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UNIVERSITÉ DU LUXEMBOURG

Fluctuations of Additive Functionals of Fractional Brownian Motion

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Joint works with:

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Fractional Brownian Motion

- The **fractional Brownian motion** with **Hurst index** $H \in (0,1)$ is the unique (in law) centered Gaussian process $B^H = \{B^H_t : t \geq 0\}$ such that

$$\mathbb{E}[B^H_s B^H_t] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$$

($B^{\frac{1}{2}} := W$ = standard Brownian motion).

- **Stationary increments**:

$$\mathbb{E}[(B^H_s - B^H_t)^2] = |t-s|^{2H}.$$

- **Self-similarity**: for all $a > 0$,

$$B^H \overset{\text{LAW}}{=} \left\{ \frac{1}{a^H} B^H_{at} : t \geq 0 \right\}.$$

- *Kolmogorov* (1940), *Mandelbrot & van Ness* (1968).

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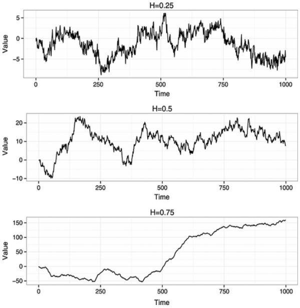

⋆ **Hölder Regularity**: for all $\gamma < H$, B^H is locally γ-Hölder continuous:

$$|B^H_s(\omega) - B^H_t(\omega)| \leq C_{\gamma,T,\omega}\, |s-t|^{\gamma},\ \forall s,t \leq T.$$

⋆ **Correlated fractional noise**: For $H \neq \frac{1}{2}$,

$$\rho_H(n) := \mathbb{E}[B^H_1(B^H_{n+1} - B^H_n)] \sim H(2H-1)n^{2H-2},$$

with:

(i) for $H > \frac{1}{2}$, $\rho_H(n) > 0$ and $\sum_n \rho_H(n) = \infty$ (*long memory*),

(ii) for $H < \frac{1}{2}$, $\rho_H(n) < 0$ (*intermittency*) and $\sum_n |\rho_H(n)| < \infty$.

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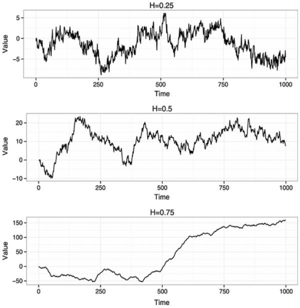

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Fractional Brownian Motion

⋆ **Local non-determinism**: for all $t_0 < \cdots < t_m$ and $a_1, ..., a_m \in \mathbb{R}$,

$$\mathbf{Var}\left(\sum_{i=1}^{m} a_i (B^H_{t_i} - B^H_{t_{i-1}})\right) \geq k_H \sum_{i=1}^{m} a_i^2 (t_i - t_{i-1})^{2H}.$$

⋆ For all $H \in (0,1)$ there exists a jointly continuous version of the **local time**

$$(t,x) \mapsto L^H_t(x) := \int_0^t \delta_0(B^H_s - x) ds,$$

$$\text{so that: } \int_0^t f(B^H_s) ds = \int_{\mathbb{R}} f(x) L^H_t(x) dx.$$

⋆ fBm as a « **canonical object** »: *Taqqu* (1975), *Sottinen* (2001), *Enriquez* (2004), *Hammond & Sheffield* (2011).

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Vignette: Breuer-Major CLTs

Theorem (Breuer & Major, 1983; Nourdin & Nualart, 2019)
Let $\{X_i : i \geq 0\}$ *be a unit variance, centered stationary Gaussian sequence, and write* $\rho_X(i) = \mathbb{E}(X_0 X_i)$ *(e.g.,* $X_i = B^H_{i+1} - B^H_i$*). Let* $f : \mathbb{R} \to \mathbb{R}$ *be such that* $\mathbb{E}[f(X_0)] = 0$, $\mathbb{E}[|f(X_0)|^p] < \infty$, $p > 2$, *and* f *has* **Hermite rank** $R \geq 1$. *Then, if*

$$\sum_i |\rho_X(i)|^R < \infty,$$

one has that

$$t \mapsto V_n(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} f(X_i), \quad t \in [0,1],$$

converges weakly in $D[0,1]$ *to a multiple of a standard Brownian motion.*

Papanicolaou, Stroock and Varadhan (1977)

⋆ Consider an integrable $f : \mathbb{R} \to \mathbb{R}$. One has that, a.s.-$\mathbb{P}$,

$$\sqrt{n} \int_0^t f(\sqrt{n}\, W_s) ds$$
$$= \sqrt{n} \int f(\sqrt{n} x) L_t(x) dx \longrightarrow L_t(0) \int f(x) dx, \quad n \to \infty.$$

⋆ **[Papanicolaou, Stroock and Varadhan, 1977]** Under some additional integrability assumptions, if $\int f(x)dx = 0$, then

$$n^{3/4} \int_0^t f(\sqrt{n} W_s) ds \left(\overset{\text{LAW}}{=} n^{-1/4} \int_0^{nt} f(W_u) du \right) \overset{\text{LAW}}{\Longrightarrow} c\beta_{L_t(0)},$$

where β is a Brownian motion independent of W, $c < \infty$ depends on f, and the convergence is functional and stable. (*Proof*: martingale methods)

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Case of Fractional Brownian Motion

⋆ For all $\lambda \in \mathbb{R}$, one has that, a.s.-$\mathbb{P}$,

$$n^H \int_0^t f(n^H(B_s^H - \lambda))ds \longrightarrow L_t^H(\lambda) \int f(x)dx, \quad n \to \infty,$$

⋆ **[Nualart & Xu, 2014]** If $\frac{1}{3} < H < 1$, and f is integrable against $|x|^{1/H-1}dx$ and such that $\int f(x)dx = 0$, then

$$n^{\frac{H+1}{2}} \int_0^t f(n^H B_s^H)ds \left(\stackrel{\mathbf{LAW}}{=} n^{\frac{H-1}{2}} \int_0^{nt} f(B_u^H)du \right) \stackrel{\mathbf{LAW}}{\Longrightarrow} c\, \beta_{L_t^H(0)},$$

where β is a Brownian motion independent of B, $c < \infty$ depends on f and H, and the convergence is functional and stable. (*Proof*: method of moments (!)).

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JEGANATHAN, 2004–2008

- ⋆ **[Jeganathan, 2004]** Assume $f \in L^1 \cap L^2$. Then, for every $H \in (0,1)$ and every $\lambda \in \mathbb{R}$, as $n \to \infty$,

$$n^{H-1} \sum_{i=1}^{\lfloor nt \rfloor} f(n^H(B^H_{\frac{i-1}{n}} - \lambda)) \xrightarrow{L^2(\mathbb{P})} L^H_t(\lambda) \int f(x)dx.$$

- ⋆ **[Jeganathan, 2008]** Assume $\int |f(x)|^p + |xf(x)| < \infty$, for $p = 1,2,3,4$ and $\int f(x)dx = 0$. Then, for every $\frac{1}{3} < H < 1$, as $n \to \infty$,

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where β is a Brownian motion independent of B^H, $c < \infty$ depends on f and H, and the convergence is in the sense of finite-dimensional distributions and stable.

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where β is a Brownian motion independent of B^H, $c < \infty$ depends on f and H, and the convergence is in the sense of finite-dimensional distributions and stable.

QUESTIONS

1. Can one remove the assumption that $\int f(x)dx = 0$?
2. What happens in the « **critical case** » $H = \frac{1}{3}$?
3. What happens in the « **rough range** » $0 < H < \frac{1}{3}$?

MAIN RESULTS: $H > \frac{1}{3}$

Theorem (Jaramillo, Nourdin, Nualart & Peccati, 2023)
Suppose $\frac{1}{3} < H < 1$ *and fix* $\lambda \in \mathbb{R}$. *Then,*

$$A_n(t) := n^{\frac{1-H}{2}} \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t^H(\lambda) \int f(x)dx \right), \; t \geq 0.$$

converges towards

$$C(f,H) \times \beta_{L_t^H(\lambda)}, \quad t \geq 0,$$

in the sense of **finite-dimensional distributions**, *where* β *is a* **standard Brownian motion** *independent of* B^H, *the convergence is stable and* $C(f,H)$ *is an absolute constant.*

MAIN RESULTS: $H = \frac{1}{3}$

Theorem (Jaramillo, Nourdin, Nualart & Peccati, 2023)
Suppose $H = \frac{1}{3}$ and fix $\lambda \in \mathbb{R}$. Then,

$$B_n(t)$$
$$:= \frac{n^{\frac{1-H}{2}}}{\sqrt{\log n}} \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t^H(\lambda) \int f(x)dx \right)$$
$$= \frac{n^{1/3}}{\sqrt{\log n}} \left(n^{1/3} \int_0^t f(n^{1/3}(B_s^{1/3} - \lambda))ds - L_t^{1/3}(\lambda) \int f(x)dx \right)$$

converges stably towards

$$C(f) \times \beta_{L_t^{1/3}(\lambda)}, \quad t \geq 0,$$

in the sense of **finite-dimensional distributions**.

IDEA OF PROOF ($H > 1/3$)

⋆ Start by representing B^H as a **Volterra process**:

$$B_t^H = \int_0^t K_H(t,s)dW_s,$$

where W is a standard Brownian motion (see e.g. *Mandelbrot & van Ness*, 1968).

⋆ Represent local time by using the **Clark-Ocone formula**:

$$L_t^H(\lambda) = \mathbb{E}[L_t^H(\lambda)] + \int_0^t \mathbb{E}[D_r L_t^H(\lambda) \,|\, W_u : u \leq r] dW_r,$$

where D is the **Malliavin gradient**.

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Idea of Proof ($H > 1/3$)

⋆ Write

$$A_n(t) = n^{\frac{1-H}{2}} \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t^H(\lambda) \int f(x)dx \right)$$
$$= n^{\frac{1-H}{2}} \left(n^H \int L_t(x) f(n^H(x-\lambda))dx - L_t(\lambda) \int f(y)dy \right)$$
$$= n^{\frac{1-H}{2}} \left(\int_{\mathbb{R}} f(y)[L_t(\lambda + \frac{y}{n^H}) - L_t(\lambda)]dy \right).$$

⋆ Use Clark-Ocone to write

$$A_n(t) = \int_0^t G(t,s,n)dW_s + R(t,n),$$

where $R(t,n)$ is negligible and $M_u^{(t,n)} := \int_0^u G(t,s,n)dW_s$, $u \le t$ is a **Brownian martingale**.

IDEA OF PROOF

- Prove that

$$\langle M^{(t,n)}, M^{(t,n)} \rangle_u \xrightarrow{\mathbb{P}} \sqrt{C(f,H)} L^H_{t\wedge u}(\lambda),$$

and that

$$\langle M^{(t,n)}, W \rangle_u \xrightarrow{\mathbb{P}} 0 \quad \text{(uniformly)}.$$

- Conclude by using a version of the **asymptotic Knight's Theorem**.

Proposition (Jaramillo, Nourdin & Peccati (2021))

Fix $0 < H < \frac{1}{3}$, and denote by ϕ_ε the centered Gaussian density with variance ε. For every $t \geq 0$ and $\lambda \in \mathbb{R}$, as $\varepsilon \to 0$ the random variables

$$L_{t,\varepsilon}^{(1,H)}(\lambda) := \int_0^t \frac{d}{d\lambda}\phi_\varepsilon(B_s^H - \lambda)ds,$$

converge in $L^2(\mathbb{P})$ to a limit $L_t^{(1,H)}(\lambda)$, as $\varepsilon \to 0$. The limit random variable $L_t^{(1,H)}(\lambda)$ can be written in Fourier form as

$$L_t^{(1,H)}(\lambda) = -\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^t (\boldsymbol{i}\xi)e^{\boldsymbol{i}\xi(B_s^H-\lambda)}dsd\xi,$$

where the integral converges in $L^2(\Omega)$. The constraint on H is sharp.

Proposition (Jaramillo, Nourdin & Peccati (2021))

For $0 < H < \frac{1}{3}$,

$$L_t^{(1,H)}(\lambda) = \lim_{h \to 0} \frac{1}{h}(L_t^H(\lambda + h) - L_t^H(\lambda)),$$

where the limit is in $L^2(\mathbb{P})$. *In addition, for fixed* λ, *the process*

$$\{L_t^{(1,H)}(\lambda) : t \geq 0\}$$

has a modification with γ*-Hölder continuous paths (in the variable* t*) for all* $\gamma < 1 - 2H$.

More Thresholds

- ⋆ In **Jaramillo, Nourdin and Peccati (2021)**: for general $\ell = 1, 2, \ldots$, if

$$0 < H < \frac{1}{2\ell + 1},$$

  then the ℓth spatial derivative $\{L_t^{(\ell,H)}(\lambda) : t \geq 0\}$ exists, with similar regularity properties.
- ⋆ The existence of the ℓth spatial derivative of the local time of B is proved in **Geman & Horowitz (1981)**, as an application of results by **Berman (1971)**.
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- ⋆ Sharpness of the restriction $0 < H < \frac{1}{3}$ and time regularity are new.

Main Results: $H < \frac{1}{3}$

Theorem (Jaramillo, Nourdin & Peccati, 2021; Jaramillo, Nourdin, Nualart & Peccati, 2023)

For f s.t. $\int |f(y)|(1+|y|^{\nu})dy < \infty$ and $\lambda \in \mathbb{R}$, one has that both

$$C_n(t) := n^H \left(n^{H-1} \sum_{i=1}^{\lfloor nt \rfloor} f(n^H(B^H_{\frac{i-1}{n}} - \lambda)) - L^H_t(\lambda) \int f(x)dx \right),$$

and

$$D_n(t) := n^H \left(n^H \int_0^t f(n^H(B^H_s - \lambda))ds - L^H_t(\lambda) \int f(x)dx \right)$$

converge in $L^2(\mathbb{P})$ to

$$L^{(1,H)}_t(\lambda) \times \int y f(y)dy, \quad \text{as } n \to \infty.$$

Idea of the Proof

- The discrete-time setting is the most difficult to deal with.
- Substantial technical contribution: showing that

$$\mathbb{E}[(C_n(t) - D_n(t))^2] \to 0$$

(with some uniformity). Techniques: *Fourier Analysis*, *Malliavin Calculus*.

- Once this is done, we exploit the representation

$$D_n(t) = n^H \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t^H(\lambda) \int f(y)dy \right)$$

$$= n^H \left(n^H \int L_t(x) f(n^H(x - \lambda))dx - L_t^H(\lambda) \int f(y)dy \right)$$

$$= n^H \left(\int_{\mathbb{R}} f(y)[L_t^H(\lambda + \frac{y}{n^H}) - L_t^H(\lambda)]dy \right).$$

NOTATION

- ⋆ To fix ideas, from now on we fix $f : \mathbb{R} \to \mathbb{R}$ continuous, with compact support, and such that $\int f(x)dx = 0$.

- ⋆ Then f admits a unique antiderivative F such that $F \in L^1$. We write

$$\mu[F] := \int F(x)dx = -\int xf(x)dx.$$

Quantitative Versions: First Order

Theorem (Jaramillo, Nourdin & Peccati, 2021)
For every $0 < H < 1/3$,

$$\mathbb{E}\left[\left(n^{2H-1}\sum_{i=1}^{\lfloor nt \rfloor} f(n^H(B_{\frac{i-1}{n}} - \lambda)) + L_t^{(1,H)}(\lambda)\mu[F]\right)^2\right] \leq Cn^{-2H\kappa},$$

for every $\kappa < \frac{1}{2}(\frac{1}{H} - 3) \wedge \frac{1}{2}$, *where* C *depends on* t.

If $0 < H < 1/4$, *then the convergence is uniform on compact intervals.*

Quantitative Versions: Second Order

Theorem (Jaramillo, Nourdin & Peccati, 2021)

Fix $0 < H < 1/5$ *and assume that* $\widetilde{F} \in L^1(\mathbb{R})$, *where* $\widetilde{F}(x) := xF(x)$.
Then, for every $t > 0$ *and* $\lambda \in \mathbb{R}$,

$$\mathbb{E}\Big[n^H\Big(\Big(n^{2H-1}\sum_{i=1}^{\lfloor nt \rfloor} f(n^H(B_{\frac{i-1}{n}} - \lambda)) + L_t^{(1,H)}(\lambda)\mu[F]\Big) - L_t^{(2,H)}(\lambda)\mu[\widetilde{F}]\Big)^2\Big] \leq C(n^{-2H\kappa}),$$

for every $\kappa < \frac{1}{2}(\frac{1}{H} - 5) \wedge \frac{1}{2}$.

For $0 < H < 1/6$, *the above convergence is uniform on compact intervals.*

The General Picture

Our results deal with arbitrary derivatives. For instance, if $0 < H < \frac{1}{2\ell+1}$ and g is smooth and such that $\int g^{(i)}(x)dx = 0$, for all $i = 1, ..., \ell$,

$$\mathbb{E}\left[\left(n^{H(\ell+1)-1}\sum_{i=1}^{\lfloor nt\rfloor} g^{(\ell)} n^{H}(B_{\frac{i-1}{n}} - \lambda)) + (-1)^{1+\ell} L_t^{(\ell,H)}(\lambda)\mu[g]\right)^2\right] \leq Cn^{-2H\kappa\wedge\kappa},$$

for every $\kappa \in (0, \frac{1}{2})$ such that $H(1 + 2\ell + 2\kappa) < 1$.

Starting Point

- Use Fourier inversion to write

$$n^{a(\ell+1)-1}\sum_{i=2}^{\lfloor nt\rfloor} g^{(\ell)}(n^a(B_{\frac{i-1}{n}}-\lambda))$$

$$=\frac{1}{2\pi n}\int_{\mathbb{R}^2}\sum_{i=2}^{\lfloor nt\rfloor}(\mathbf{i}\xi)^{\ell}e^{\mathbf{i}\xi(B_{\frac{i-1}{n}}-\lambda-\frac{y}{n^a})}g(y)dyd\xi$$

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- Exploiting the Fourier representation of $L_t^{(\ell)}(\lambda)$,

$$n^{a(\ell+1)-1}\sum_{i=2}^{\lfloor nt\rfloor} g^{(\ell)}(n^a(B_{\frac{i-1}{n}}-\lambda))-\mu[g]L_t^{(\ell)}(\lambda)$$

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Opening

- ⋆ Critical case $H = \frac{1}{3}$ in the **discrete-time setting**.
- ⋆ **Tightness**.
- ⋆ "**Third order results**", e.g. in the range $\frac{1}{5} < H < \frac{1}{3}$.
- ⋆ Inclusion of **fractional noise** (Podolskij & Rosenbaum, 2017)
- ⋆ Arbitrary **dimension**.
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THANK YOU FOR YOUR ATTENTION!