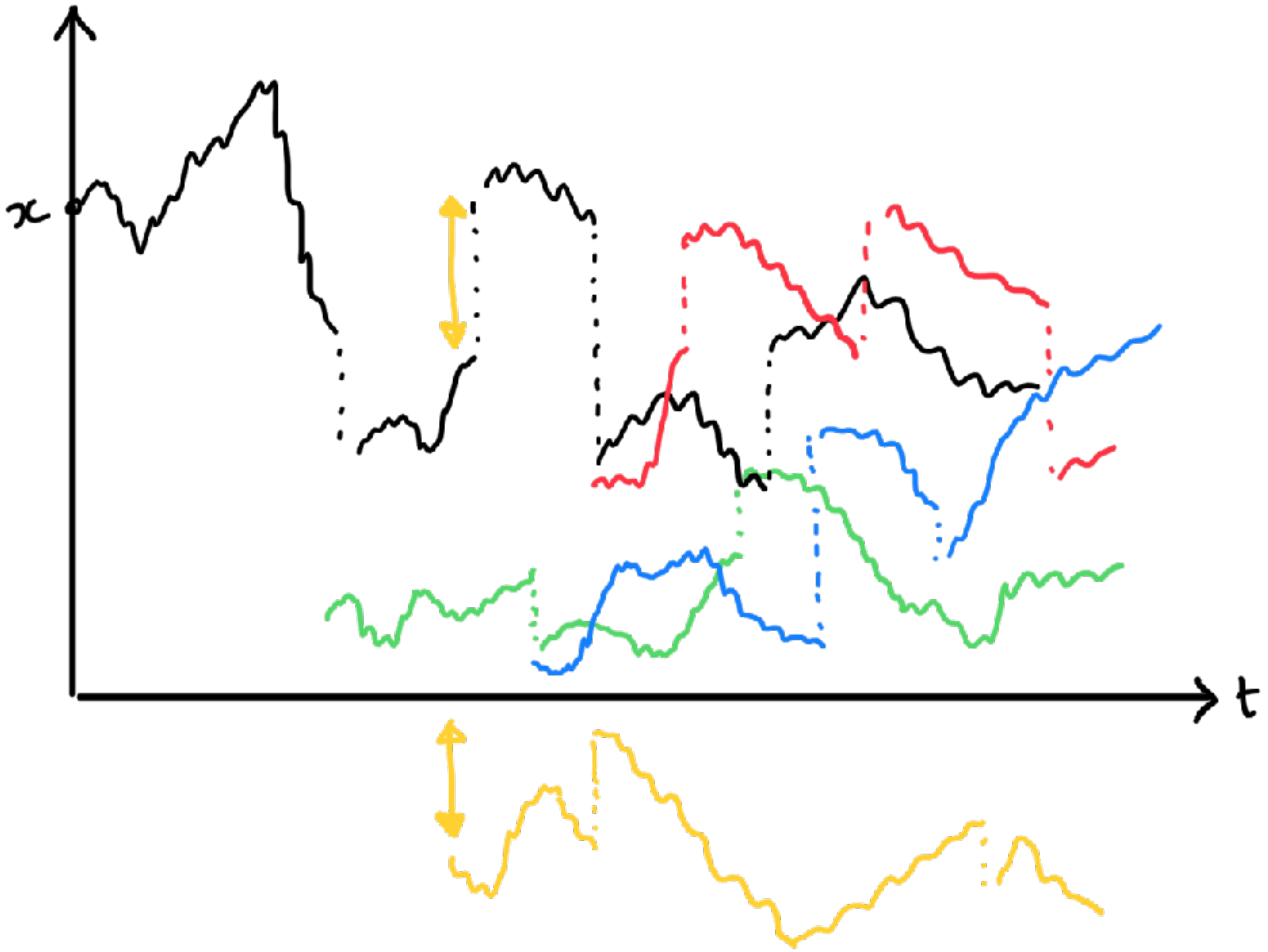
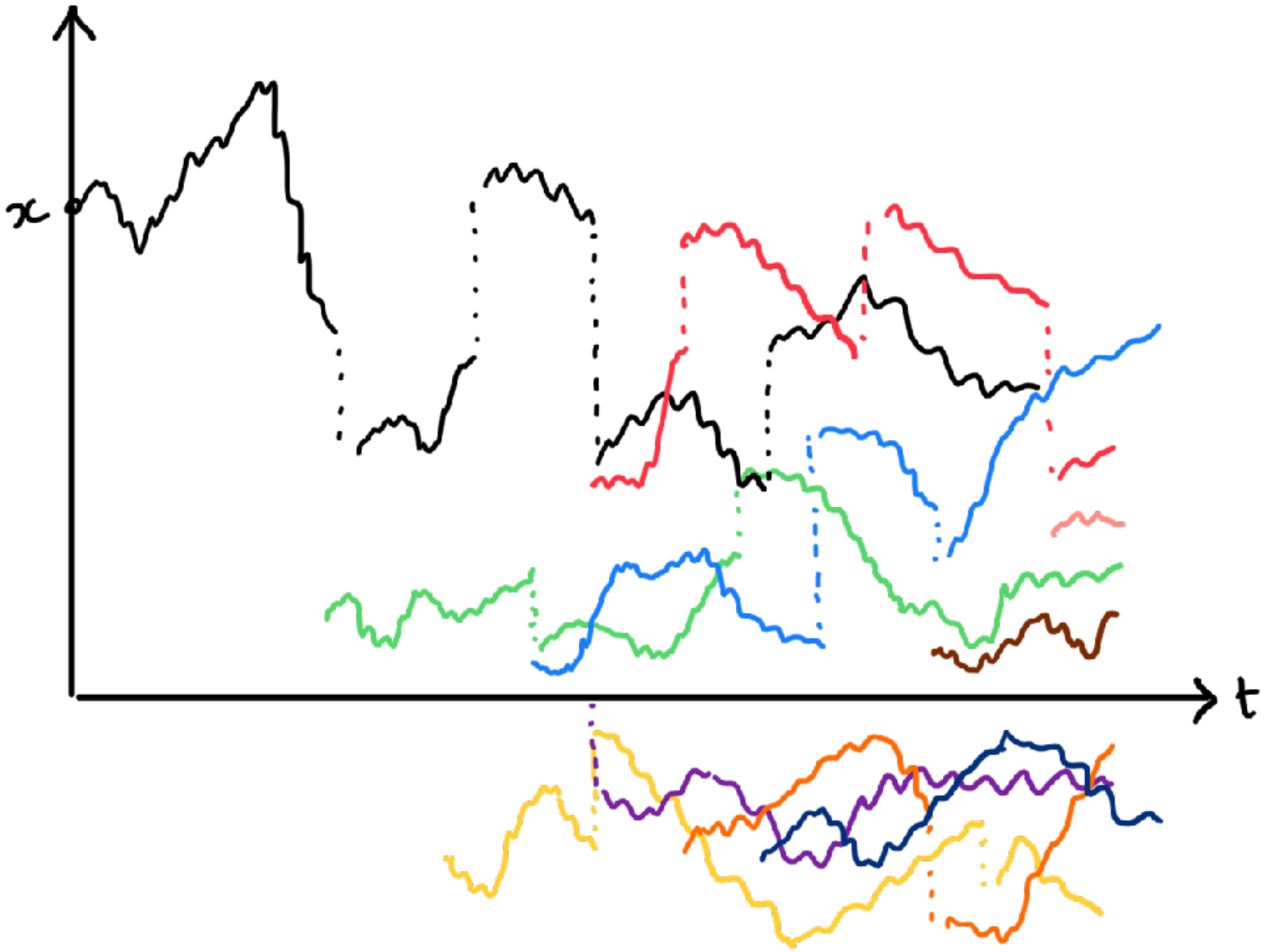
- **Signed version:** positive jumps $\rightsquigarrow$ new particles of negative mass



Growth Fragmentations

Bertoin, Bertoin-Budd-Curien-Kortchemski, Aïdékon-Da Silva, Da Silva …

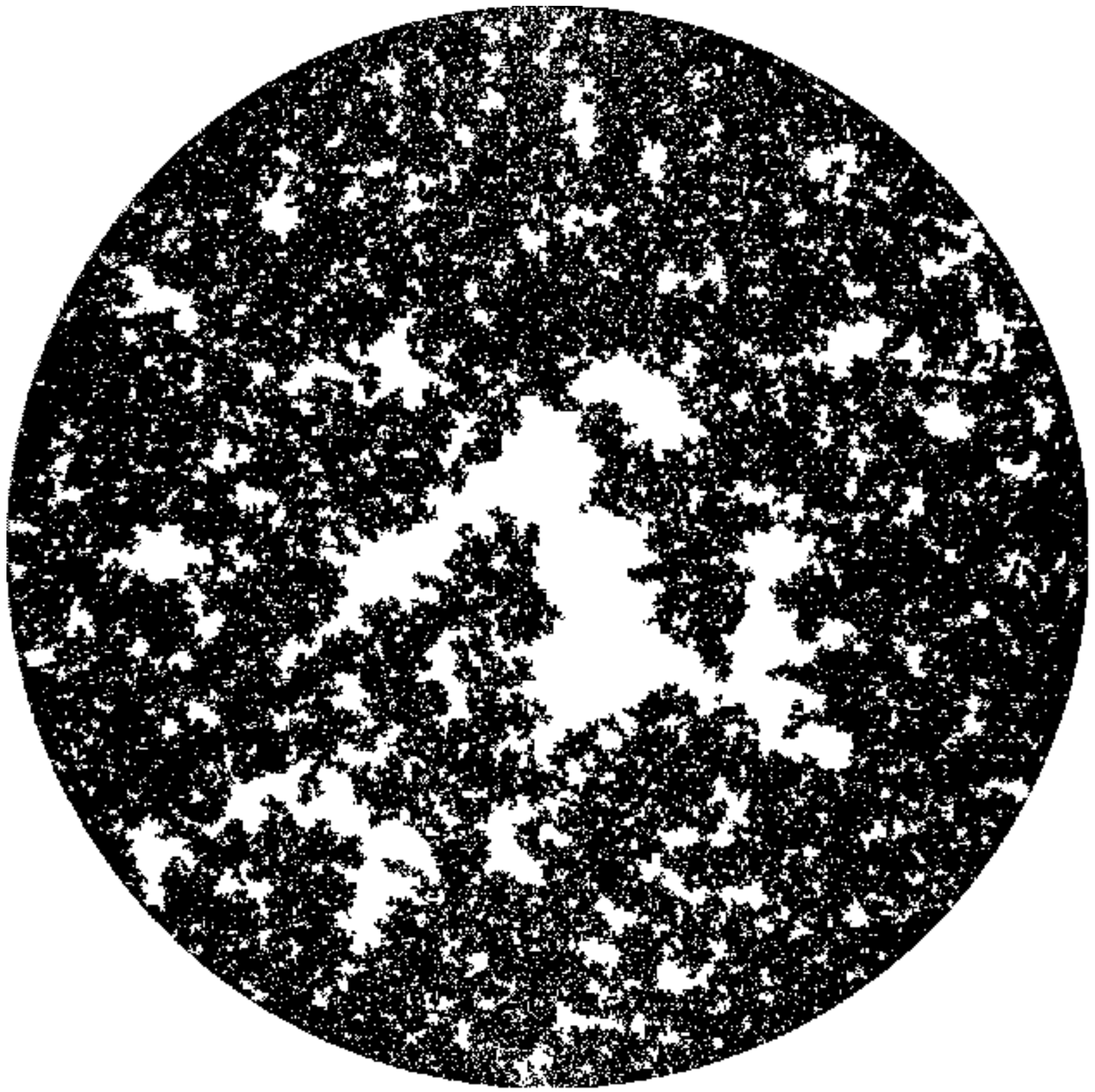
- $X =$ positive self-similar Markov process, some initial value x
- **Growth (or shrinking) of cells:** evolution of X
- **Fragmentation:** negative jump - Δ of $X \rightsquigarrow$ new particle with initial size Δ
- At time $t \geq 0$, system = collection of particles with (signed) masses



Conformal Loop Ensembles

Conformal Loop Ensembles $\kappa \in (8/3,4]$

- Simple CLE_κ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield-Werner)



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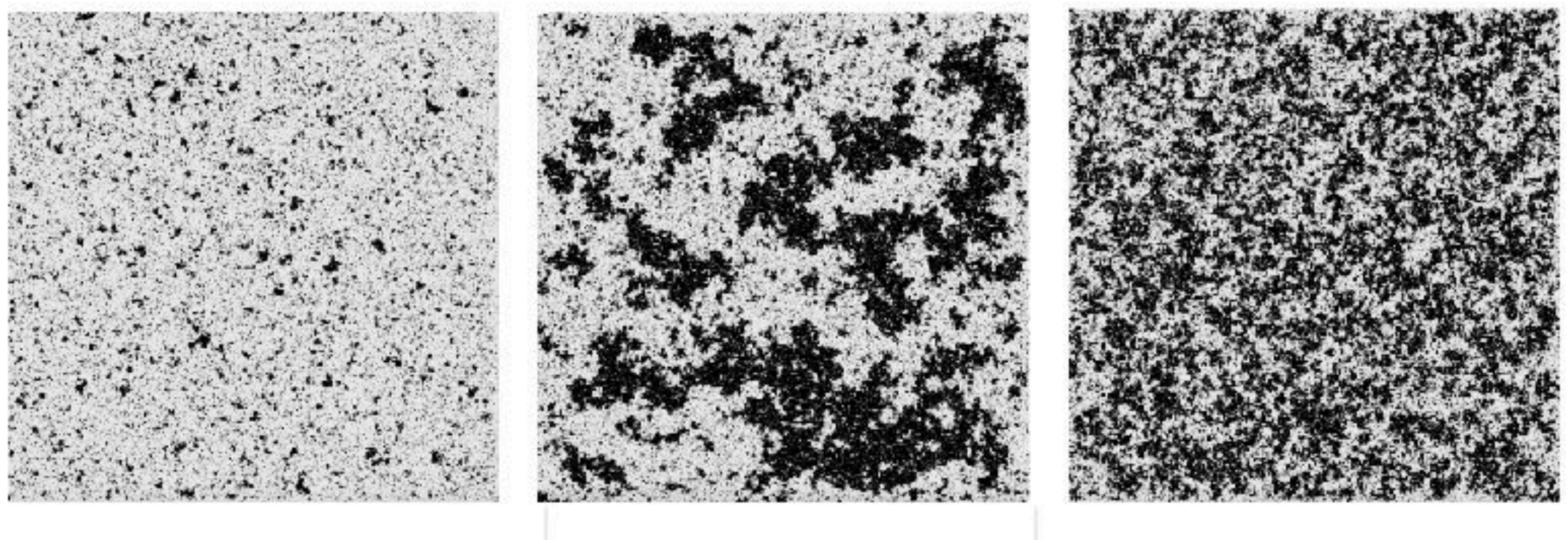
- Simple CLE_κ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield—Werner)

- (Conjectured) **scaling limit of interfaces** in discrete models

- CLE_3 (top, bottom left): Chelkak—Duminil-Copin—Hongler—Smirnov, Benoist—Hongler



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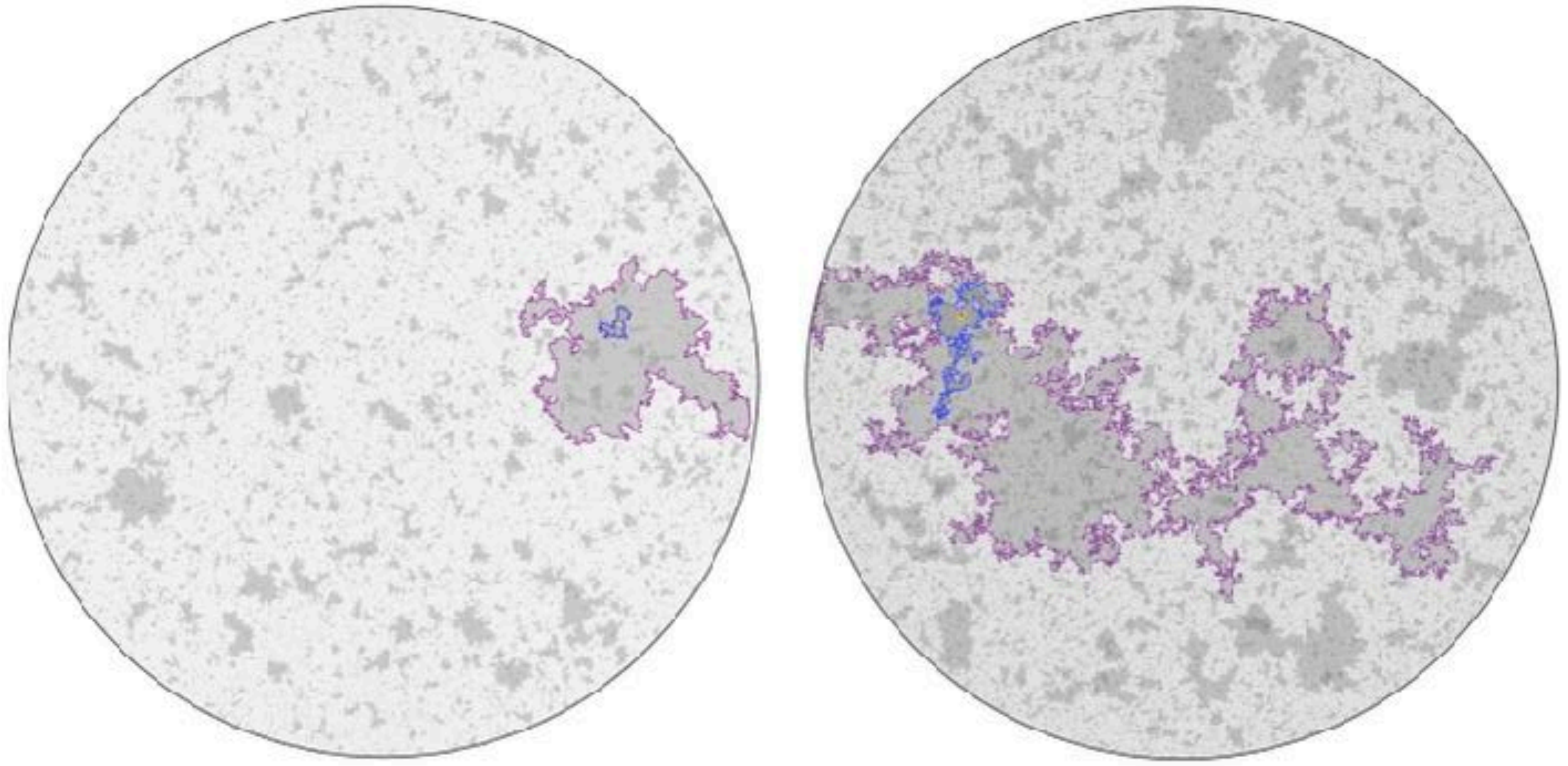

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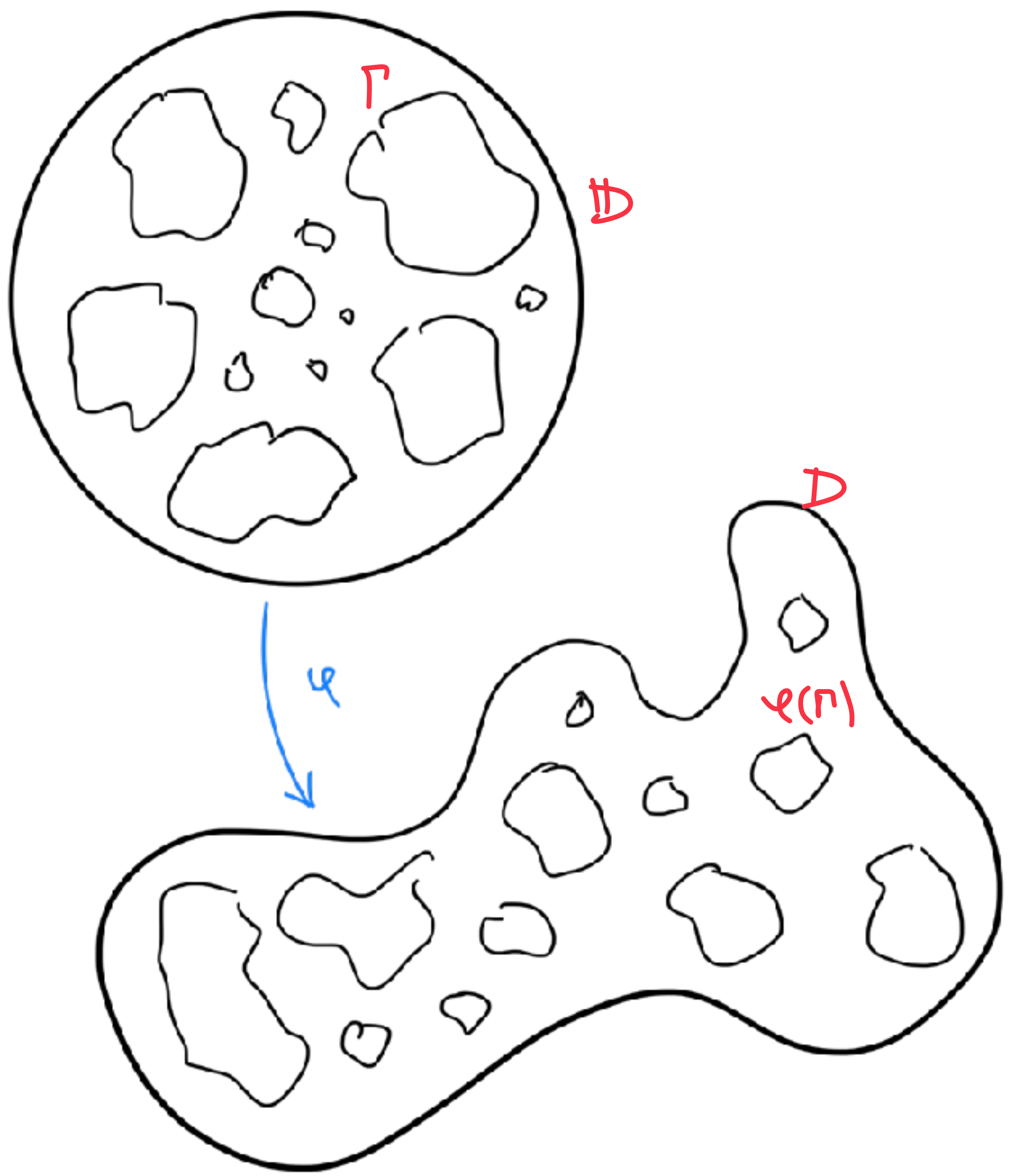
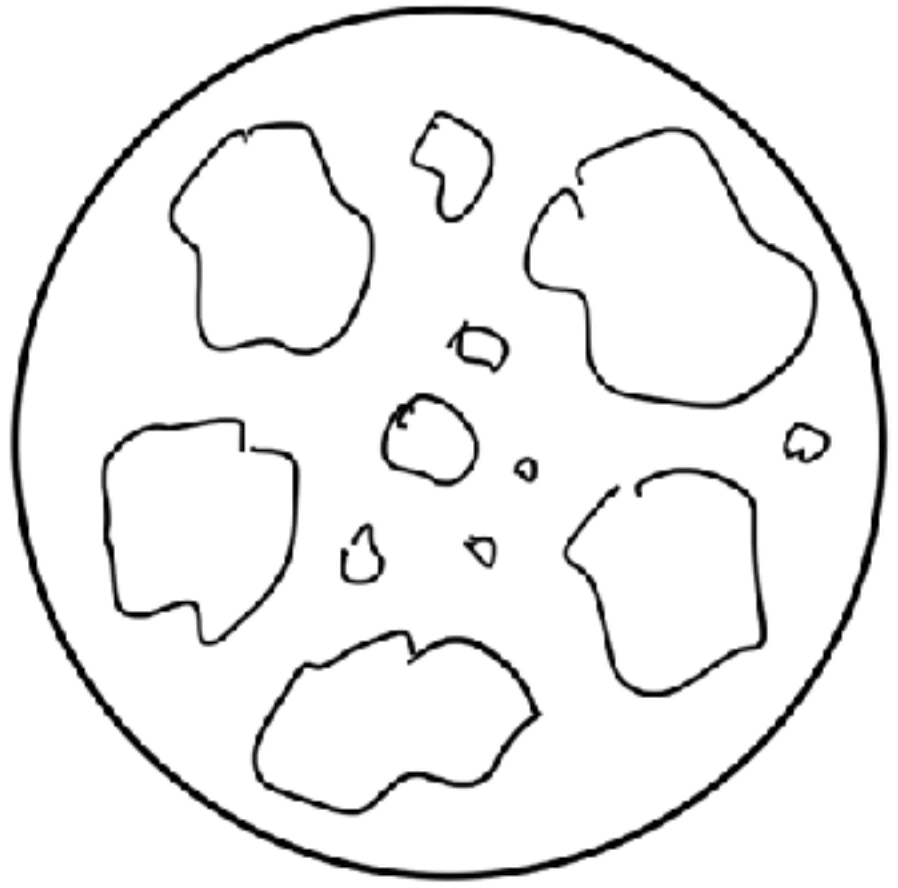
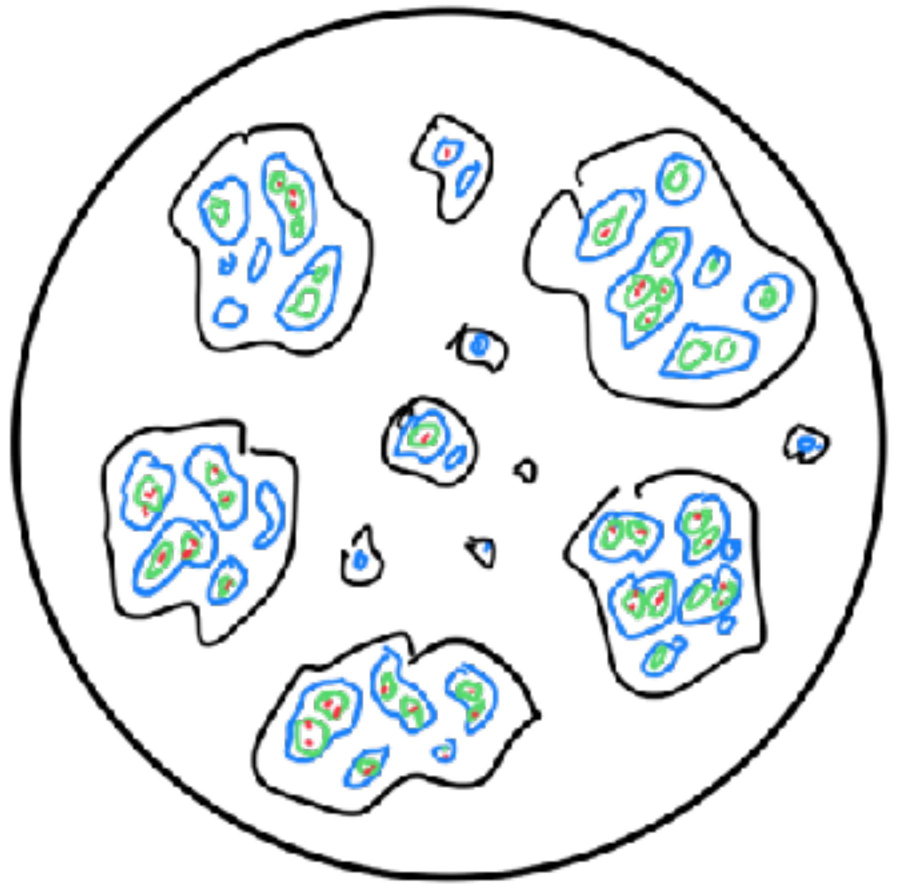
- (Conjectured) **scaling limit of interfaces** in discrete models
- **Conformally invariant**
- $\Gamma \overset{(d)}{=} \text{CLE}_\kappa \text{ in } D \Rightarrow \varphi(\Gamma) \overset{(d)}{=} \text{CLE}_\kappa \text{ in } D'$



Conformal Loop Ensembles $\kappa \in (8/3,4]$

- Simple CLE$_\kappa$ = random collection of disjoint simple loops in a simply connected domain of $\mathbb{C}$, introduced by (Sheffield-Werner)
- (Conjectured) **scaling limit of interfaces** in discrete models
- **Conformally invariant**
- **Nested version** defined by iteration

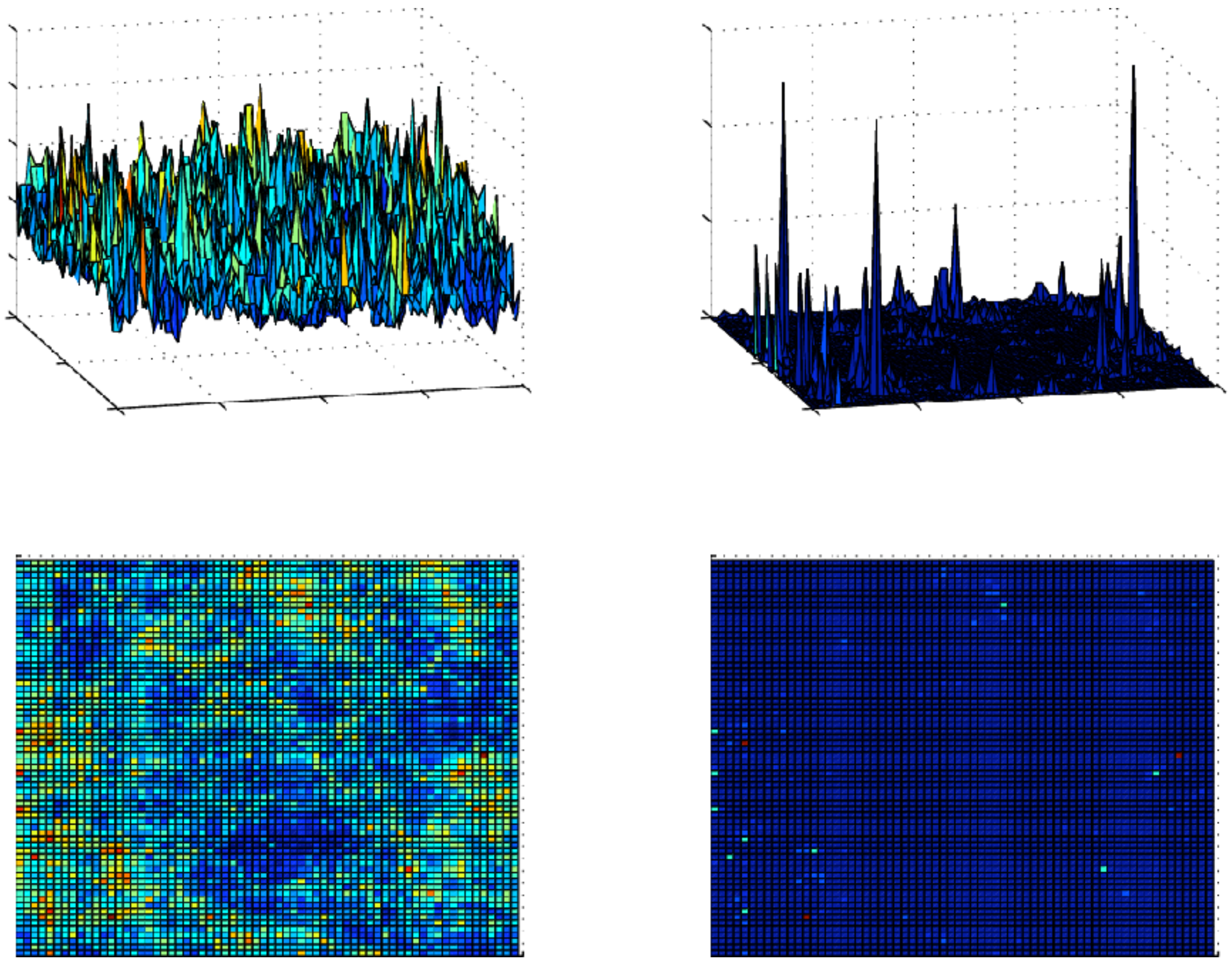




Gaussian Multiplicative Chaos

Gaussian Multiplicative Chaos/ Liouville Quantum Gravity

- Family of measures on $D \subset \mathbb{R}^d$,
- Parameter $\gamma \in (0, \sqrt{2d})$
- $\mu_\gamma(dx)$ " = " $\exp(\gamma h(x))\, dx$, h a **Gaussian log-correlated field on** D



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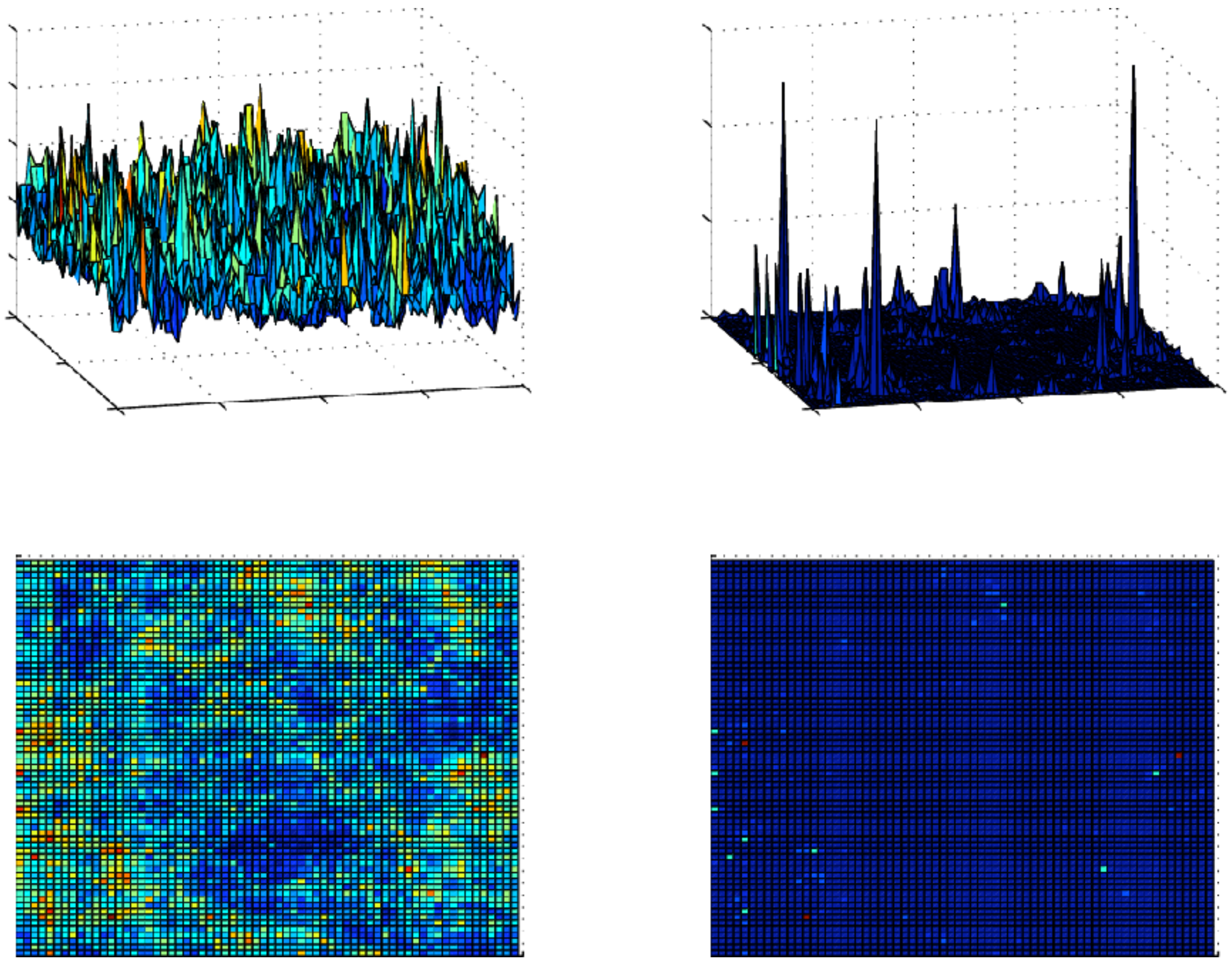
- Constructed by regularisation
- Defines **areas** of regions and **lengths** of (some) curves (Kahane, Duplantier-Sheffield, Robert-Vargas, Rhodes-Vargas, Berestycki, Shamov, Junnila-Saksman …)



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Gaussian Multiplicative Chaos/ Liouville Quantum Gravity

- Family of measures on $D \subset \mathbb{R}^2$,
- Parameter $\gamma \in (0,2)$
- $\mu_\gamma(dx)$ " $=$ " $\exp(\gamma h(x))\, dx$, with h a **Gaussian free field on** D
- Constructed by regularisation
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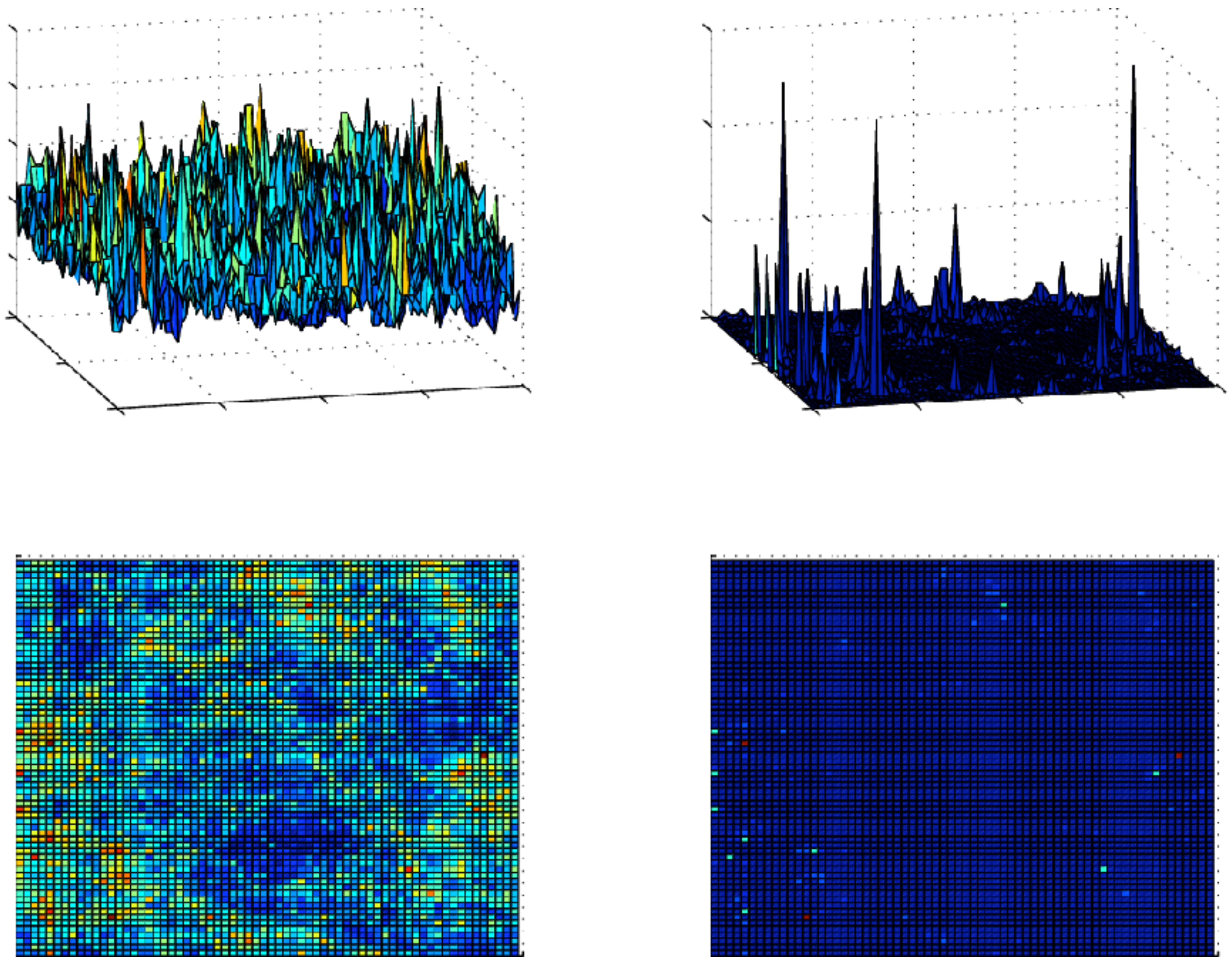

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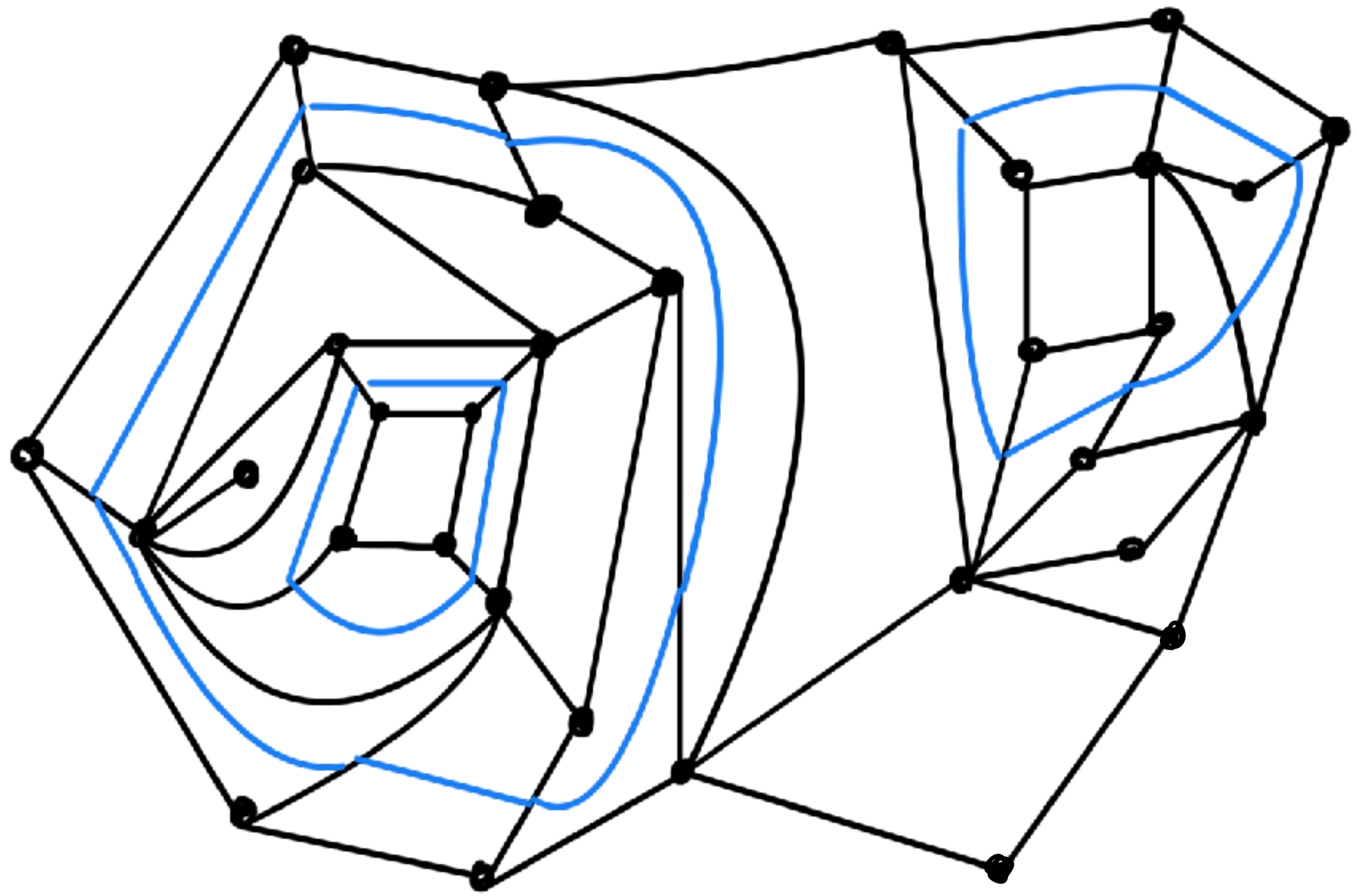
Loops and Chaos

Growth Fragmentations and Random Quadrangulations

- **Example:** O(n) model of random quadrangulation with fixed perimeter p plus loops

$$(q, l)$$

Borot-Bouttier-Guittier



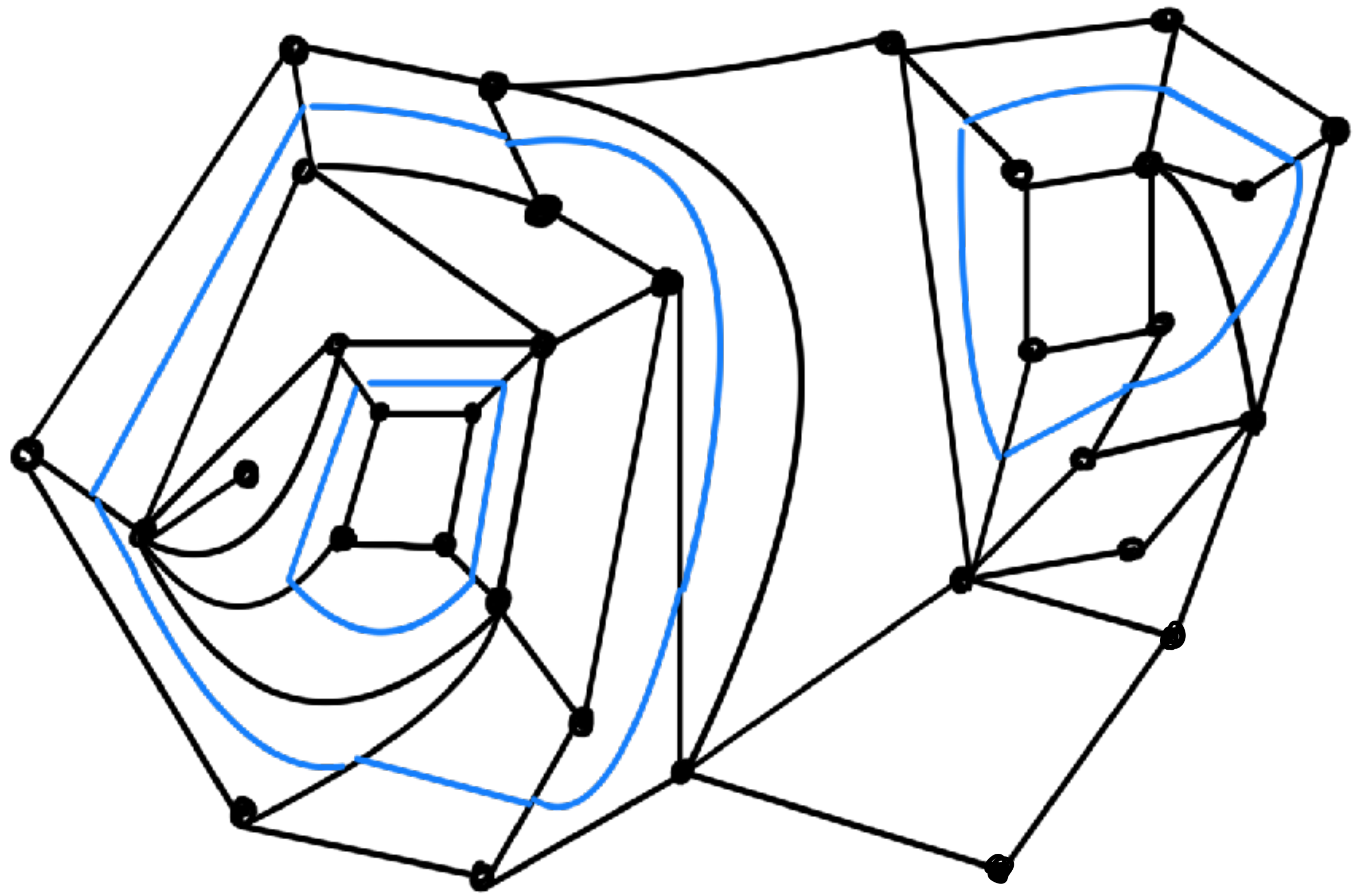
Growth Fragmentations and Random Quadrangulations

- **Example:** O(n) model of random quadrangulation with fixed perimeter p plus loops

$$(q, l)$$

- $\mathbf{P}((q,l)) \propto g^{\#\text{faces}\, q}\, h^{\text{total length}\, l} n^{\# l}$

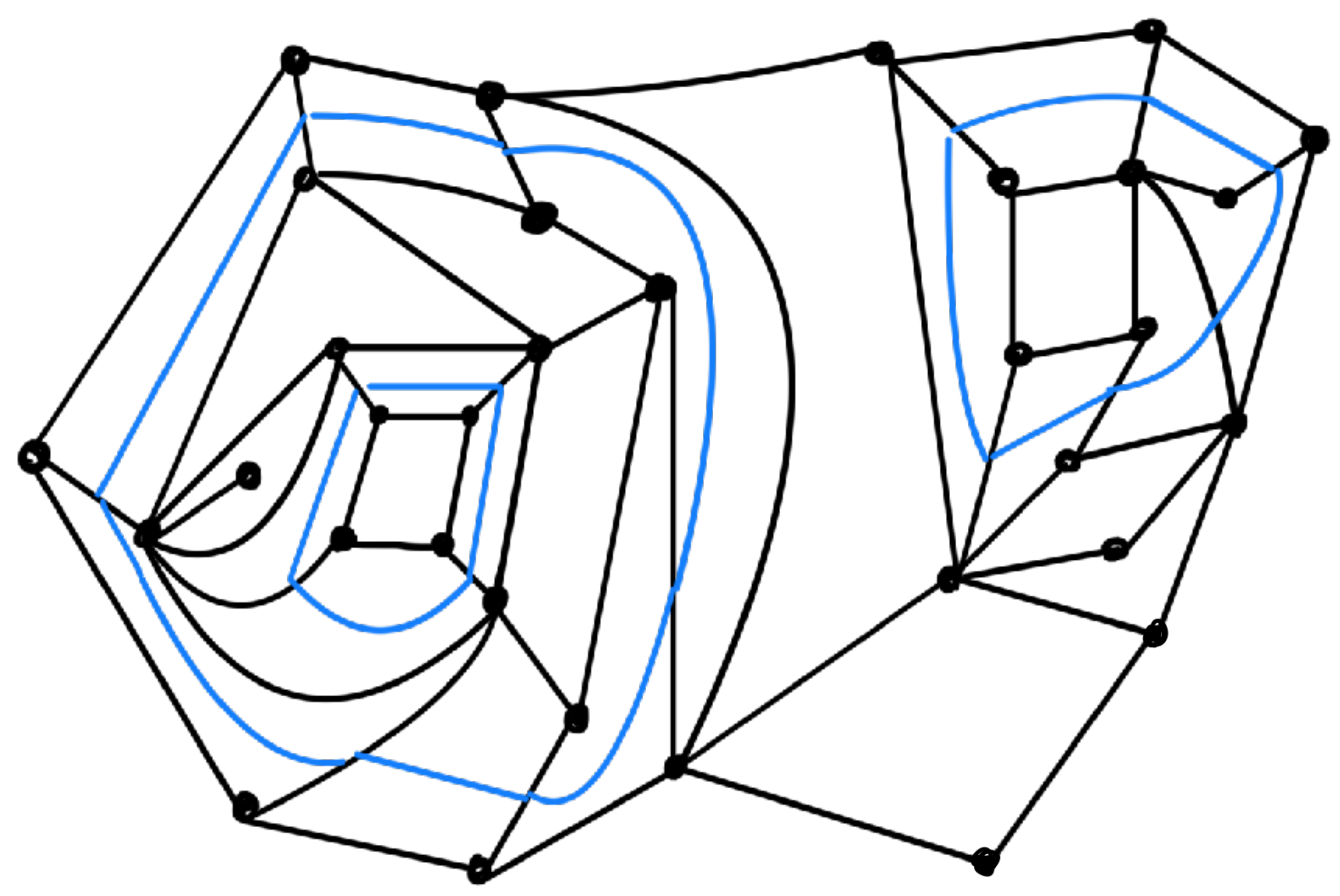
Borot-Bouttier-Guittier



Loop decorated random planar maps

- **Example:** O(n) model of quadrangulation with fixed perimeter p plus loops (q, l)

- $\mathbf{P}_p((q, l)) \propto g^{\#\text{faces}\, q}\, h^{\text{total length}\, l}\, n^{\#l}$



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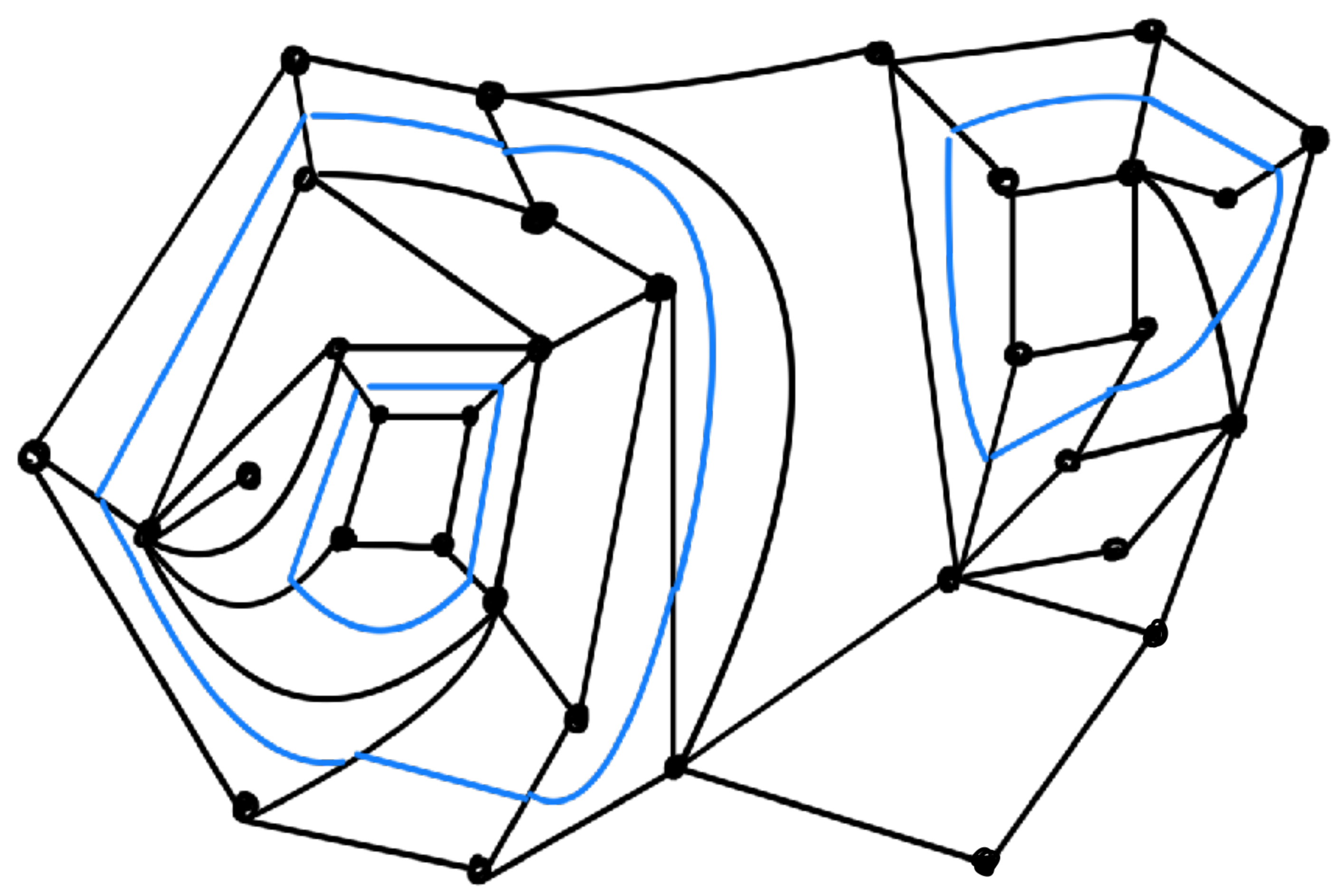

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- **Conjecture** $(n \in (0,2])$

$\exists (g^*, h^*)$ "dilute **critical**" values s.t large p scaling limit of (q, l) embedded in $\mathbb{D}$

=independent CLE_κ plus γ-GMC measure

$$\kappa = \gamma^2 = 2 - \frac{1}{\pi}\arccos\left(\frac{n}{2}\right) \in \left(\frac{8}{3}, 4\right]$$

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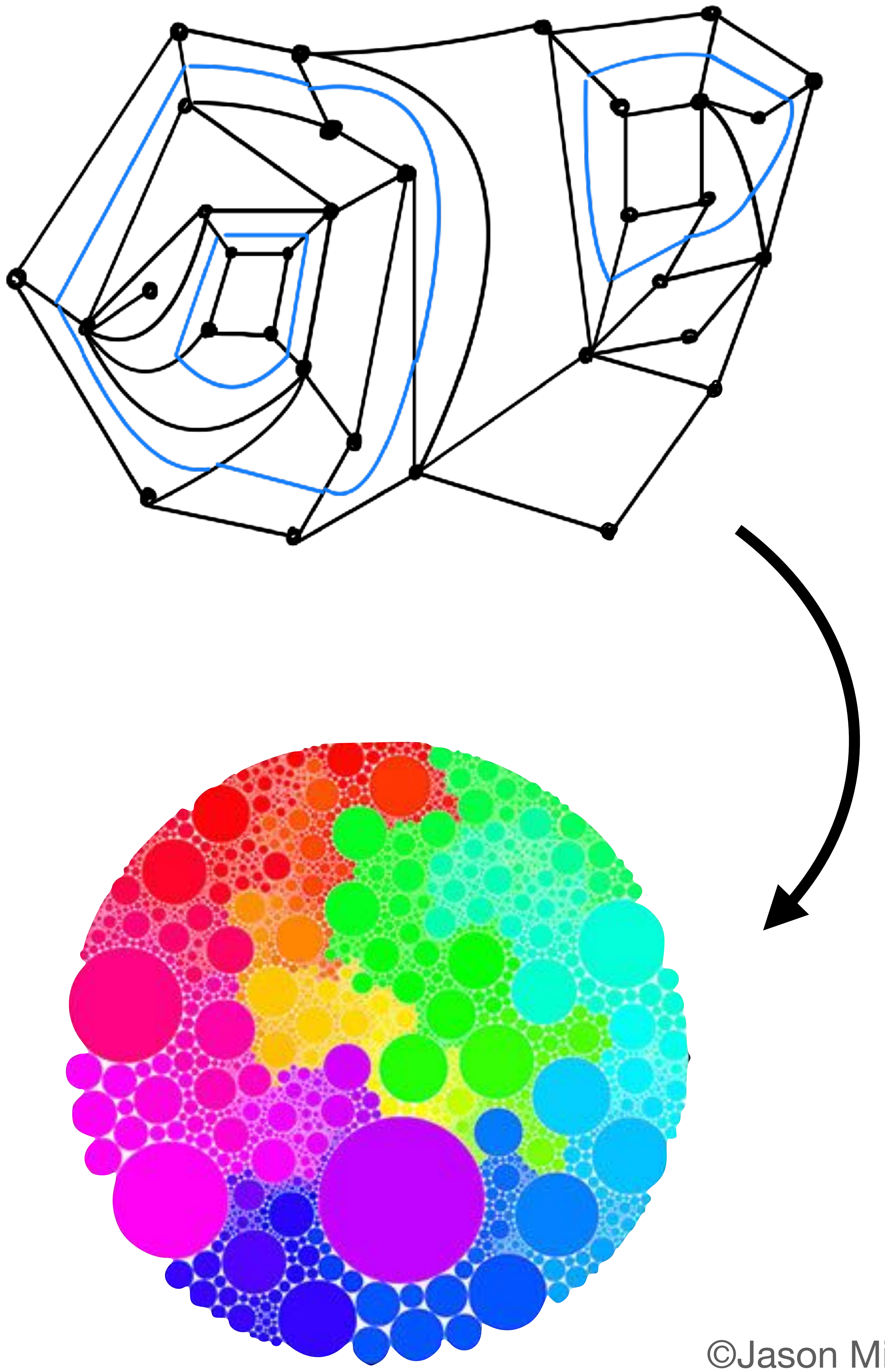

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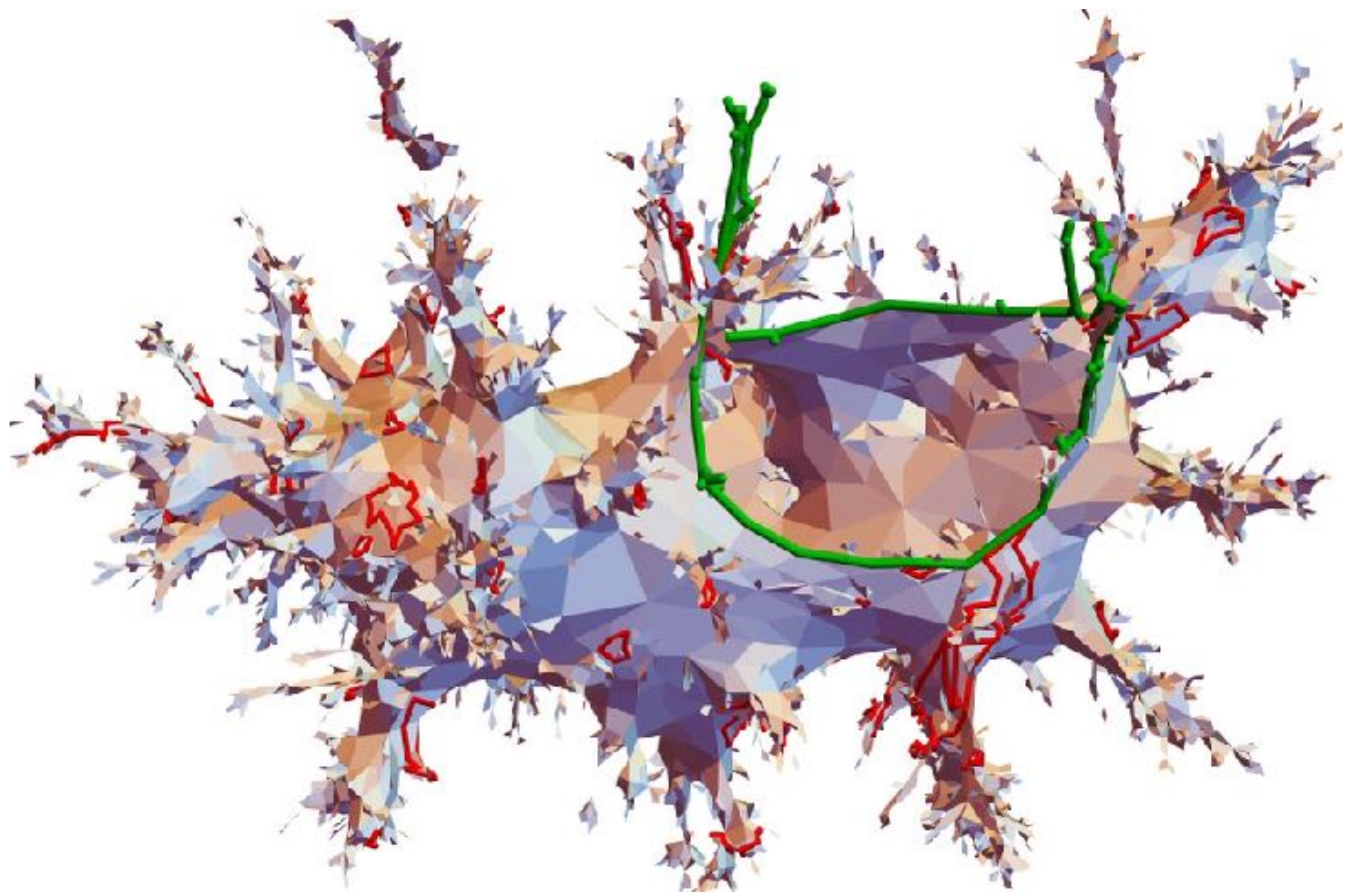

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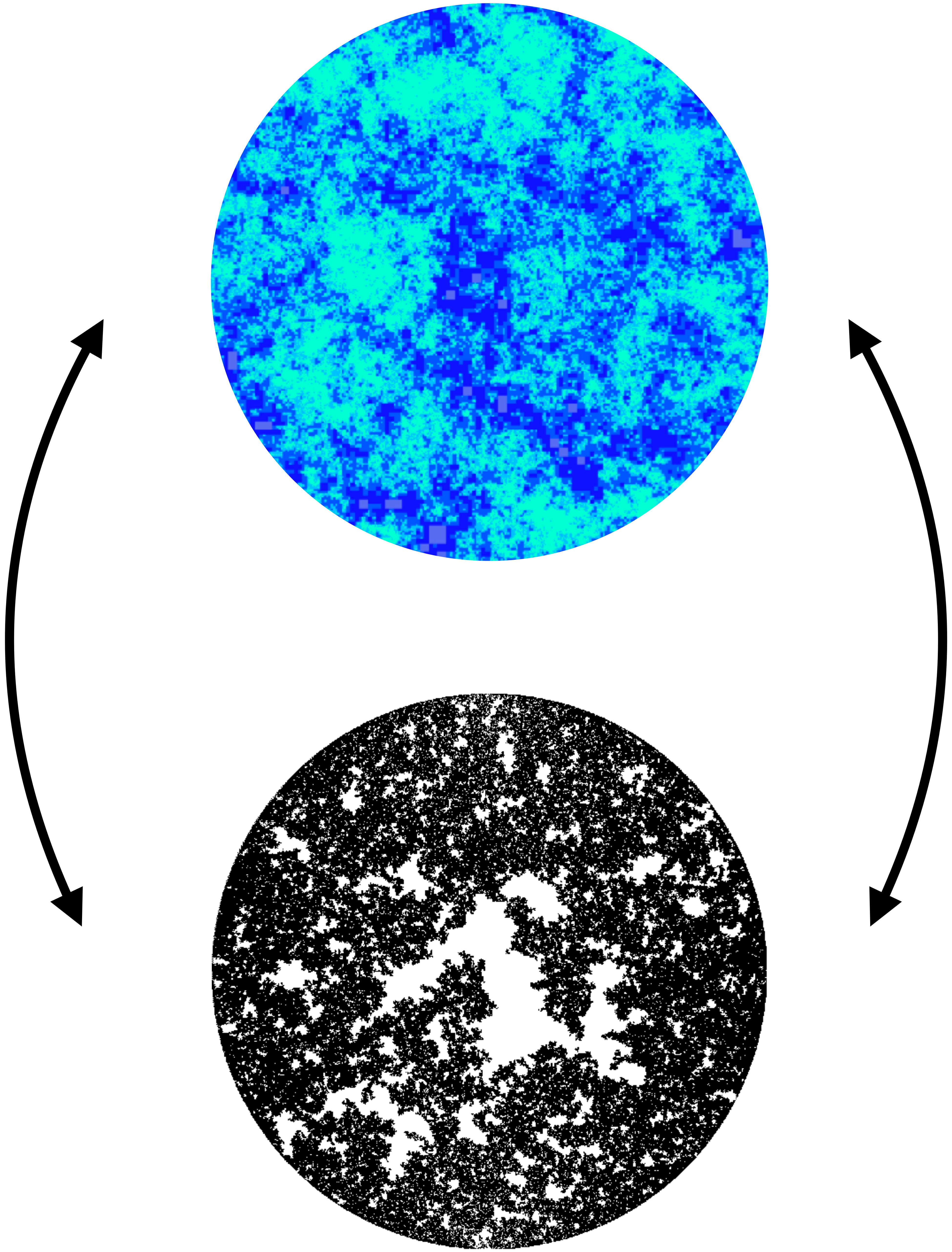
CLE decorated GMC

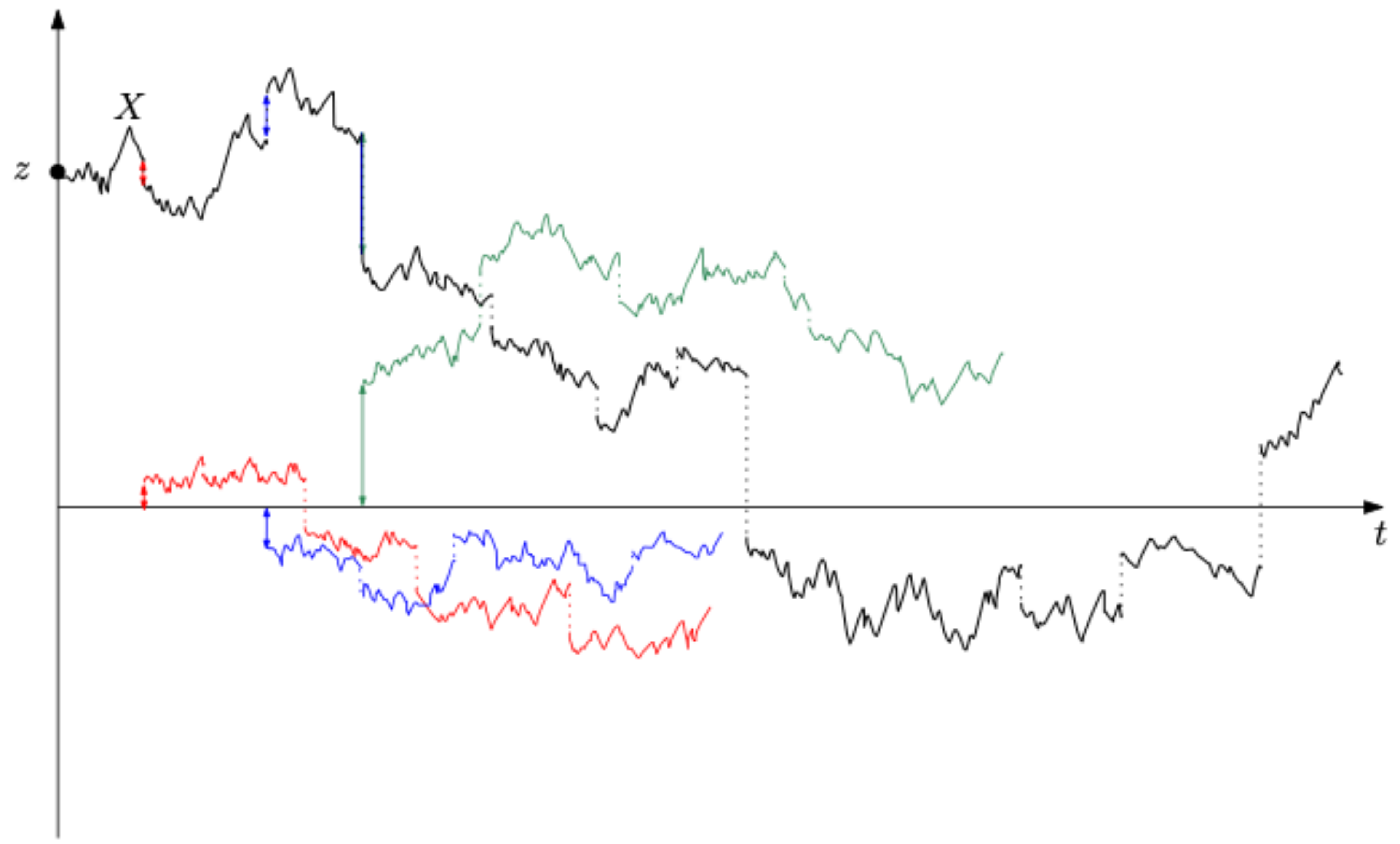
Therefore natural to study in the continuum (on $\mathbb{D}$):

- a conformal loop ensemble with parameter $\kappa \in (8/3, 4]$
- a $(\gamma = \sqrt{\kappa}) - \mathrm{GMC}$ measure
- **independent** of one another



So far…

Growth Fragmentations



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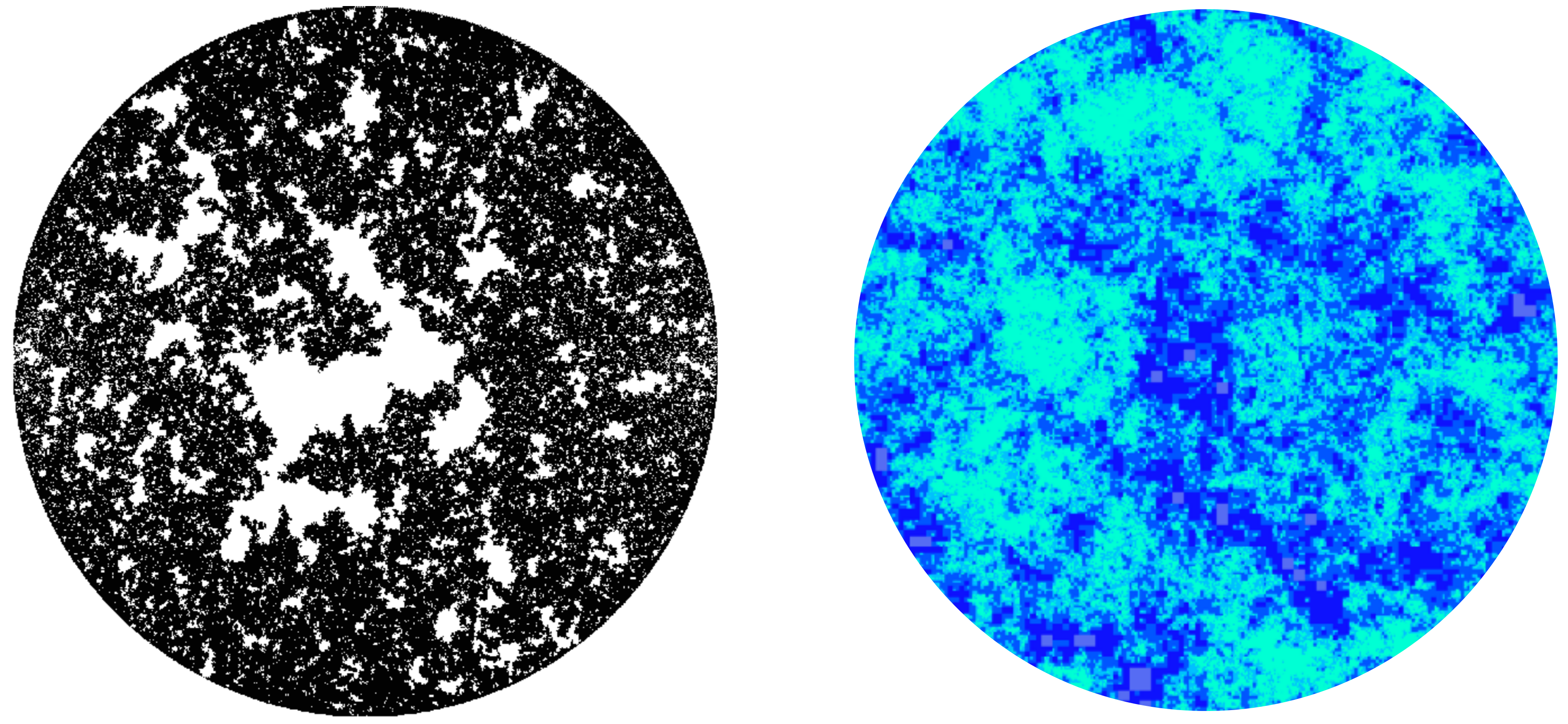
Gaussian Multiplicative Chaos & Conformal Loop Ensembles



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Existing connection with growth fragmentations

$n \in (0,2)$

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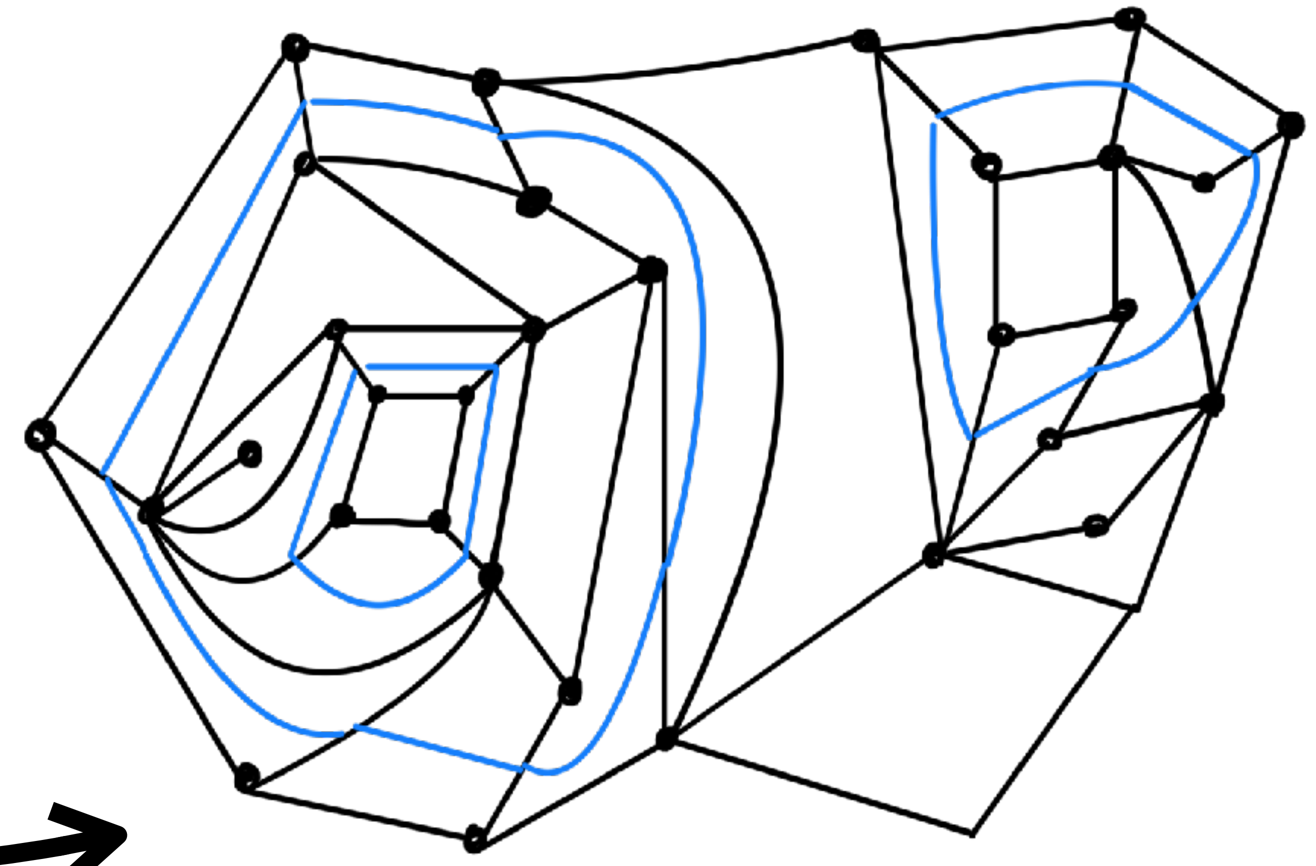
$n \in (0,2)$

Peeling explorations of random planar maps
- growth fragmentations in large volume limit



Existing connection with growth fragmentations $n \in (0,2)$

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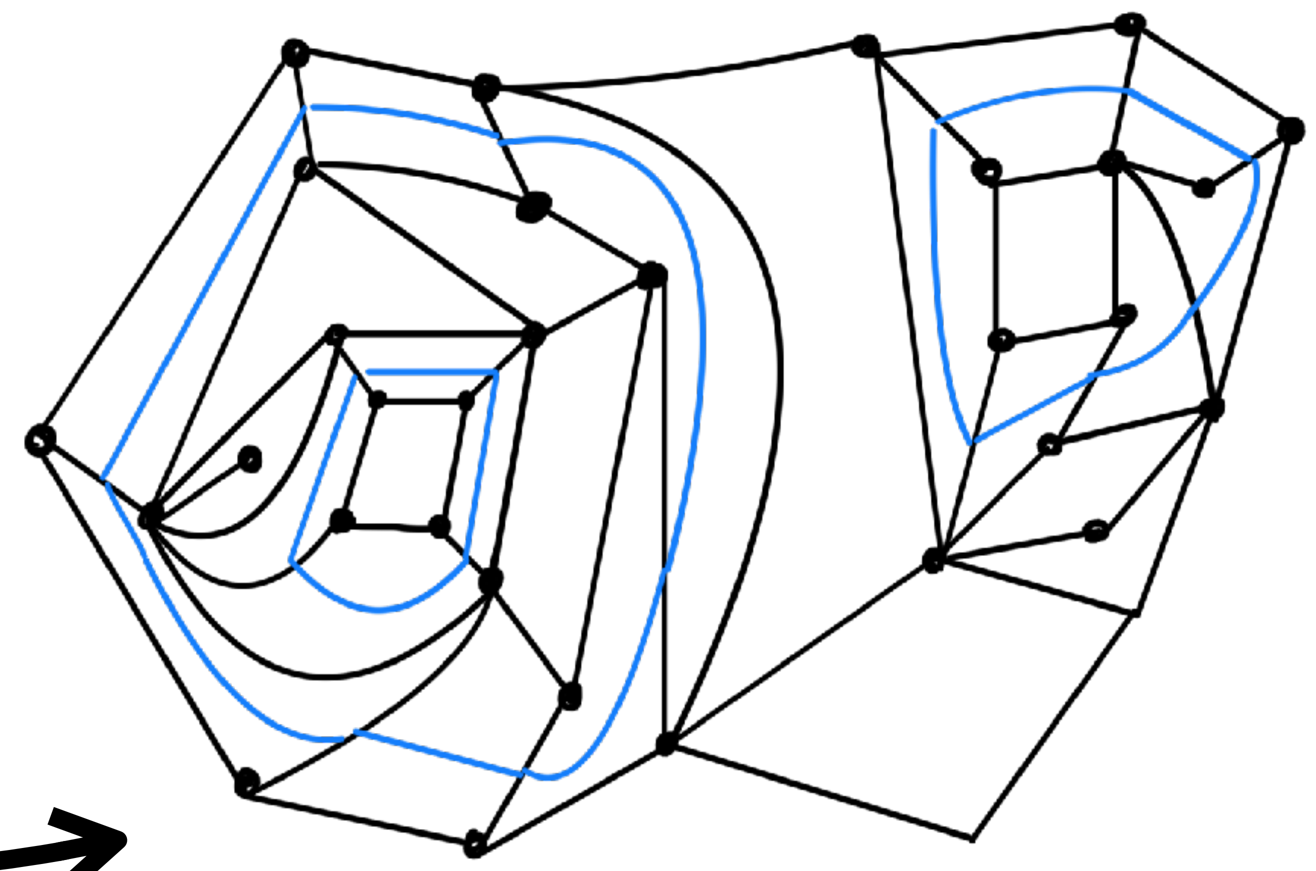
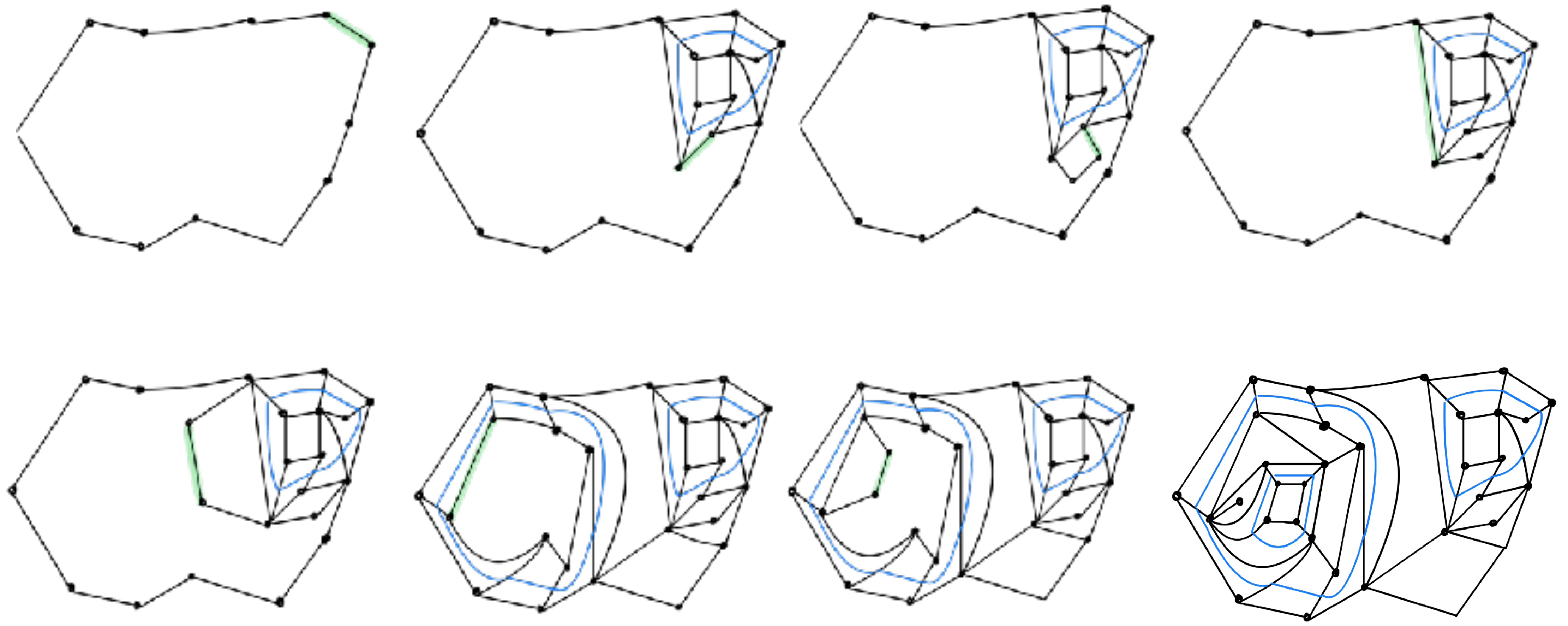


Angel, Bertoin, Budd, Chen, Curien, Kortchemski, Maillard, Le Gall …



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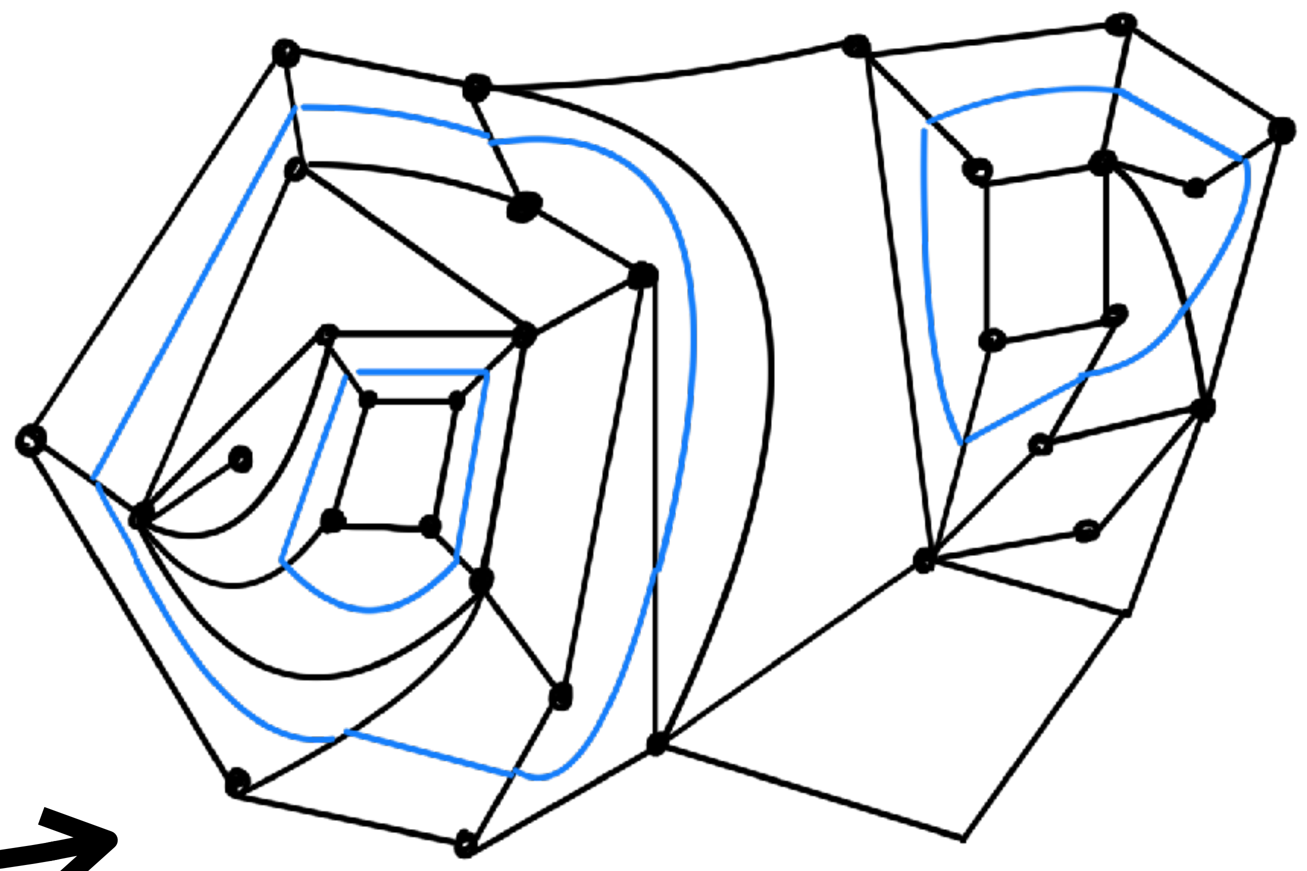

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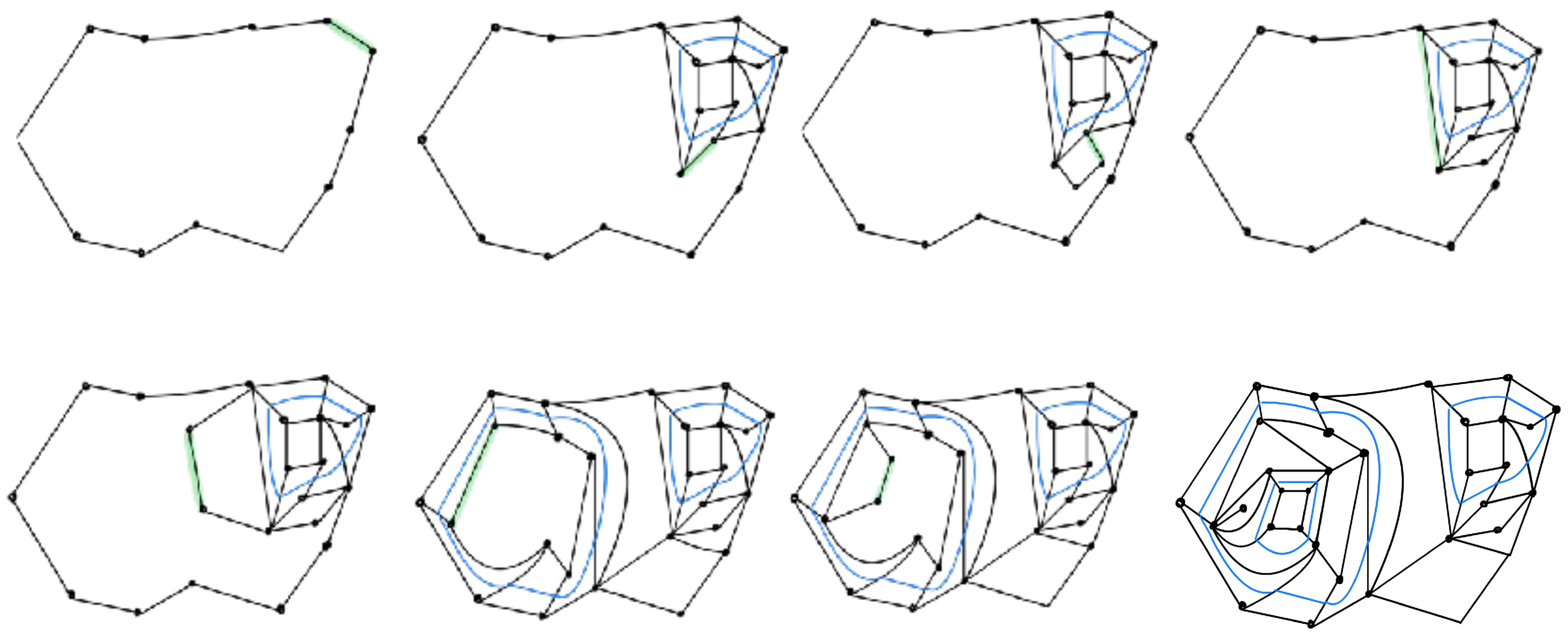

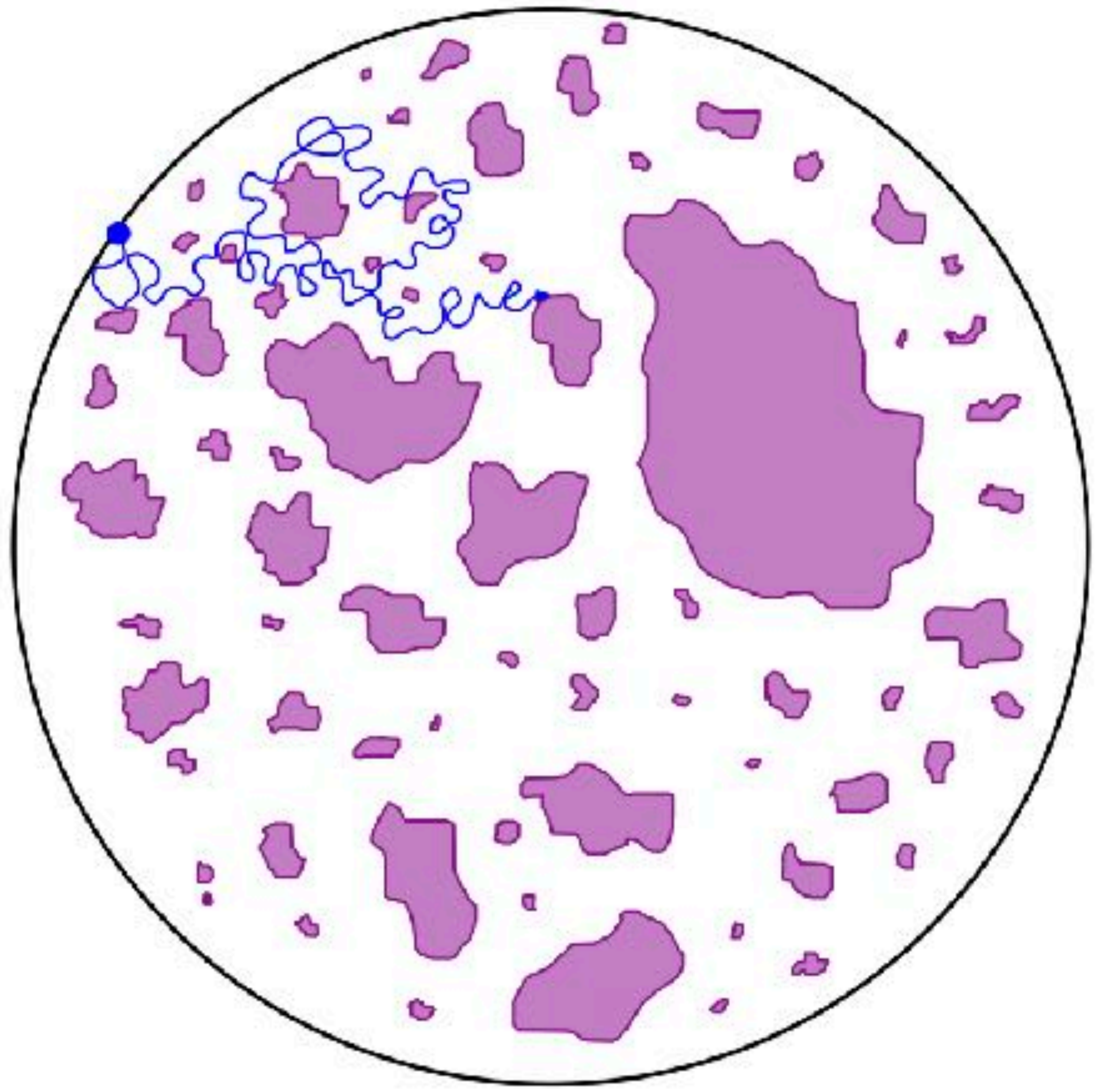

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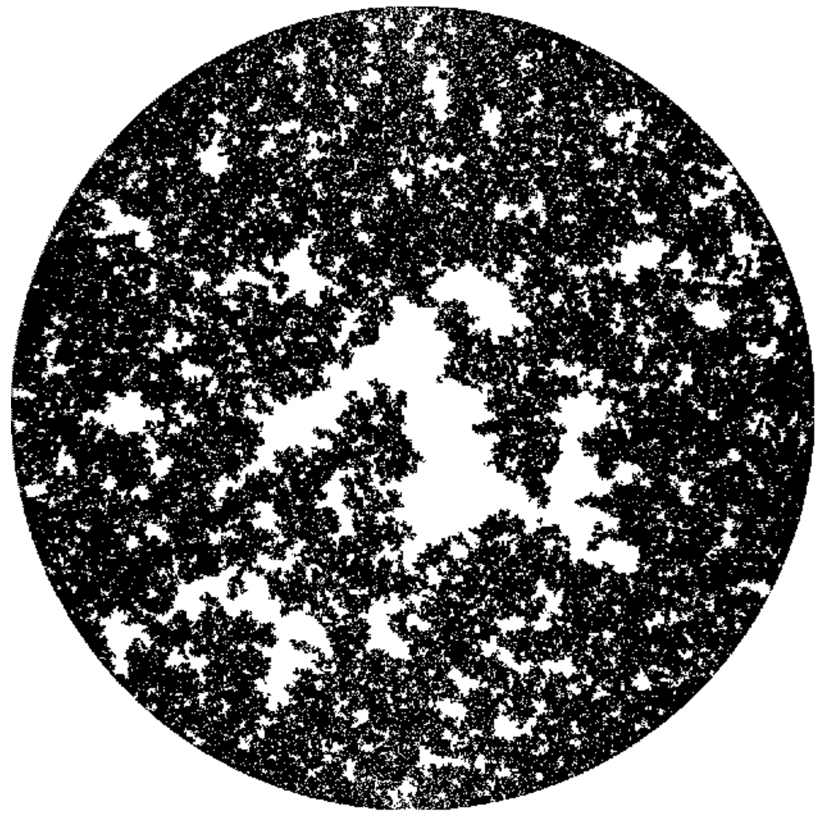
CLE percolations
(continuum analogue)
Miller-Sheffield-Werner

The $\gamma = 2, \kappa = 4$ case

Special case $(\gamma = 2, \kappa = 4)$



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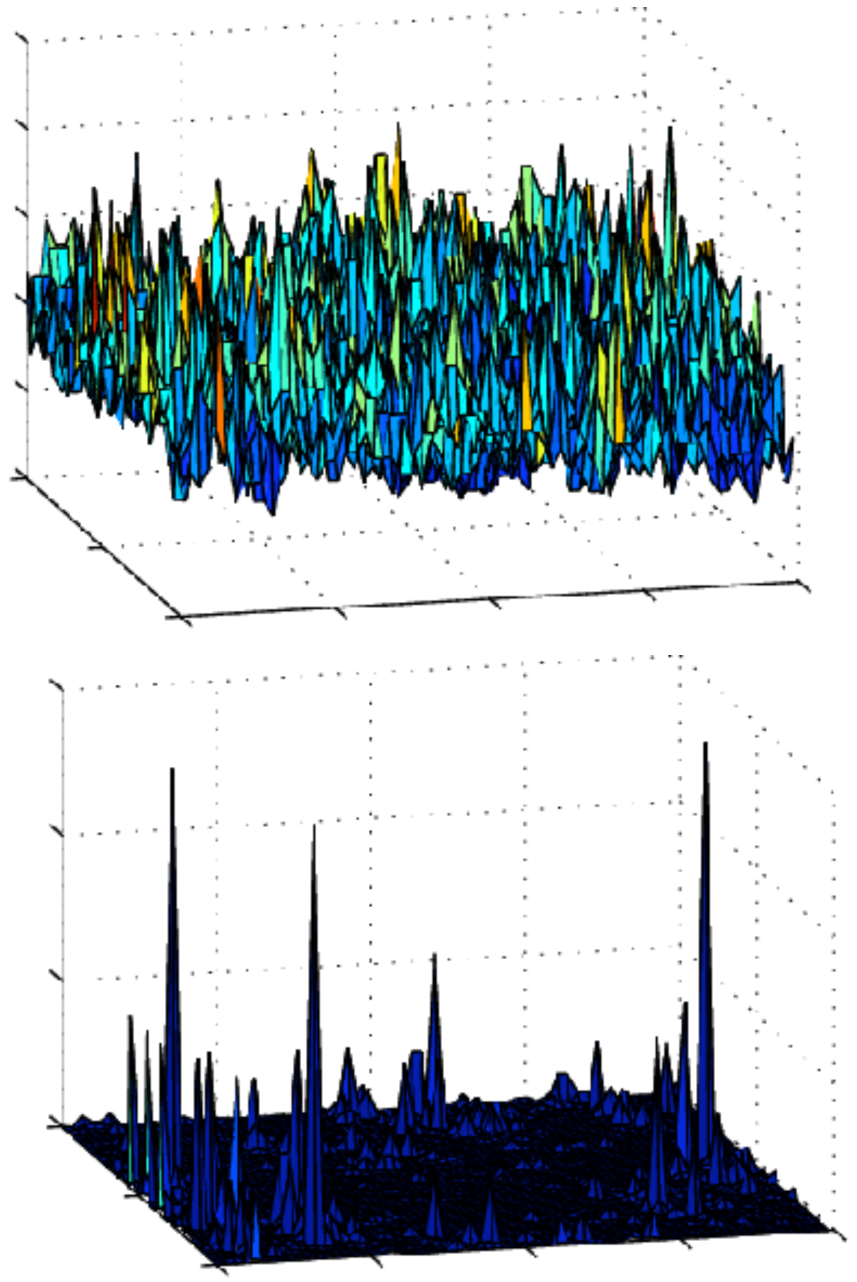

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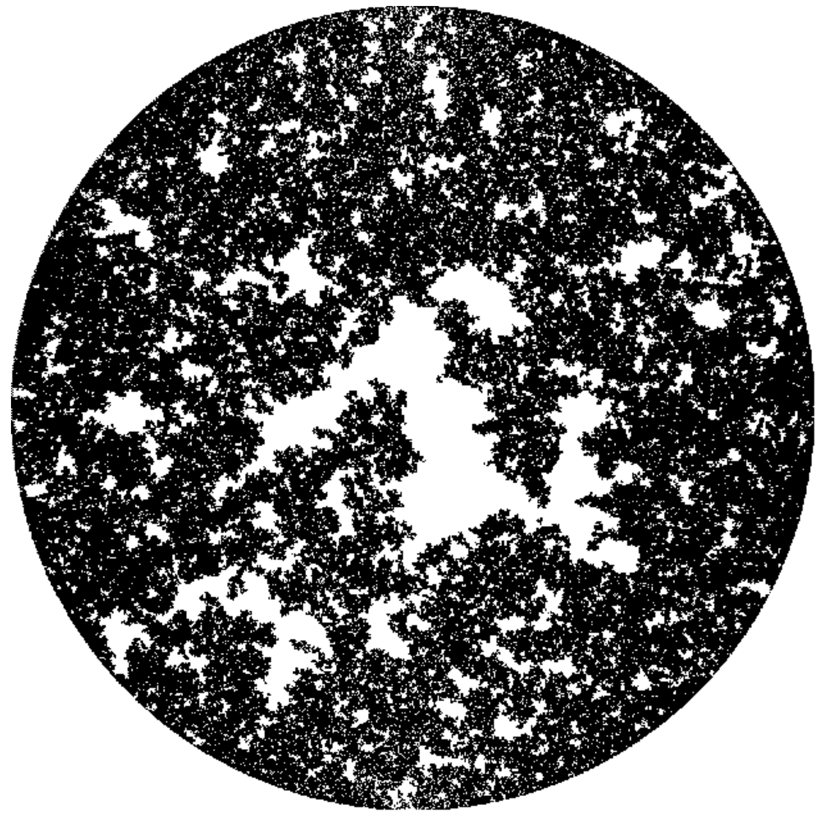
Special case $(\gamma = 2, \kappa = 4)$

- $\kappa = 4$ is a **critical value** for SLE and CLE; SLE$_\kappa$ is simple for $\kappa \leq 4$ but self-touching for $\kappa > 4$



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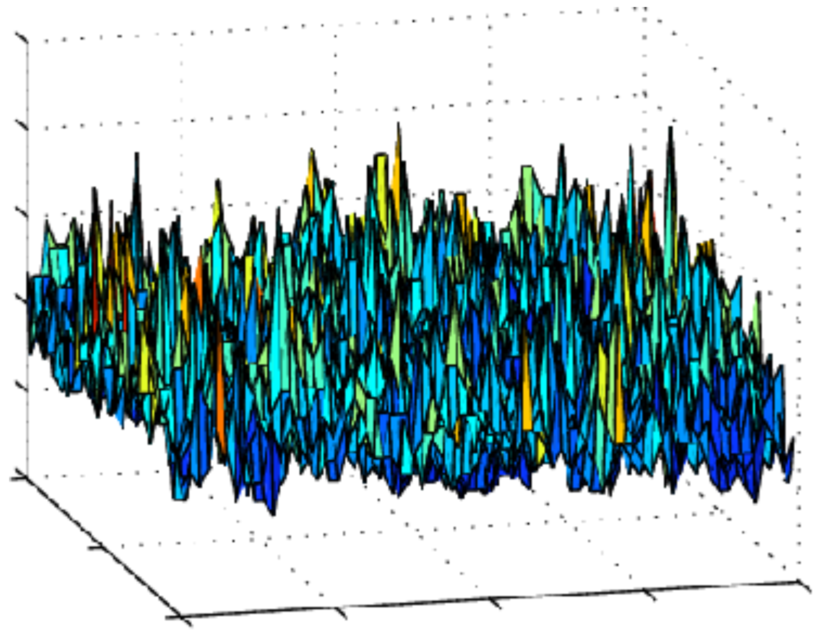

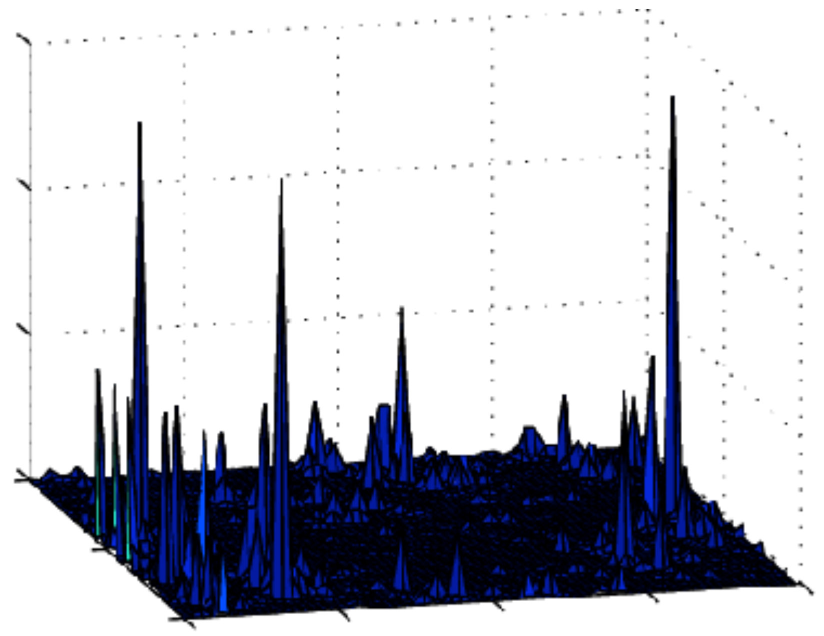

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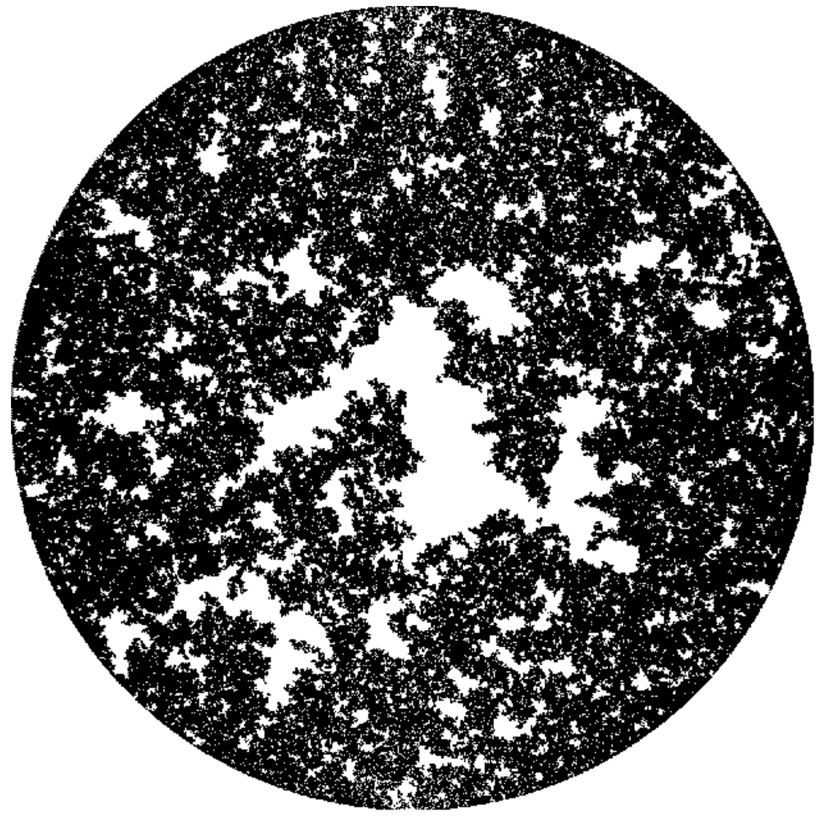
- $\gamma = 2$ is **critical** for GMC in the plane; usual definition **doesn't work**.



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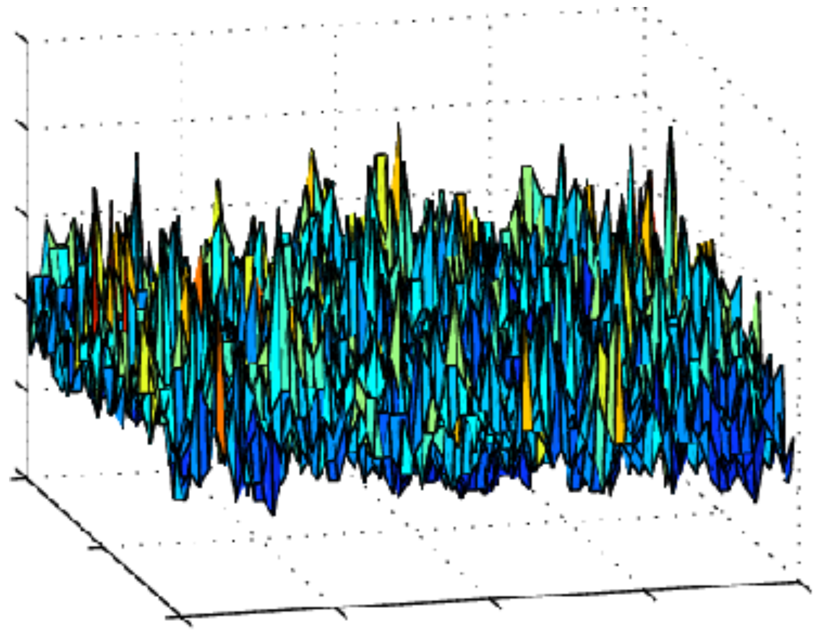

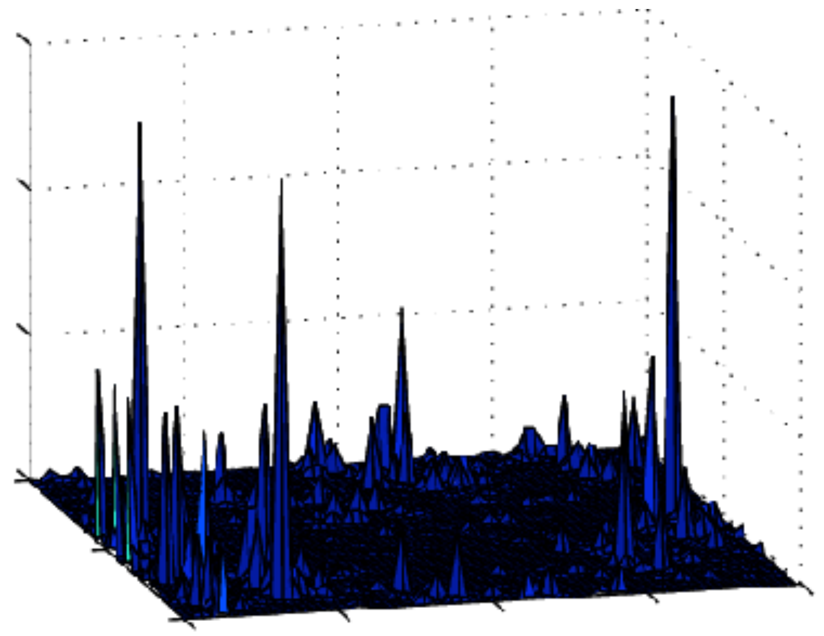

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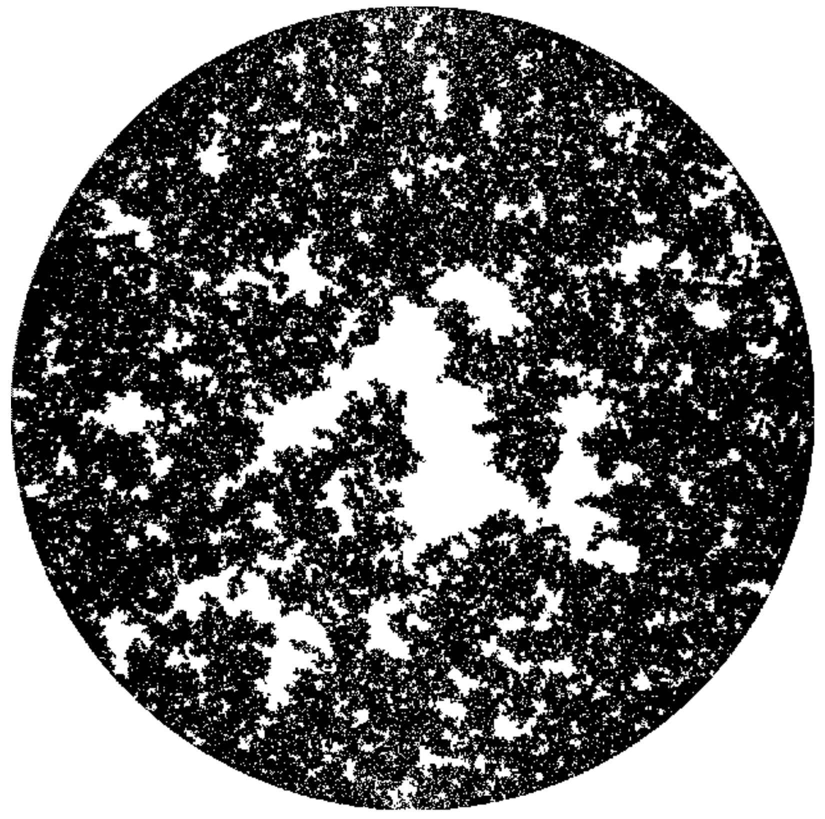
- $(\gamma = 2)$-GMC can be defined from $(\gamma < 2)$-GMC, but need to **blow up** measures by $1/(2 - \gamma)$



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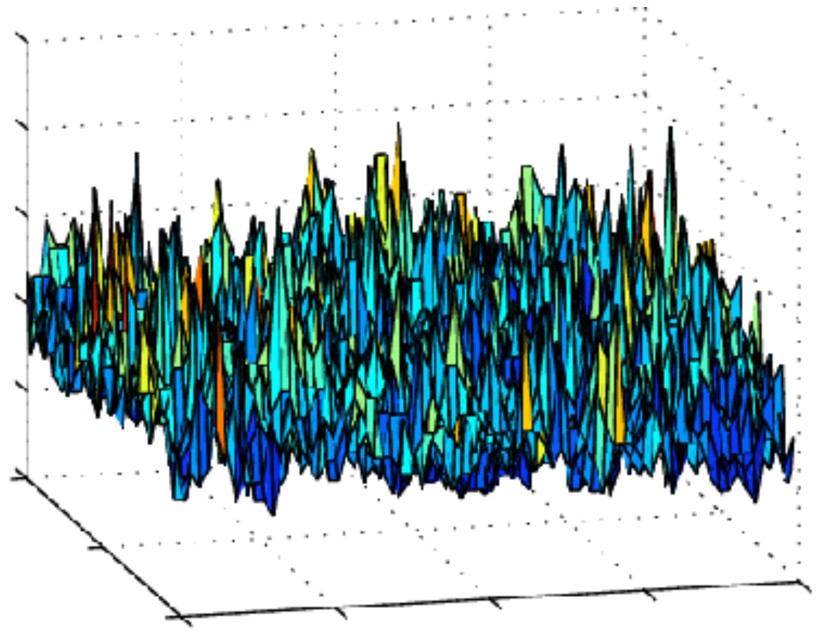

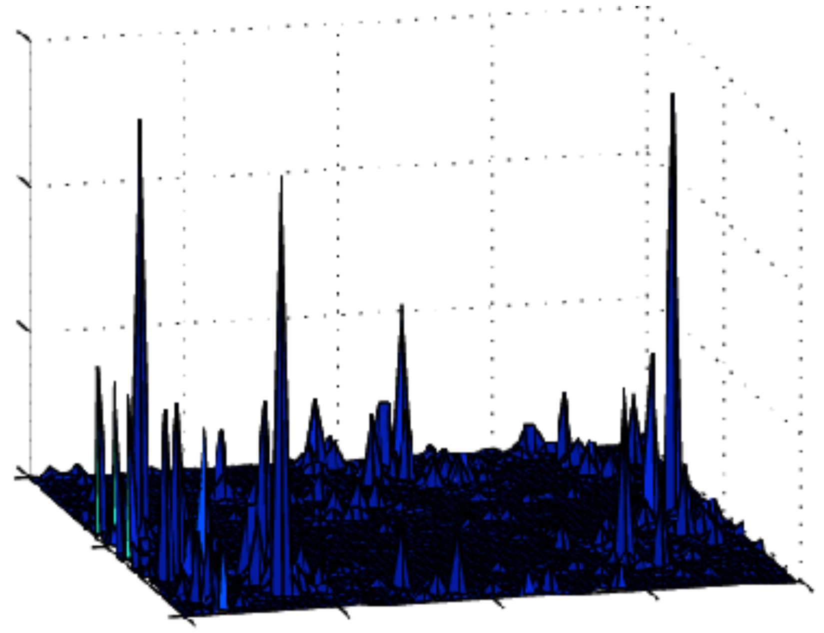

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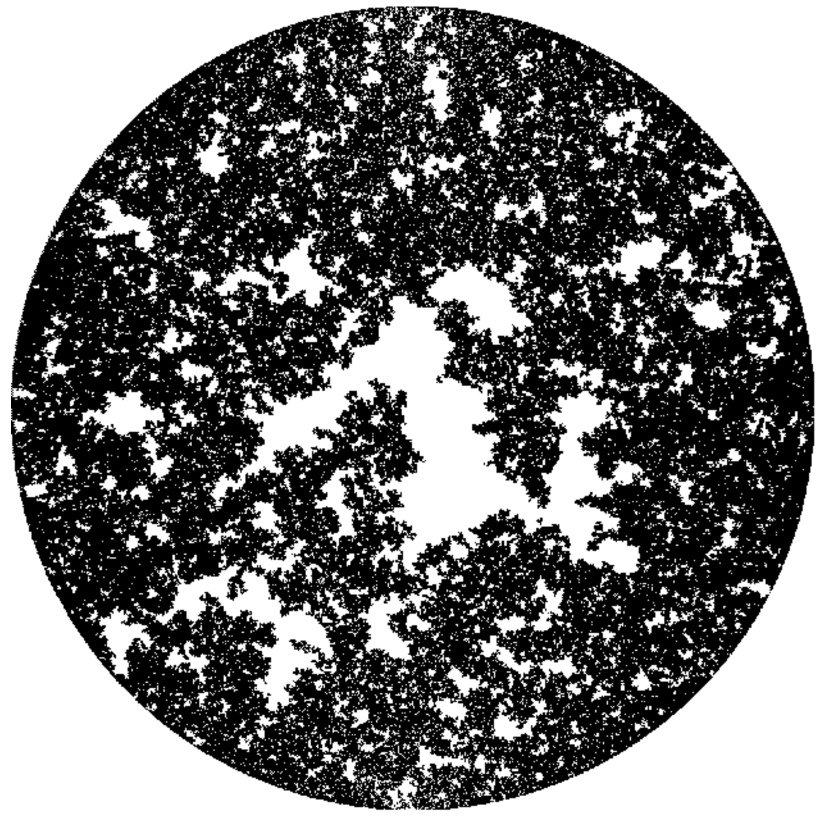
- $(\gamma = 2)$-GMC can be defined from $(\gamma < 2)$-GMC, but need to **blow up** measures by $1/(2-\gamma)$
- Miller-Sheffield-Werner's exploration doesn't have a nice limit, but…



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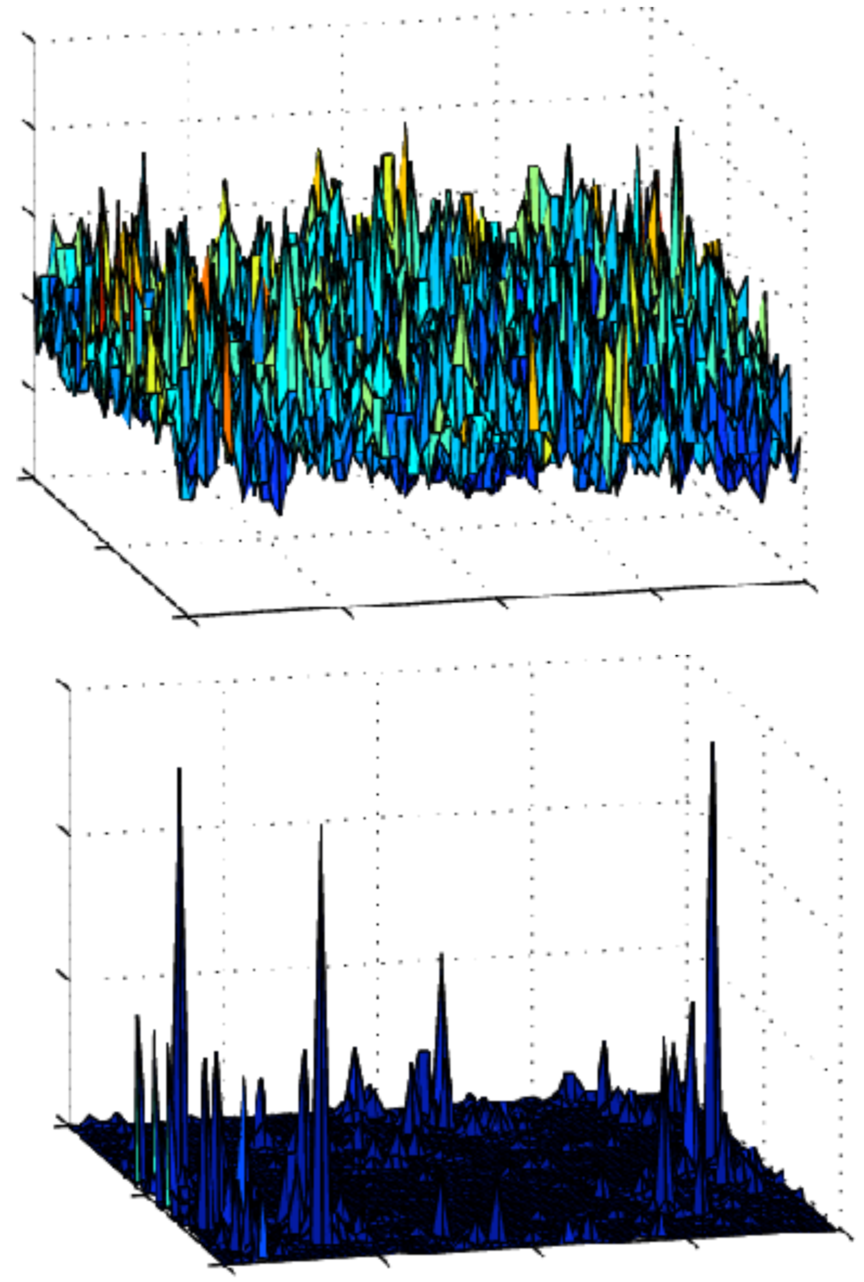

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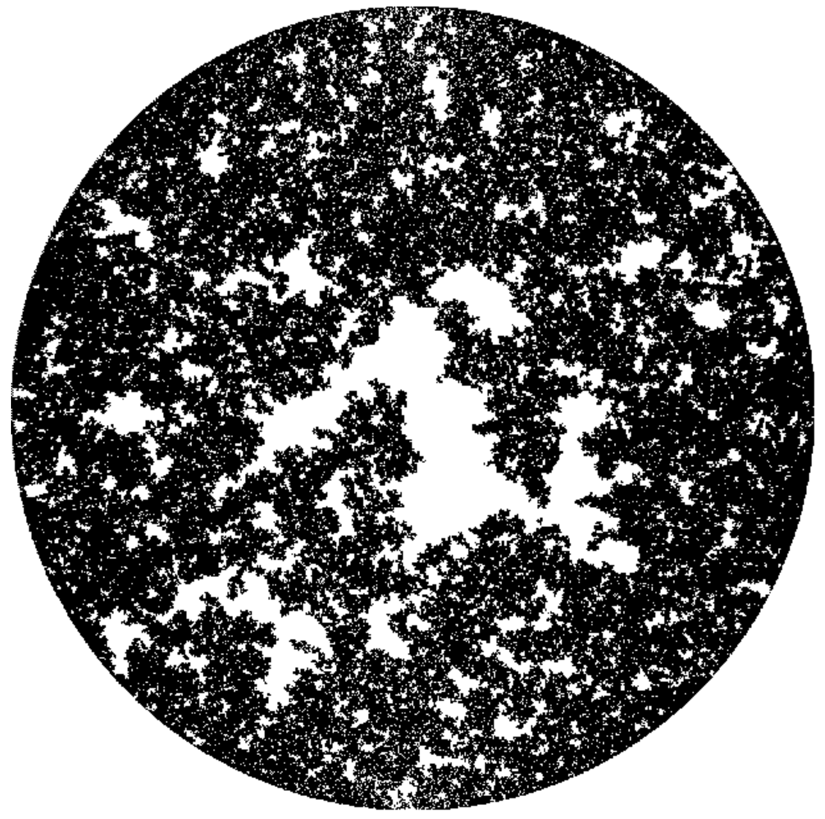
Special case $(\gamma = 2, \kappa = 4)$

- $\kappa = 4$ is a **critical value** for SLE and CLE; SLE_κ is simple for $\kappa \leq 4$ but self-touching for $\kappa > 4$
- $\gamma = 2$ is **critical** for GMC in the plane; usual definition **doesn't work**.
- $(\gamma = 2)$-GMC can be defined from $(\gamma < 2)$-GMC, but need to **blow up** measures by $1/(2-\gamma)$
- Miller-Sheffield-Werner's exploration doesn't have a nice limit, but…
- Budd-Curien-Marzouk: peeling the gasket of a critical O(2) mode -> Cauchy process



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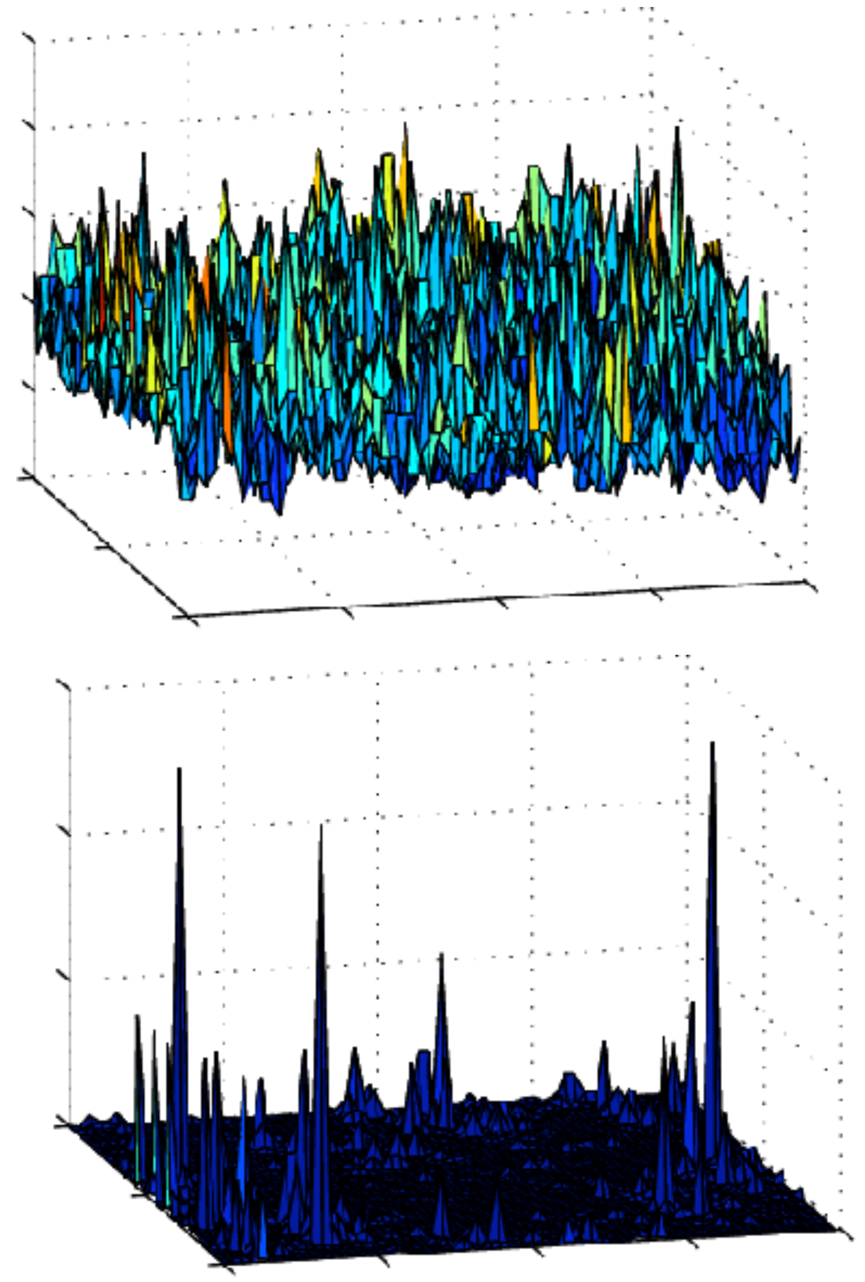

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Special case $(\gamma = 2, \kappa = 4)$

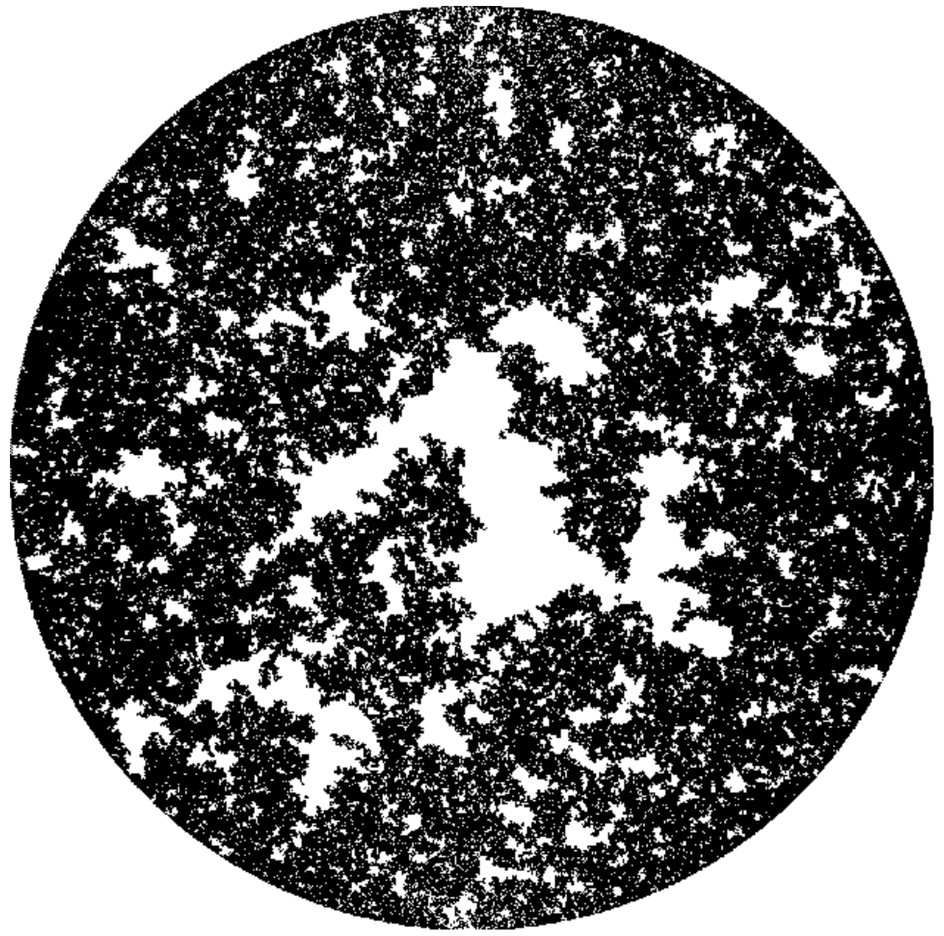
Theorem (Aru-Holden-P.-Sun)

Take a **uniform branching exploration*** of a CLE_4 in $\mathbb{D}$ and an independent GFF (variant) on $\mathbb{D}$

*see next slide!

Keeping track of critical GMC boundary lengths, as measured by the GFF

$\rightsquigarrow$ **growth fragmentation process**



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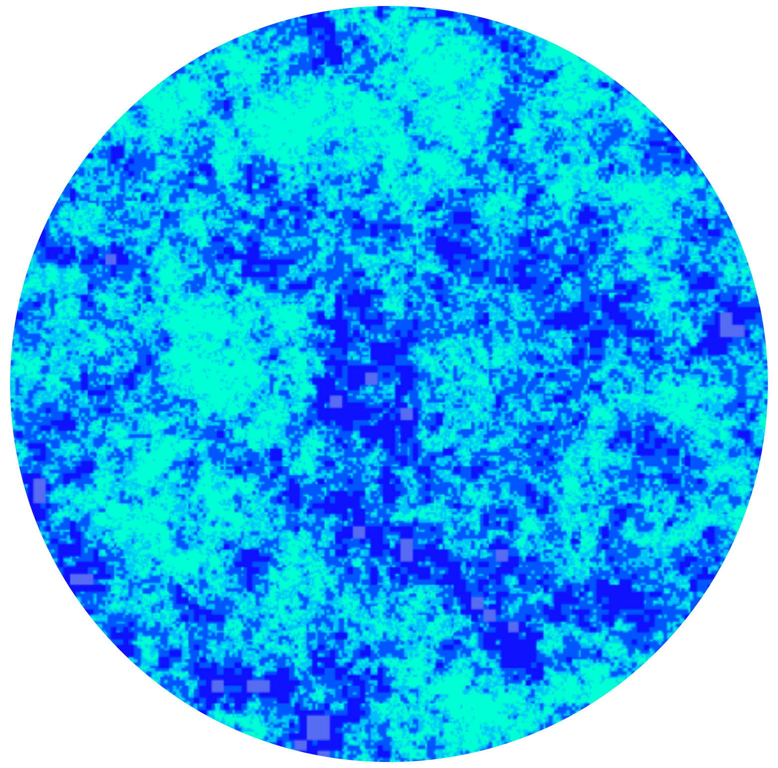

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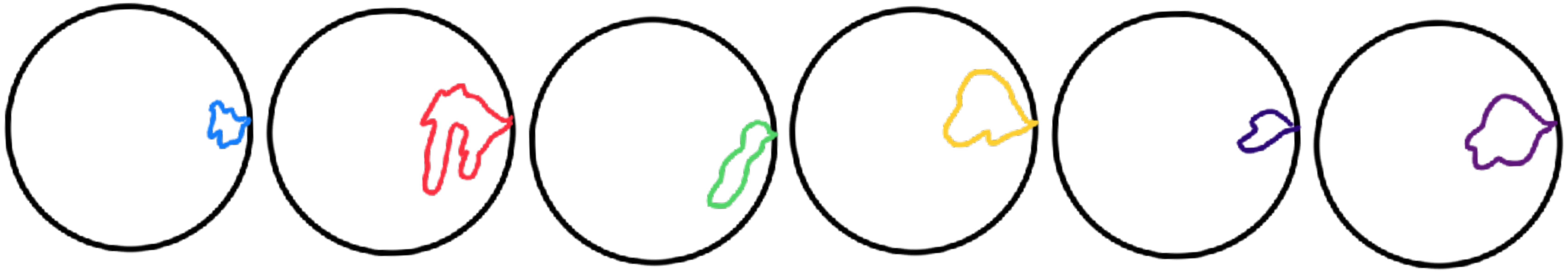
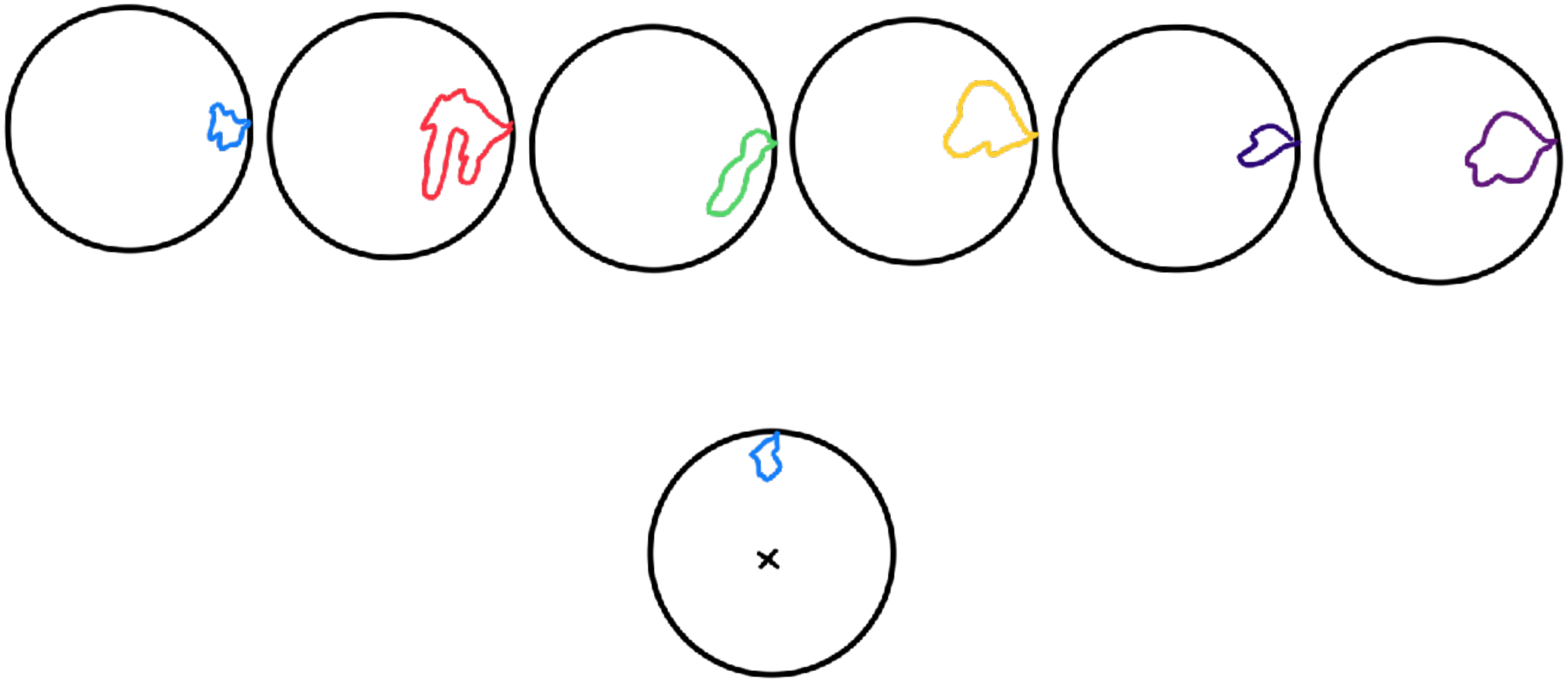
Uniform CLE_4 exploration (one branch)

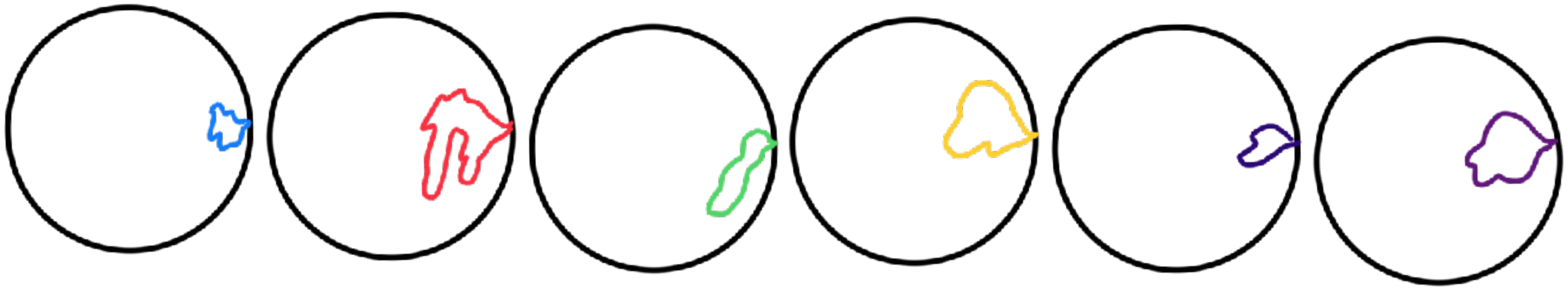


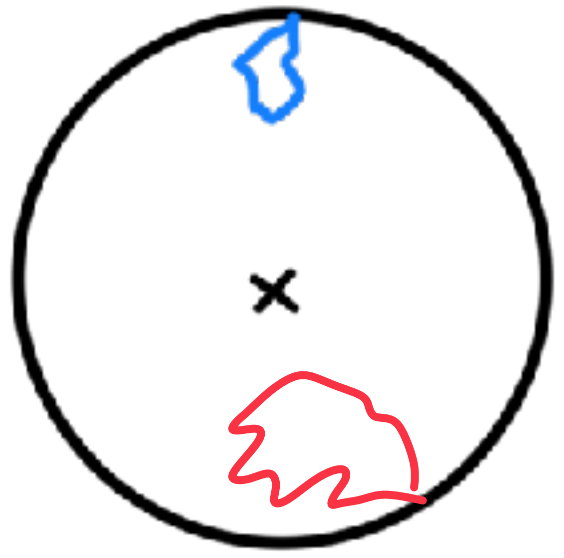
- Start with a Poisson point process of "SLE_4 bubbles"
- Add them in chronologically into the connected component containing the target with "uniform attachment point" on the boundary

Uniform CLE_4 exploration (one branch)

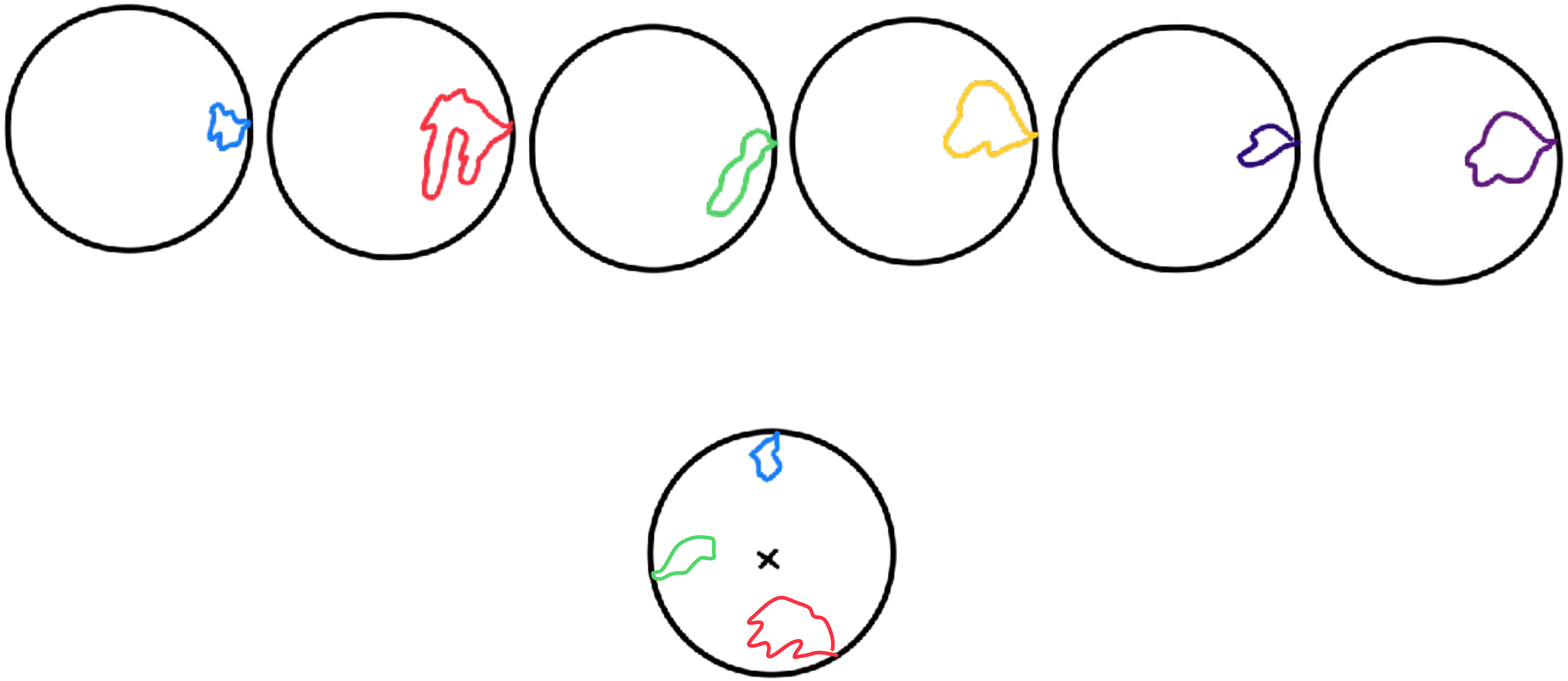


Uniform CLE_4 exploration (one branch)

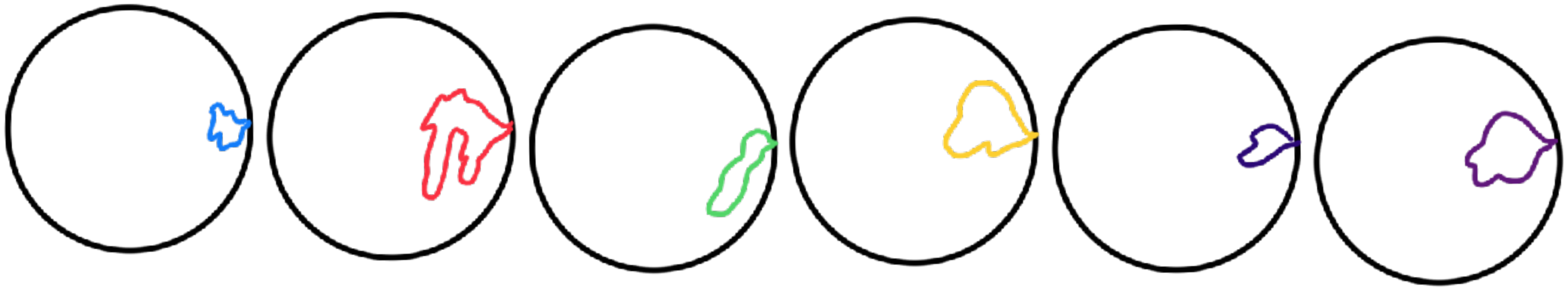




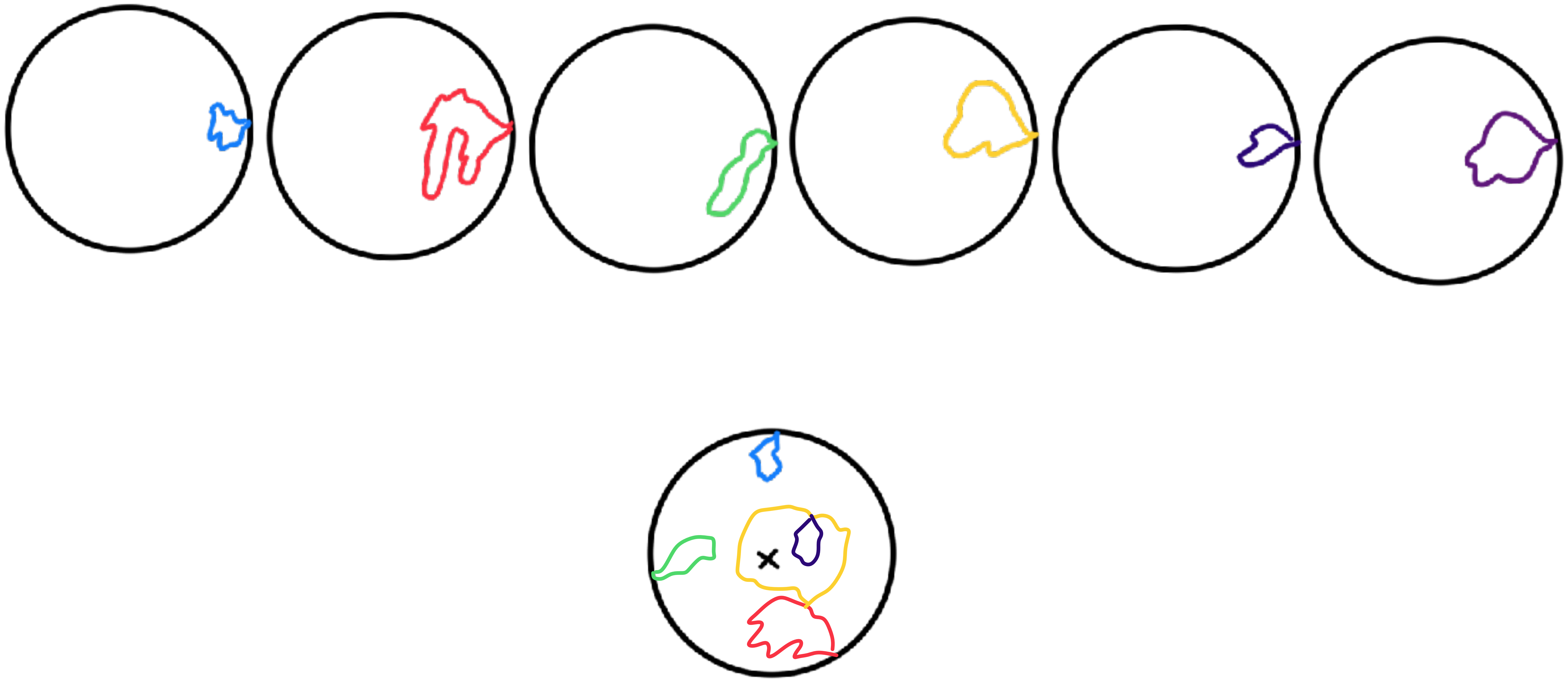
Uniform CLE_4 exploration (one branch)



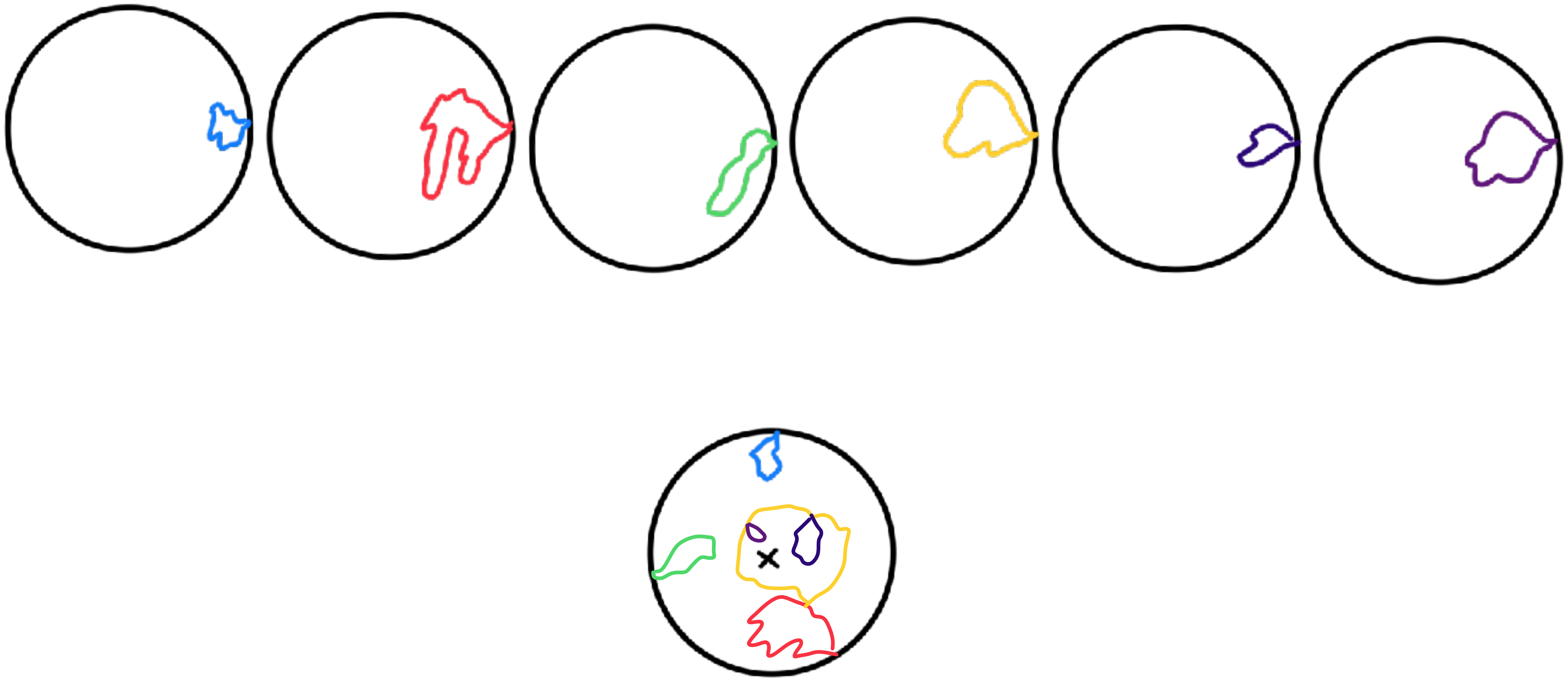
Uniform CLE_4 exploration (one branch)



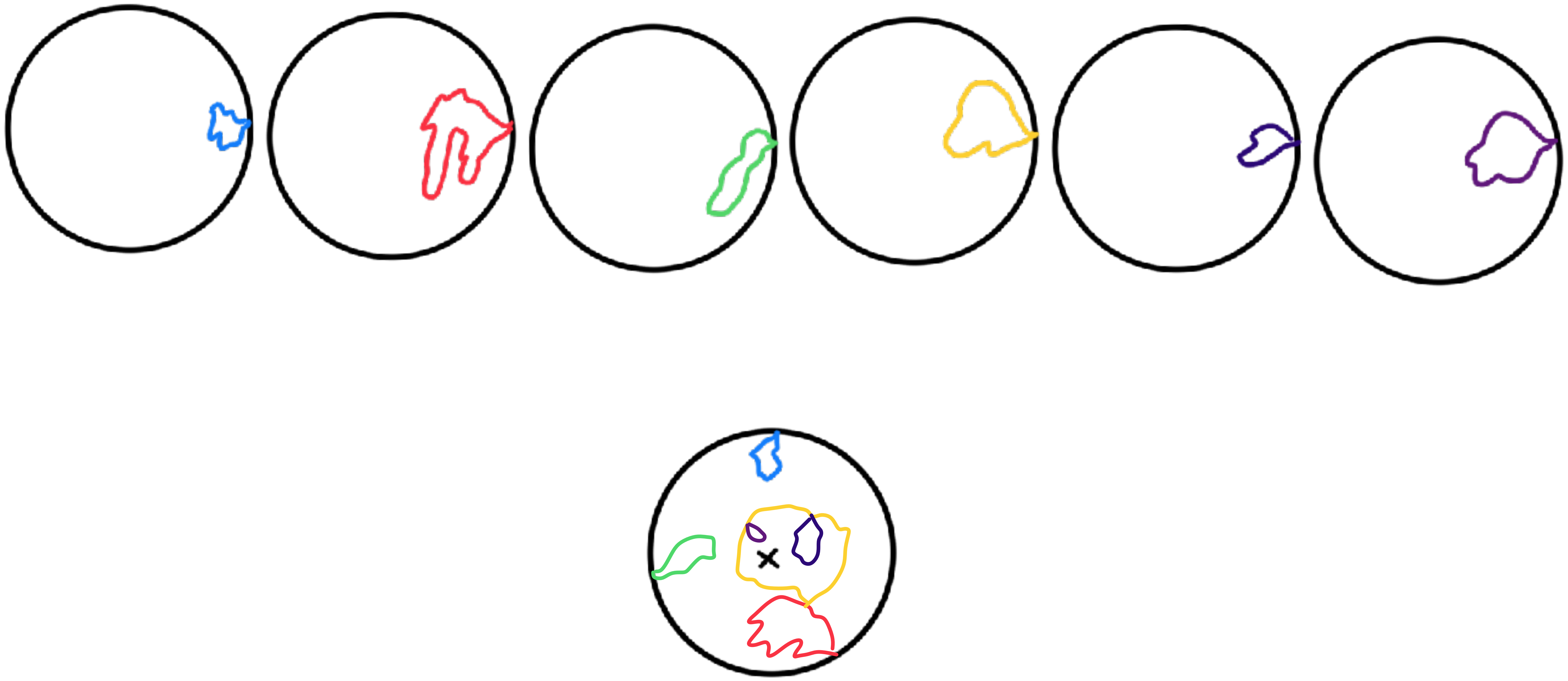
Uniform CLE_4 exploration (one branch)



Uniform CLE_4 exploration (one branch)



Uniform CLE_4 exploration



- **Branching version:** branches towards two points are independent until separated, then evolve independently

Special case $(\gamma = 2, \kappa = 4)$

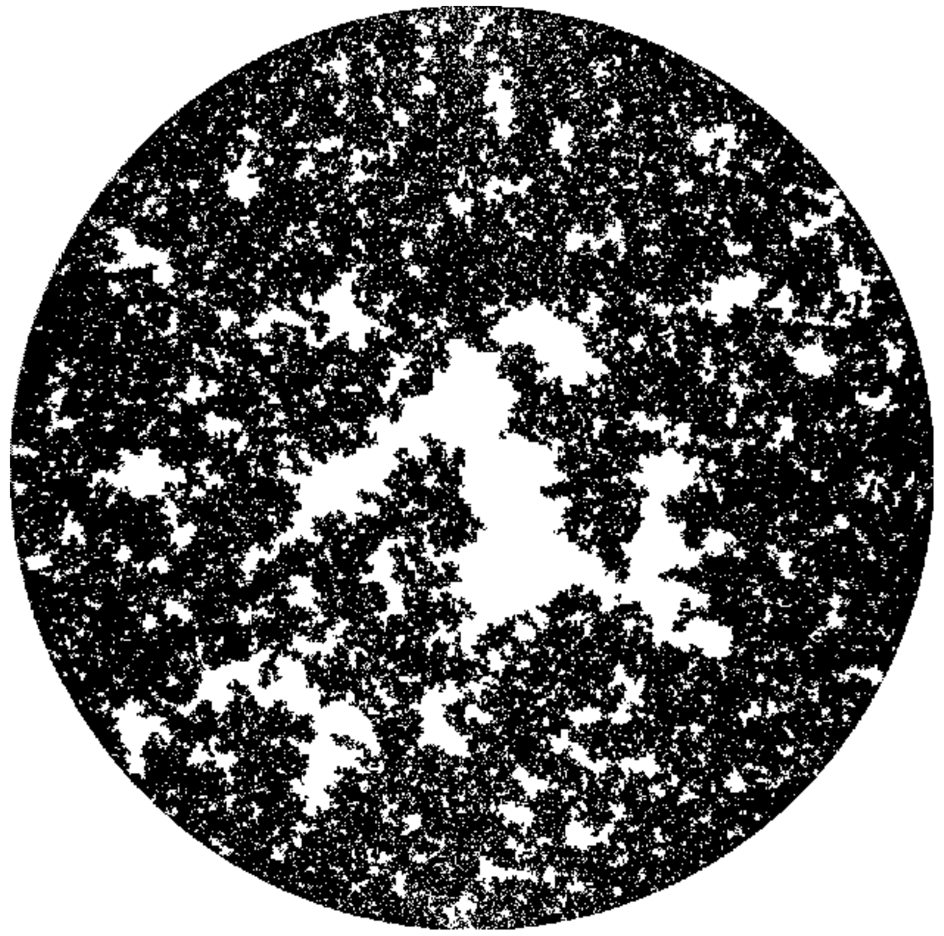
Theorem (Aru-Holden-P.-Sun)

Take a **uniform branching exploration** of a CLE_4 in $\mathbb{D}$ and an independent GFF (variant) on $\mathbb{D}$

Critical GMC lengths of **yet-to-be-explored connected components**

$\rightsquigarrow$ (explicit) signed **growth fragmentation process**

Signs: how many loops completely surround component



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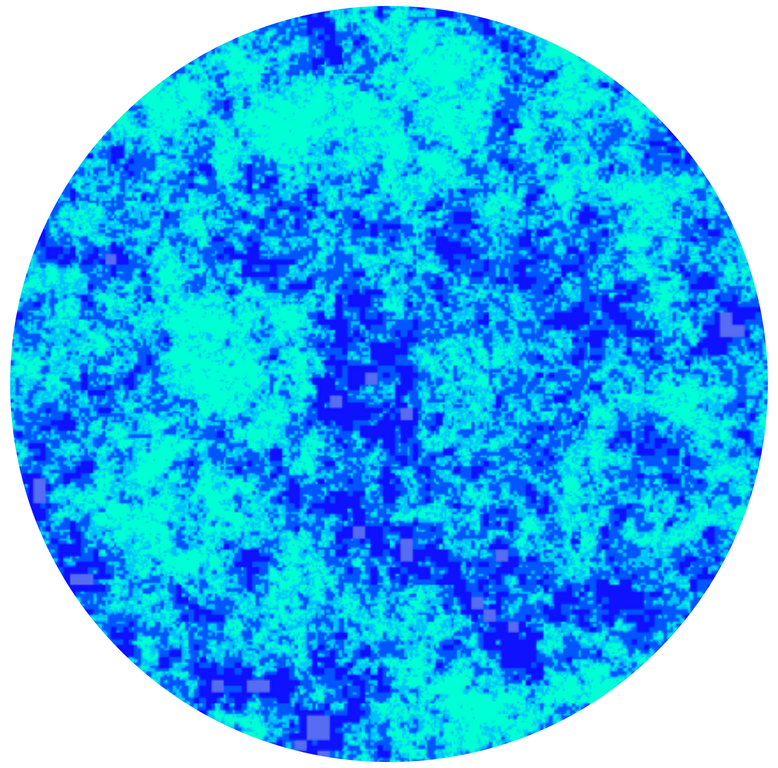

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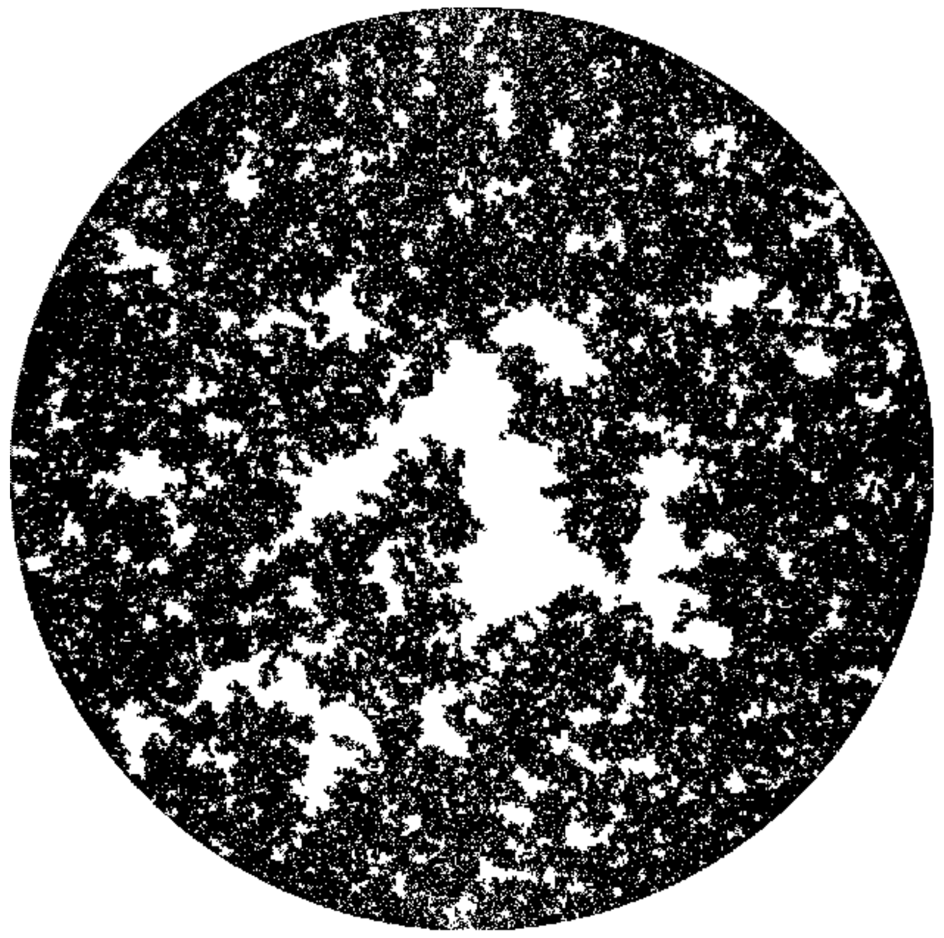
Comments

- The uniform CLE_4 exploration is different to that considered by Miller-Sheffield-Werner in the subcritical case

- “Eve cell” (pssMp X from def of GF) is a type of **Cauchy process**

- Time parameterisation = **“quantum distance”** from boundary

- It’s exactly the same the signed GF that Aïdékon-Da Silva constructed out of a **Brownian half plane excursion…**



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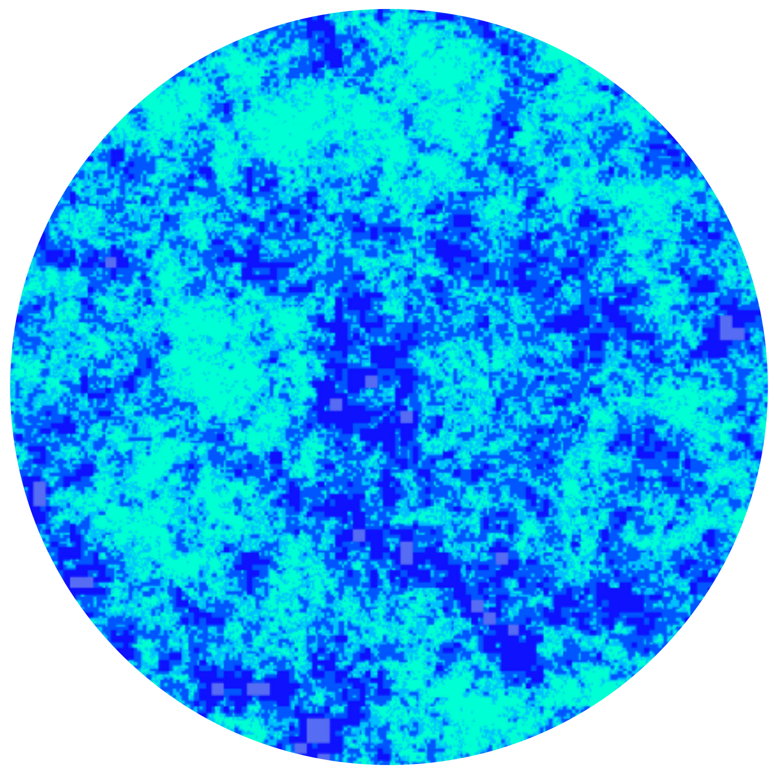

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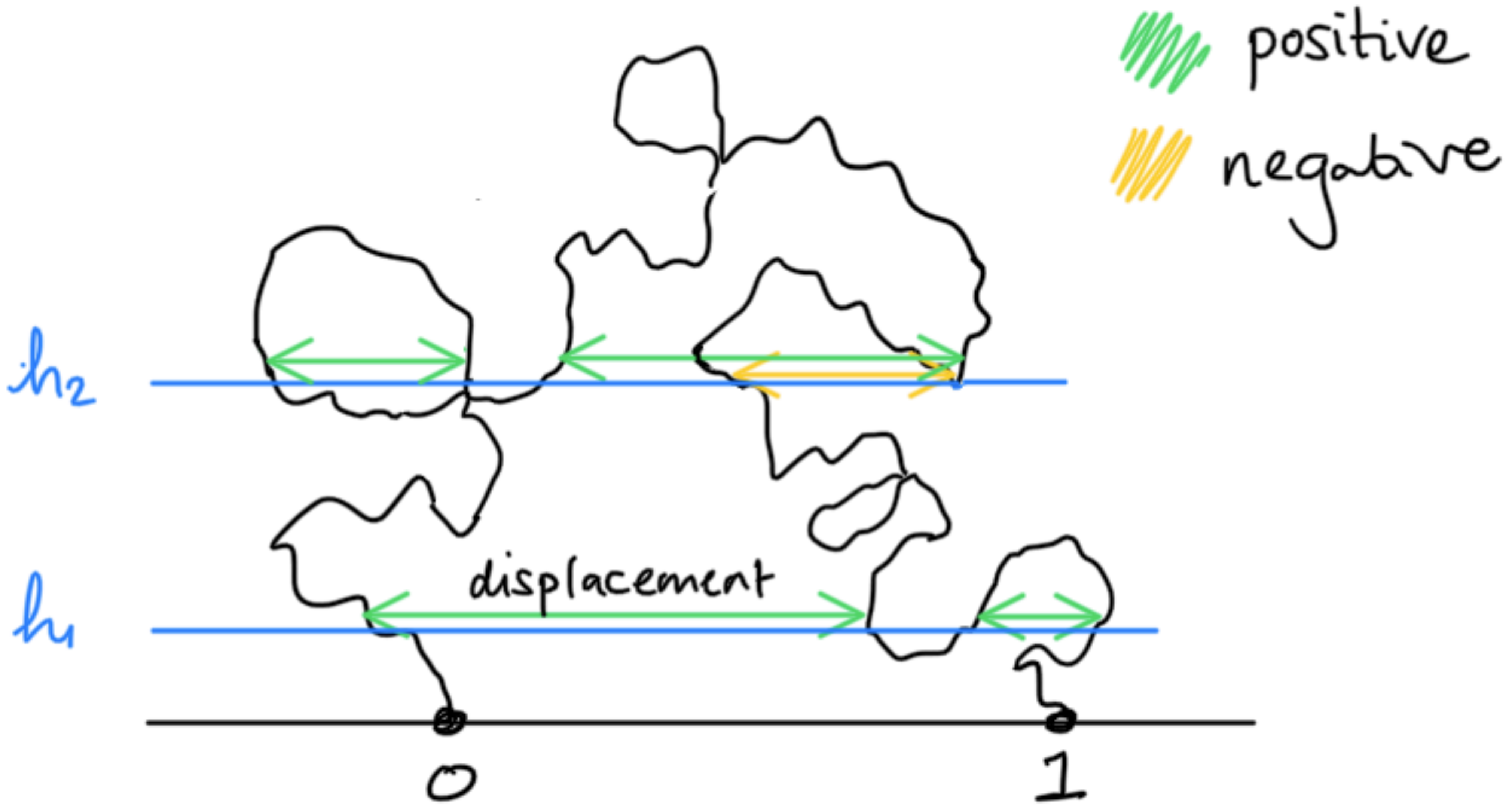
Brownian half-plane excursions

Growth fragmentations and Cauchy processes (Aïdekon & Da Silva)

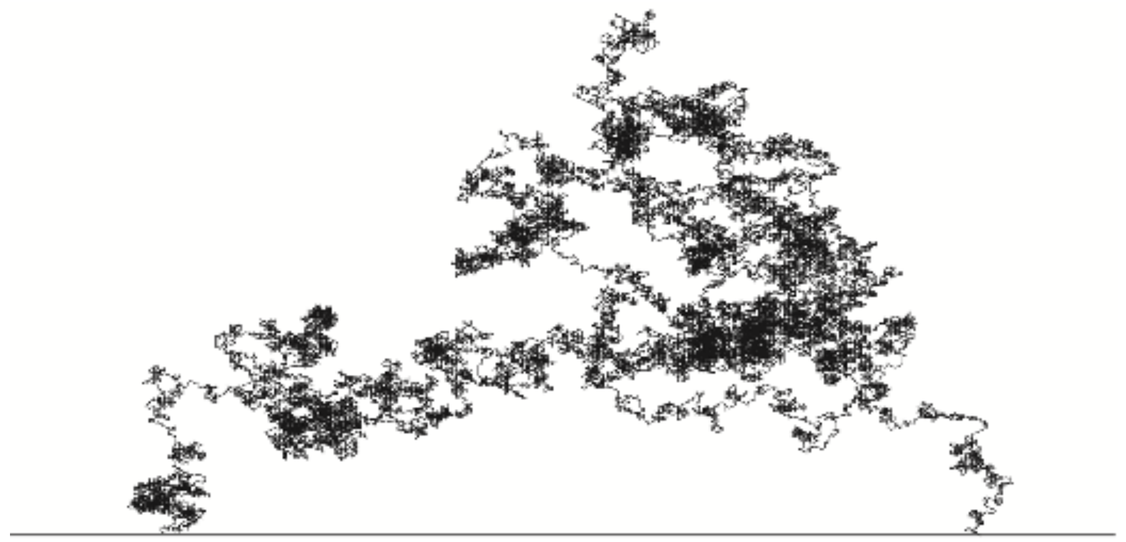


- Start with a half-planar Brownian excursion (given duration, X coordinate is Brownian bridge and Y coordinate is independent Brownian excursion)
- At each height $h \geq 0$ have countable collection of sub-excursions above h
- These have **masses** (widths) with **signs** according to direction traversed by the Brownian half-plane excursion
- Gives a **signed growth fragmentation** with the same law as in our theorem

Our Result

Correspondence:

Brownian half-plane excursion ↔
CLE_4 + “critical LQG”



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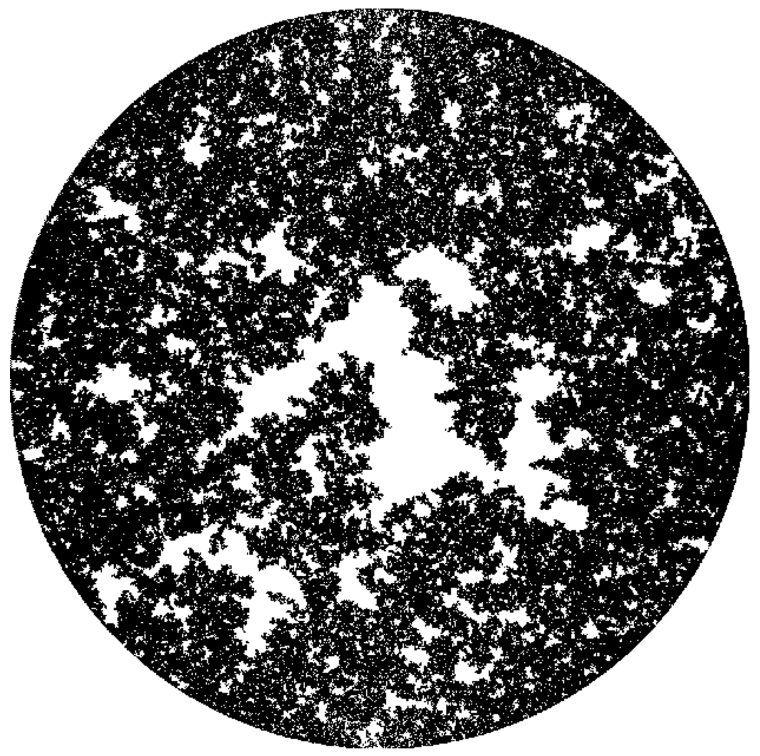

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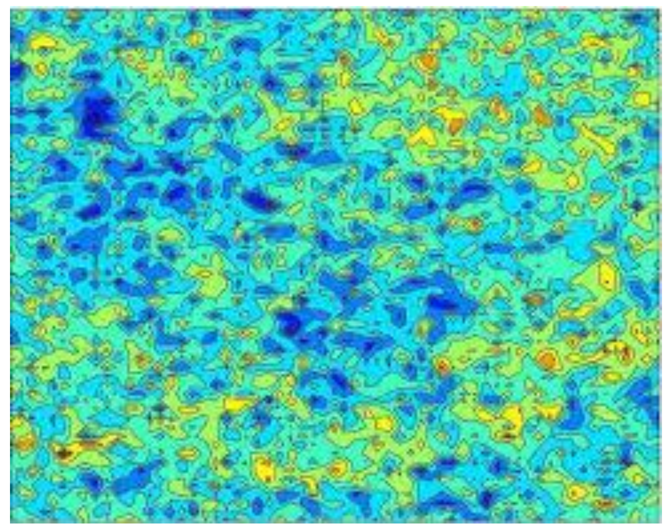
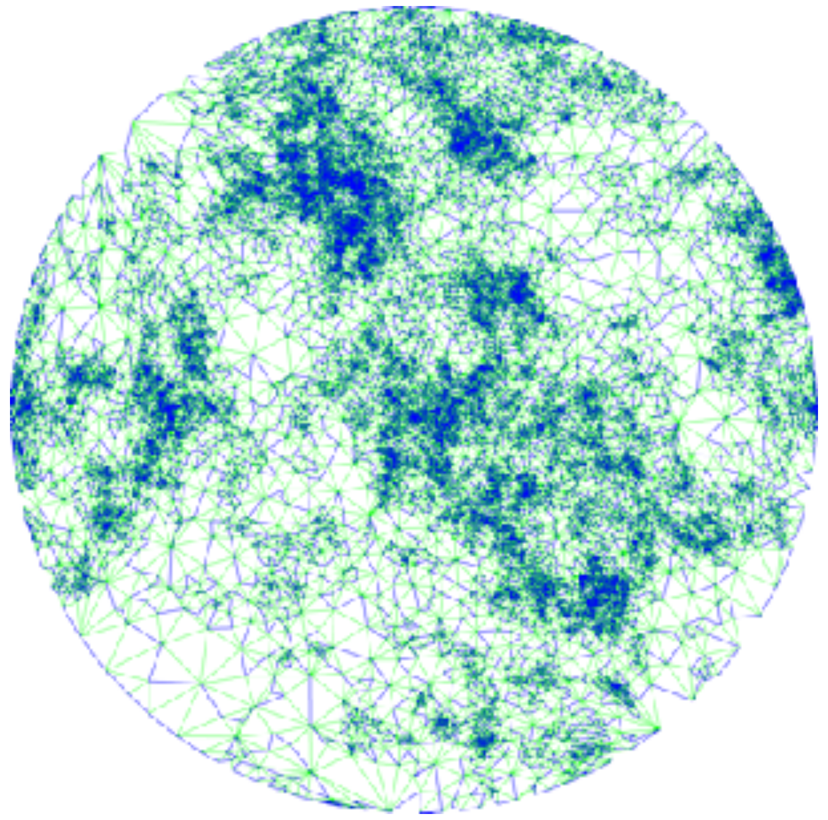
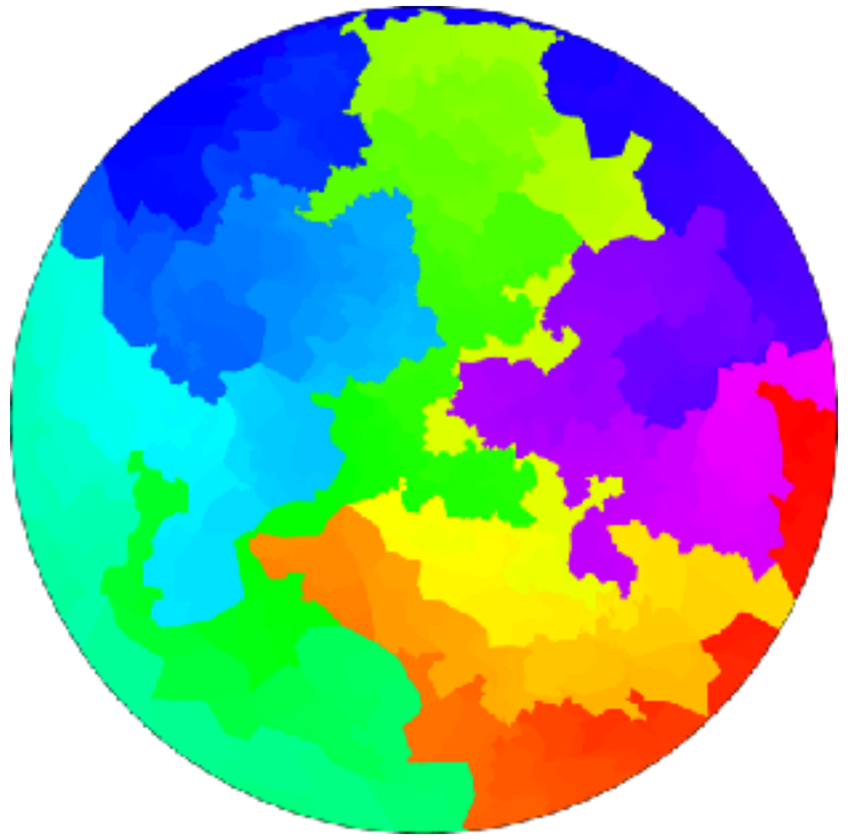
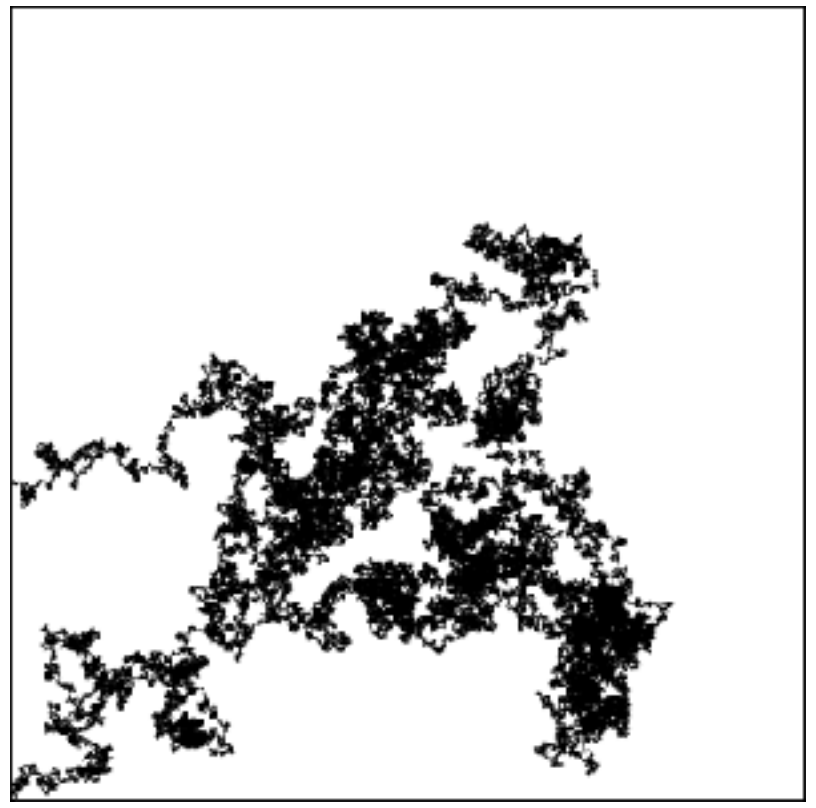


CLE_4 decorated critical quantum disk	Brownian half-plane excursion
Branching structure defined by exploration	Branching structure in the associated CRT
Boundary lengths of discovered disks	Displacements of sub-excursions above heights
Areas of discovered disk	Durations of sub-excursions above heights
Parity of nesting	Sign of subexcursion
Some notion of “quantum” distance from boundary	Height

Questions?

- For $\gamma \neq 2, \kappa \neq 4$, a correspondence between CLE_κ decorated γ-GMC and **Brownian cone excursions** is already known (Duplantier-Miller-Sheffield)
  - Can you extract a growth fragmentation process directly from correlated BM? Work in progress with Alex Watson and William Da Silva
- What can we use this to say about critical LQG?
  - Convergence results?
  - Relationship with critical LQG metrics (Ding-Gwynne) and conformally invariant CLE_4 metric (Sheffield-Watson-Wu)?

Thanks!