Genealogy of the N-particle branching random walk with polynomial tails

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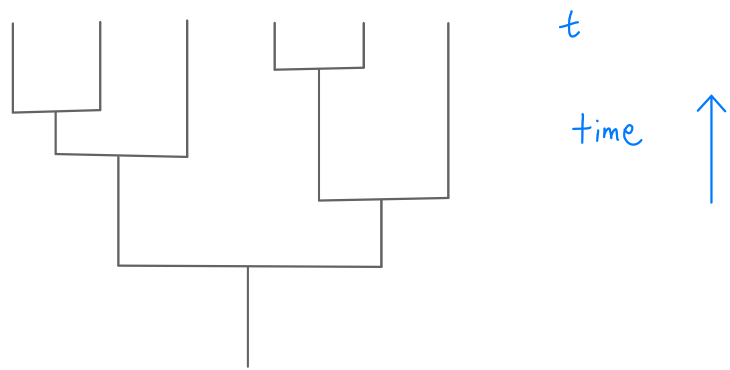
Joint work with Matt Roberts and Zsófia Talyigás

Branching-selection systems

- Particle systems: particles branch (produce offspring) and move in space
  killing rule keeps total number of particles constant.
- Toy models for a population under selection.
  Location of a particle (= individual) represents its evolutionary fitness.
- Introduced by Brunet and Derrida in 1990s.
  Recent results and open conjectures about long-term behaviour.

Genealogy:

Coalescent process



N-particle branching random walk (N-BRW)

Discrete-time branching-selection system.

N particles with locations in $\mathbb{R}$ at each timestep.

Let X be a real-valued random variable (jump distribution).

At each time $n \in \mathbb{N}_0$, each particle has two offspring.

Each of the $2N$ offspring particles makes an independent jump from its parent's location, with the same law as X.

The N rightmost particles (of the $2N$ offspring particles) form the population at time $n+1$.

Notation: $X_1^{(N)}(n) \leq X_2^{(N)}(n) \leq \dots \leq X_N^{(N)}(n)$ ordered particle positions at time n.

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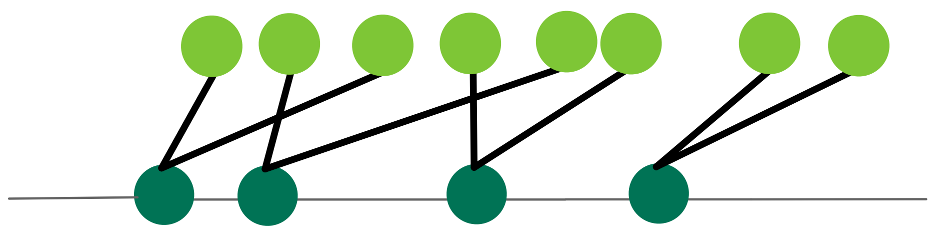

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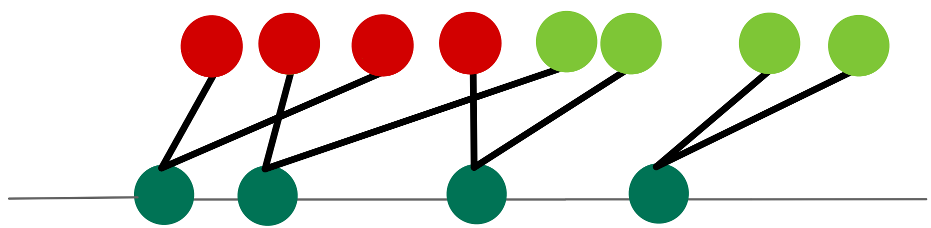

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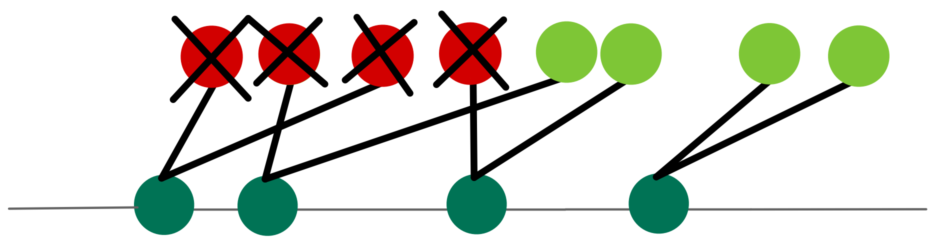

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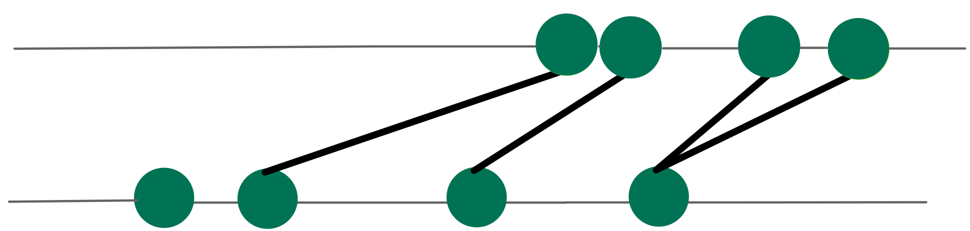

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Light-tailed jump distribution $\quad \mathbb{P}(X > x) \leq e^{-cx}, \ c > 0$

Asymptotic speed

If $\mathbb{E}[X] < \infty$ then $\exists\, v_N \in (0, \infty)$ s.t. $\displaystyle \lim_{n \to \infty} \frac{X_N^{(N)}(n)}{n} = v_N = \lim_{n \to \infty} \frac{X_1^{(N)}(n)}{n}$ a.s. and in L^1.

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<u>Theorem</u> (Bérard and Gouéré 2010) If $\mathbb{E}[e^{\lambda X}] < \infty$ for some $\lambda > 0$ (+ technical assumptions) then $\lim_{N\to\infty} v_N = v_\infty$ exists and $v_\infty - v_N \sim c\,(\log N)^{-2}$ as $N \to \infty$.

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Genealogy

Sample k particles from the N particles and trace their ancestry backwards in time $\longrightarrow$ coalescent process.

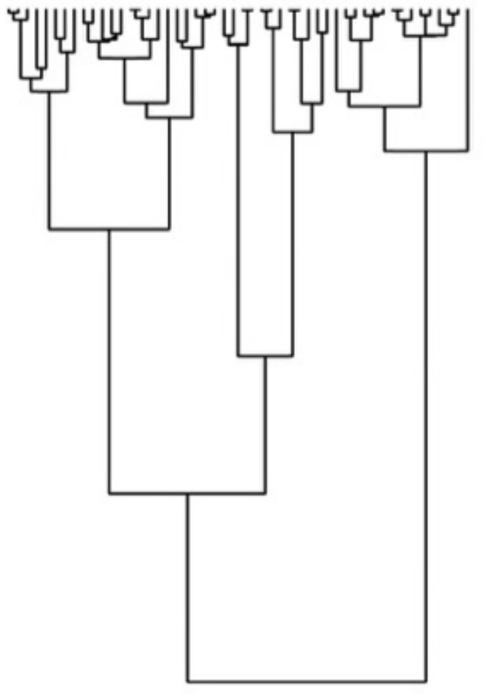
Conjecture (Brunet, Derrida, Mueller, Munier)
If X is light-tailed then the genealogy of a sample on a $(\log N)^3$ timescale converges to a Bolthausen–Sznitman coalescent as $N \to \infty$.

See Berestycki, Berestycki, Schweinsberg.

Coalescent processes

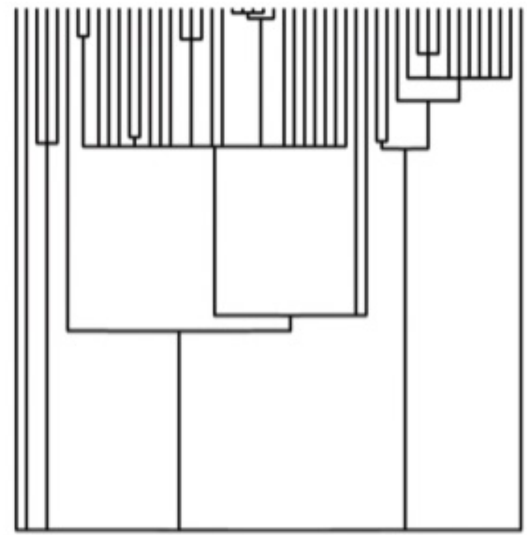
Kingman's coalescent

Neutral population: choose particles to kill uniformly at random in each generation.



Bolthausen-Sznitman coalescent

Population under selection.



Thanks to Götz Kersting and Anton Wakolbinger

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N-BRW with heavy-tailed jump distribution

Suppose $\mathbb{P}(X > x) \sim x^{-\alpha}$ as $x \to \infty$, for some $\alpha > 0$.

Asymptotic speed

<u>Theorem</u> (Bérard and Maillard 2014)

If $\mathbb{E}[X] < \infty$, $\lim_{n\to\infty} \frac{X_N^{(N)}(n)}{n} = v_N$ where $v_N \sim c_\alpha N^{1/\alpha} (\log N)^{1/\alpha - 1}$ as $N \to \infty$.

If $\mathbb{E}[X] = \infty$, cloud of particles accelerates.

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Genealogy

Conjecture (Bérard and Maillard)

The genealogy on a $\log N$ timescale is approximately given by a star-shaped coalescent when N is large.

Time and space scales

Let $\mathbb{P}(X > x) = \dfrac{1}{h(x)}$ for $x \geq 0$.

Assume h is regularly varying with index $\alpha > 0$

i.e. for any $y > 0$, $\dfrac{h(xy)}{h(x)} \longrightarrow y^{\alpha}$ as $x \to \infty$.

and $\mathbb{P}(X \geq 0) = 1$ (no negative jumps).

e.g. $h(x) = x^{\alpha}$ for $x \geq 1$

Let $\ell_N = \lceil \log_2 N \rceil$ time scale

Let $a_N = h^{-1}(2N\ell_N)$, where $h^{-1}(x) := \inf\{y \geq 0 : h(y) > x\}$. space scale

$\mathbb{E}[\#$ jumps of size $> c_1 a_N$ in a time interval of length $c_2 \ell_N]$ $= 2N \cdot c_2 \ell_N \, \mathbb{P}(X > c_1 a_N)$

$$= \frac{2N c_2 \ell_N}{h(c_1 a_N)} \sim \frac{2N c_2 \ell_N}{c_1^{\alpha} 2N \ell_N} = \frac{c_2}{c_1^{\alpha}}$$

as $N \to \infty$.

Main result

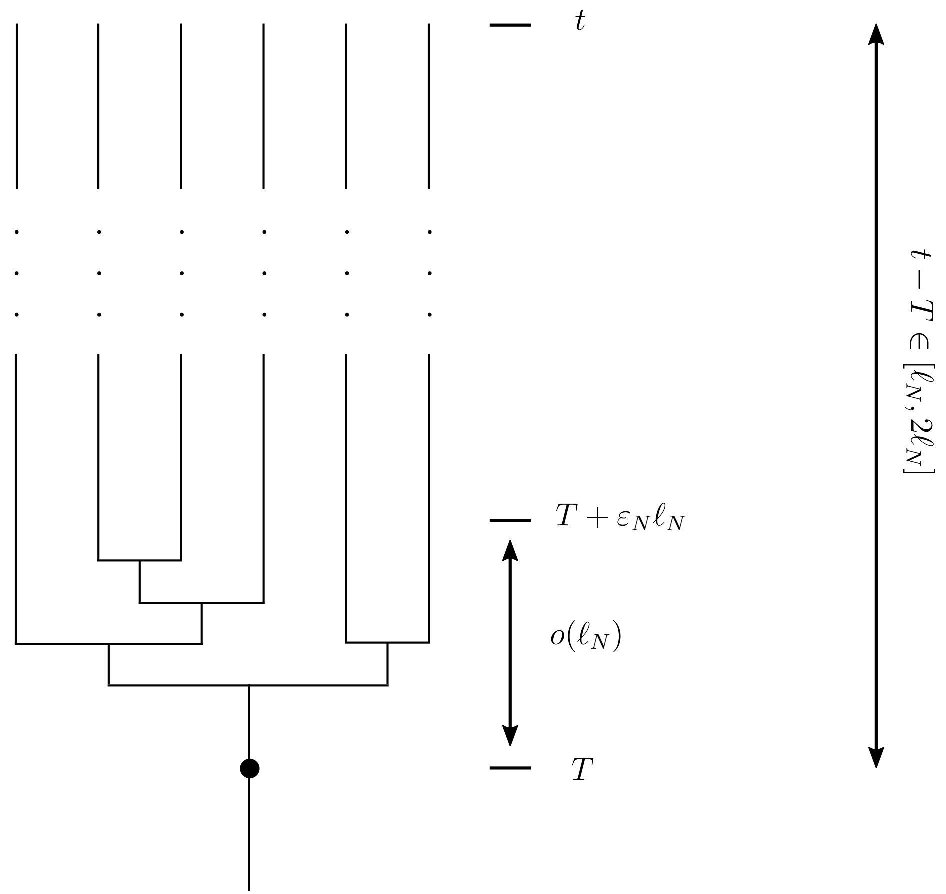
w.h.p. = with probability $\to 1$ as $N \to \infty$.

Theorem (P., Roberts, Talyigás 2021)

For $\eta > 0$, $k \in \mathbb{N}$ and $t > 4\ell_N$, the following occurs w.h.p.:

- **Spatial distribution:** At time t, there are $N - o(N)$ particles in
$$[X_1^{(N)}(t), X_1^{(N)}(t) + \eta a_N].$$
- **Genealogy:** The genealogy on an ℓ_N-timescale is asymptotically given by a star-shaped coalescent.

i.e. $\exists\, T \in [t - 2\ell_N, t - \ell_N]$ s.t. w.h.p., for a uniform sample of k particles at time t, every particle is descended from the rightmost particle at time T and no pair of particles in the sample has a common ancestor after time $T + \varepsilon_N \ell_N$, for any $(\varepsilon_N)_N$ with $\varepsilon_N \to 0$ and $\varepsilon_N \ell_N \to \infty$ as $N \to \infty$.



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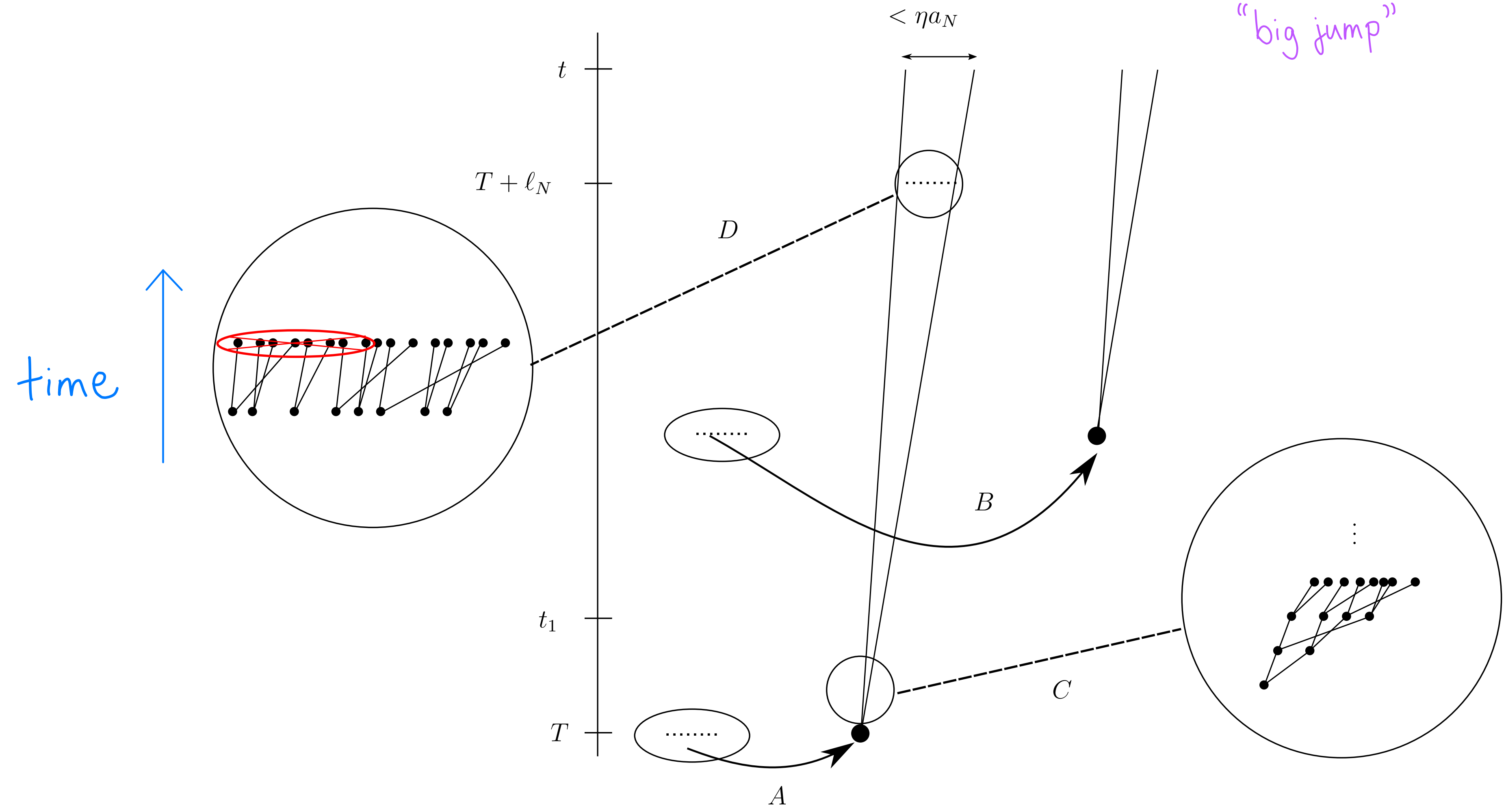
Proof heuristics $\quad t_1 := t - \ell_N.$ $\quad \rho > 0$ small constant.

Let T = last time before time t_1 when a particle makes a jump $\geq \rho a_N$ and takes the lead.

"big jump"



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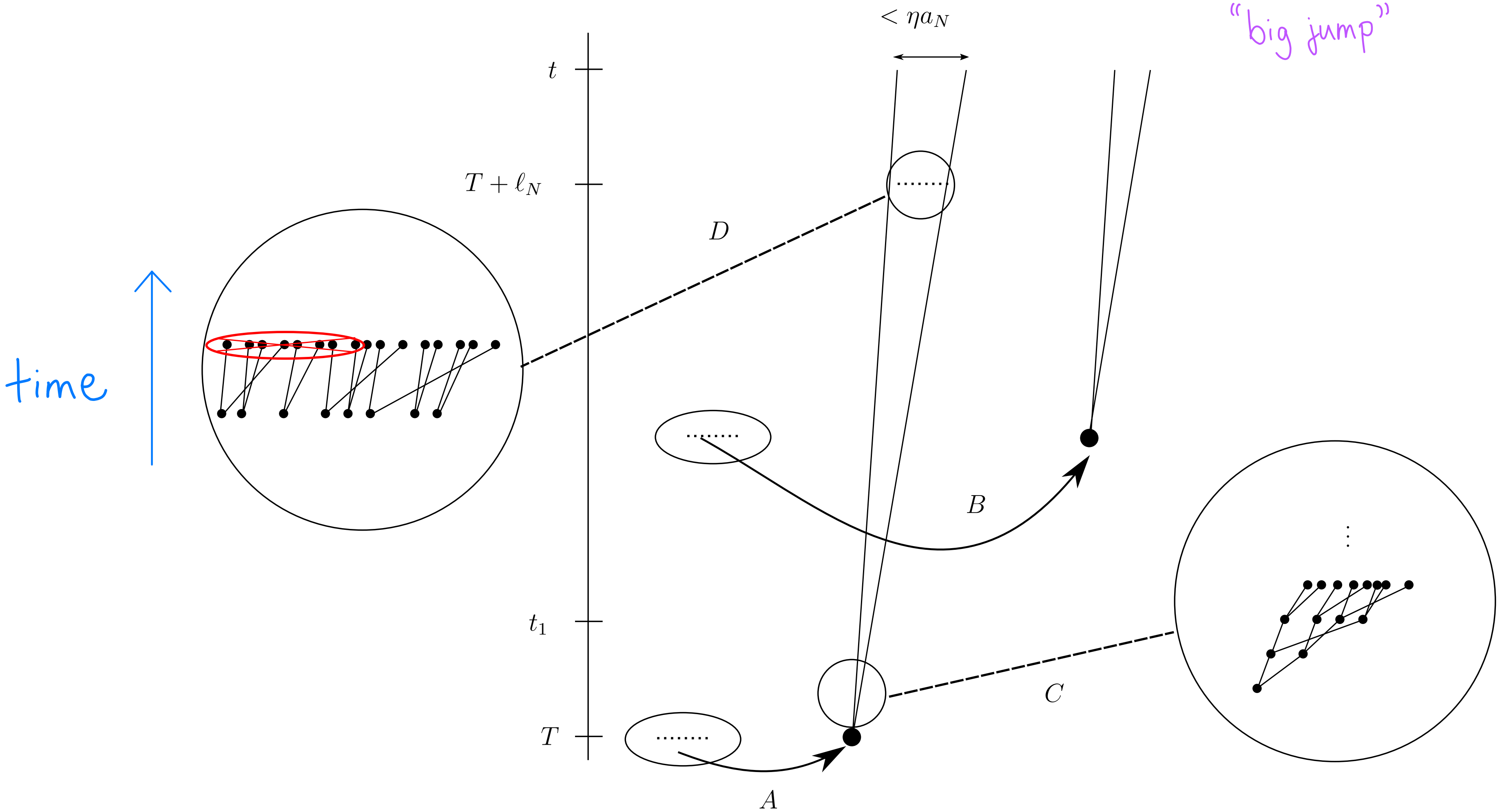

A: A particle makes a big jump at time T and takes the lead (by $\Theta(a_N)$). Its descendants stay in the lead until time t_1 (other particles can't take the lead with a big jump, and can't move far without a big jump).

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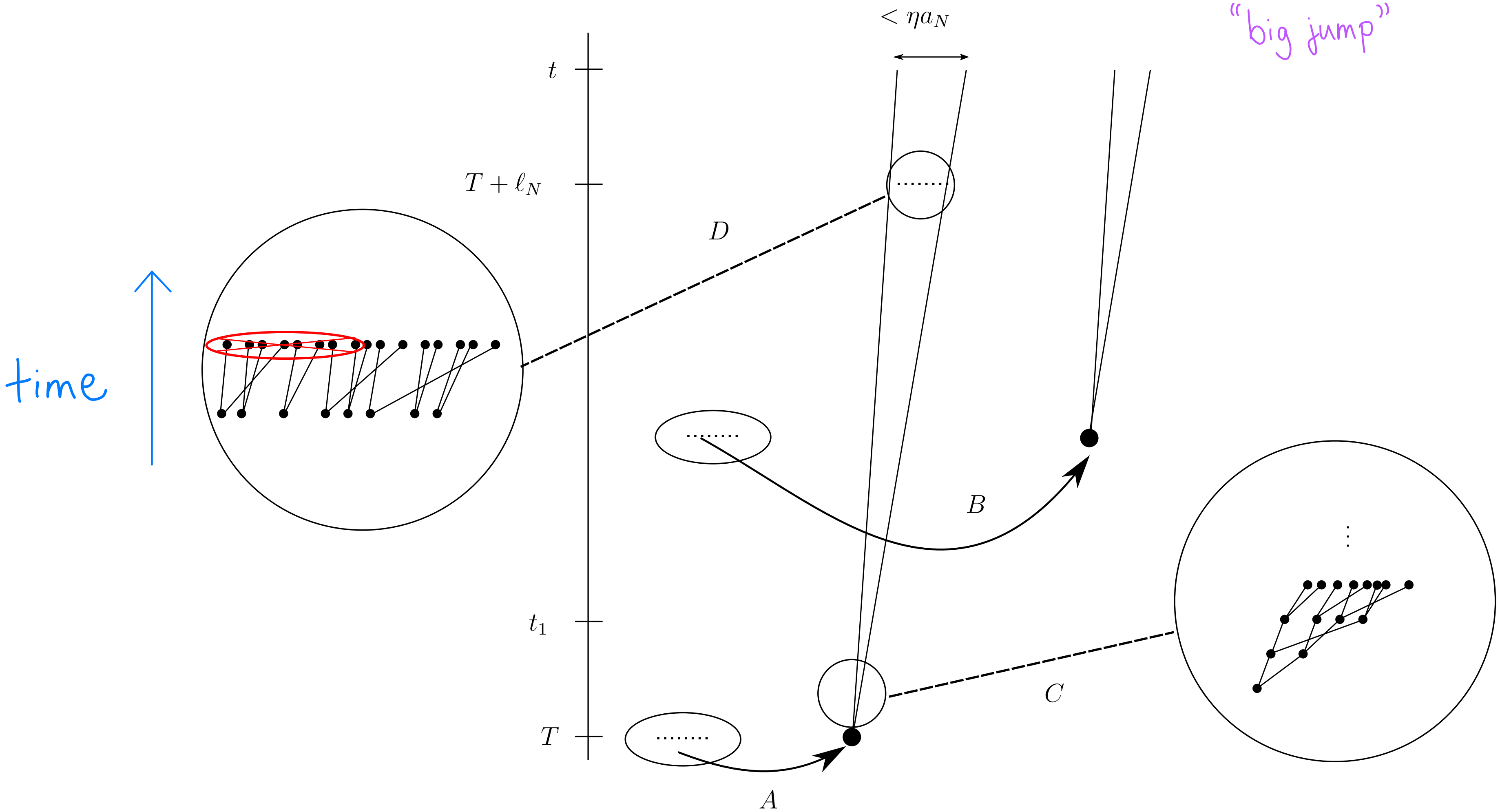

B: There are $O(1)$ big jumps in time interval $[t_1, t]$, each with $o(N)$ descendants at time t.

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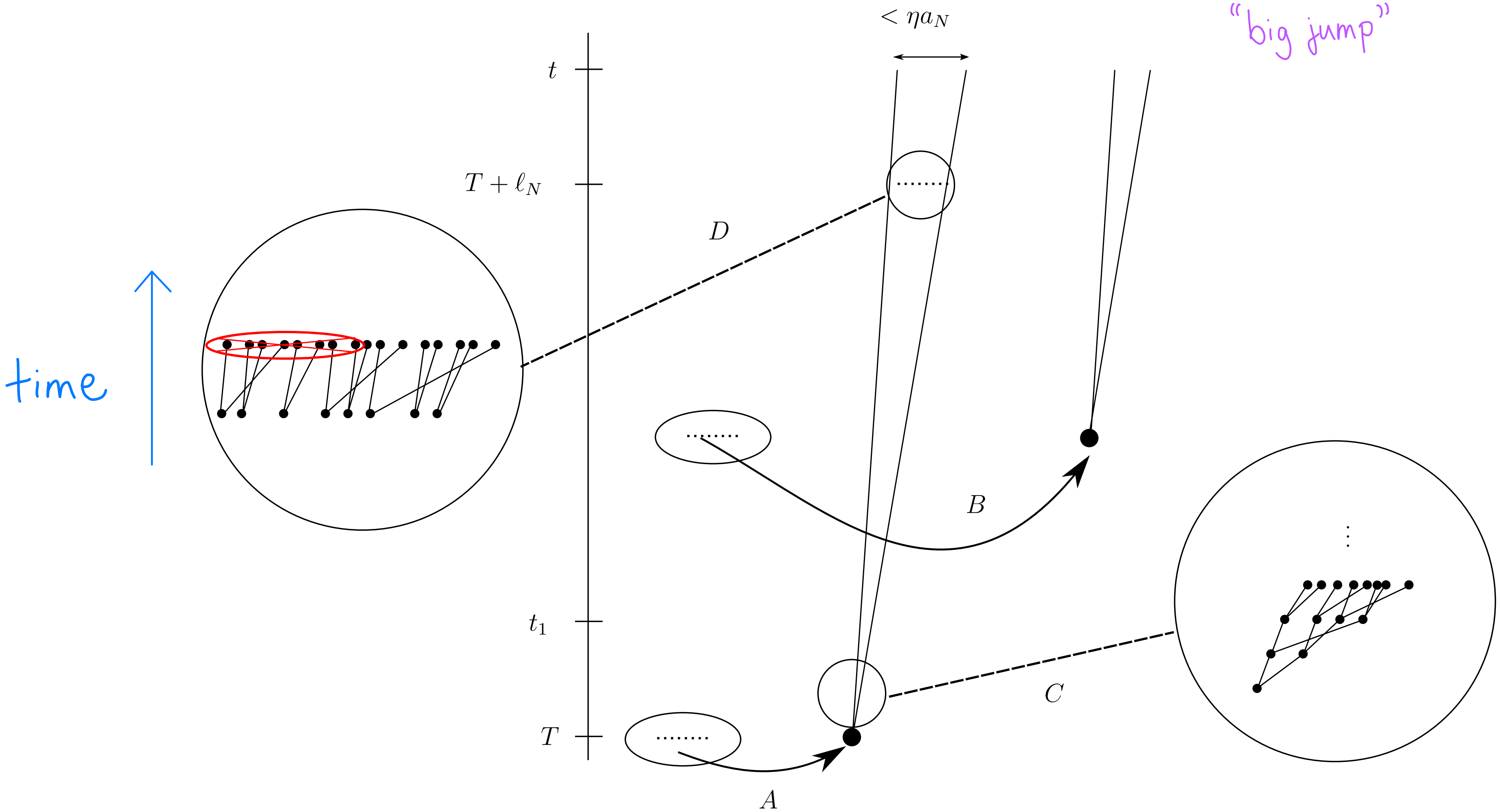

C: The tribe descended from the time-T leader doubles in size at each timestep until almost time $T + \ell_N$.

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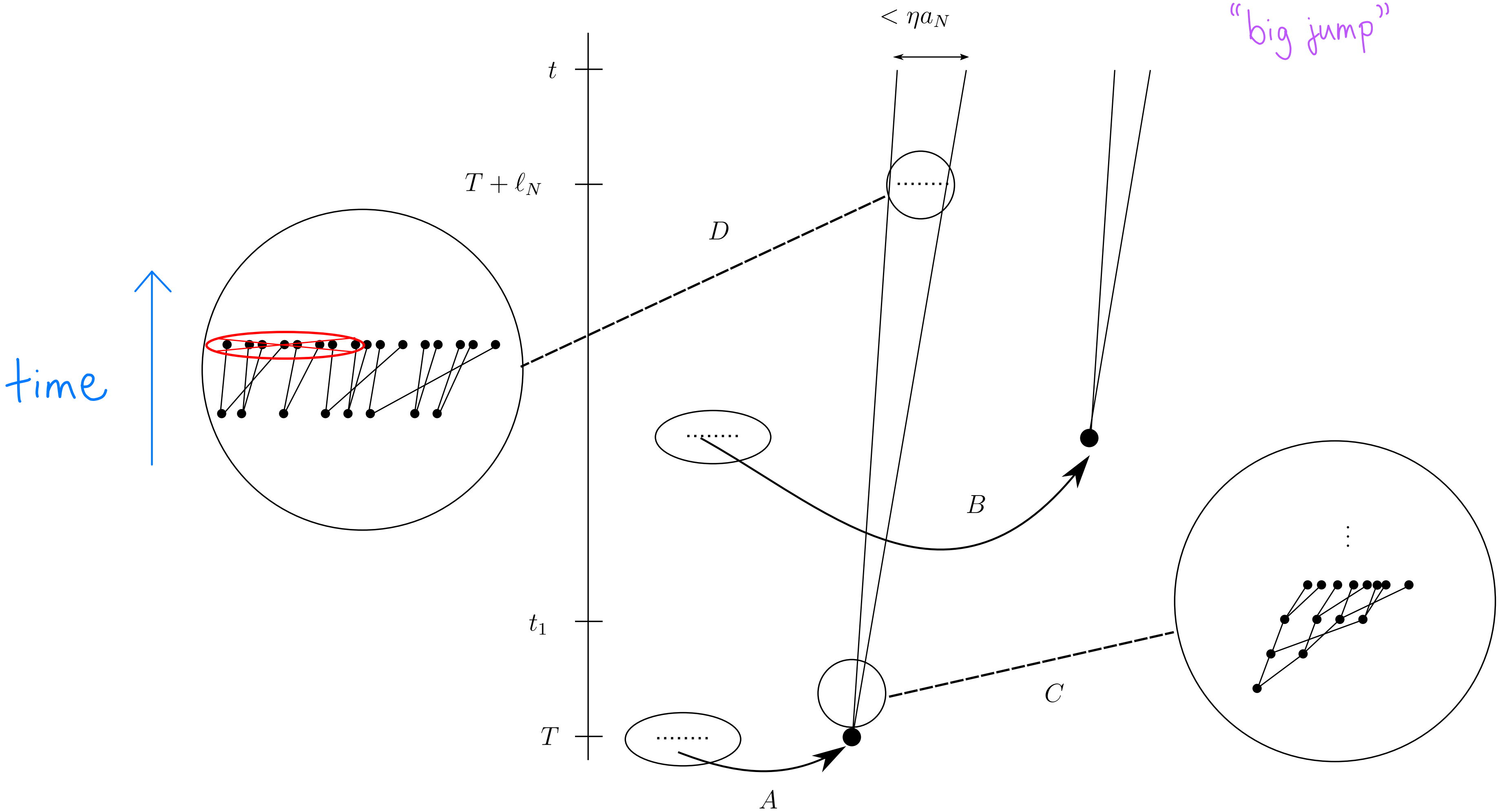

D: On the time interval $[T + \ell_N, t]$, the time-T leader's tribe has size $N - o(N)$.

N-BRW genealogy

Jump distribution X.

		Time to coalesce	Coalescent
Light-tailed	$\mathbb{P}(X > x) \leq e^{-cx}$, $c > 0$	$(\log N)^3$	Bolthausen-Sznitman
Heavy-tailed	$\mathbb{P}(X > x) \sim x^{-\alpha}$, $\alpha > 0$	$\log N$	Star-shaped

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Stretched exponential tail	$\mathbb{P}(X>x) \sim e^{-x^{\beta}}$, $\beta \in (0,1)$	??	??
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Work in progess with Z. Talyigás.

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c.f. Dyszewski, Gantert, Höfelsauer

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Simulation by Z. Talyigás.