Thick points of the planar GFF are totally disconnected for all $\gamma \neq 0$

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Joint work with Juhan Aru and Ellen Powell

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Definition of the Dirichlet GFF

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The Gaussian free field in D with Dirichlet boundary conditions is a centered Gaussian process h indexed by smooth functions with compact support in D and whose covariance is given by, for f and g two such functions,

$$\mathbb{E}[(h, f)(h, g)] = \int_{D\times D} f(x)G_D(x, y)g(y)dxdy$$

where G_D is the Green function of (minus) the Laplacian in D with Dirichlet boundary conditions normalized such that $G_D(x, y) \sim \frac{-1}{2\pi} \log(|x - y|)$ when $x \to y$.

Thick points of the GFF

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Definition

Let $\gamma \in \mathbb{R}$. The set of γ-thick points of h is

$$\mathcal{T}_\gamma(h) := \Big\{ z \in D : \lim_{r \to 0} \frac{\sqrt{2\pi}(h,\rho_r^z)}{\log 1/r} = \gamma \Big\}.$$

Known results about the geometry of the set of thick points

Theorem (Hu, Miller, Peres, 2010)

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- *for* $\gamma \in [-2, 2]$, $\mathcal{T}_\gamma(h)$ *has almost sure Hausdorff dimension* $2 - \gamma^2/2$*;*
- $\mathcal{T}_\gamma(h)$ *is conformally invariant, that is if* $\varphi : D \to \tilde{D}$ *is a conformal map, then almost surely for any* $\gamma \in [-2, 2]$, $\varphi(\mathcal{T}_\gamma(h)) = \mathcal{T}_\gamma(\tilde{h})$ *where* $\tilde{h}$ *is a GFF with Dirichlet boundary conditions in* $\tilde{D}$*.*

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Theorem (Aru, P., Powell, 2022)

Let h be a GFF with Dirichlet boundary conditions in $\mathbb{D}$. Then almost surely for any $\gamma \in (0, 2]$, $\mathcal{T}_\gamma(h)$ is totally disconnected.

Key ideas of the proof

1. Use an alternative, but natural, definition of the thick points of h via the level-line coupling between h and a nested CLE_4 Γ.

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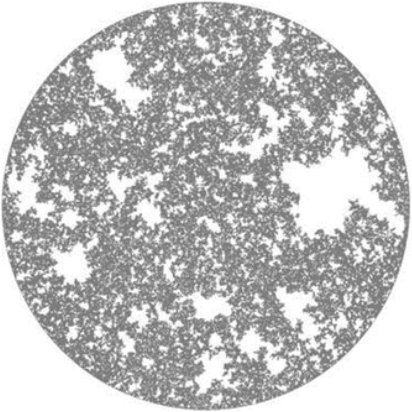

Figure 1: Simulation of non-nested CLE_4 by David Wilson

Key ideas of the proof

1. Use an alternative, but natural, definition of the thick points of h via the level-line coupling between h and a nested CLE_4 Γ. Conditionally on $\Gamma^{(n)}$,

$$h = \sum_{\ell \in \Gamma^{(n)}} h_\ell + H, \quad \text{where} \quad H_{|\mathrm{Int}(\ell)} = \sum_{j=1}^{n} \xi_{\ell(j)},$$

$$\text{with} \quad \mathrm{Int}(\ell) \subset \mathrm{Int}(\ell(n-1)) \subset \cdots \subset \mathrm{Int}(\ell(1)) \quad \text{and}$$

$$\mathbb{P}(\xi_j = -2\lambda) = \mathbb{P}(\xi_j = 2\lambda) = \frac{1}{2}, \quad (\xi_j)_j \text{ independent}, \quad \lambda = \sqrt{\pi/8}.$$

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For $z \in \mathbb{D}$ and $r > 0$, denote by $I(z,r)$ the generation of the first loop intersecting $B(z,r)$. For $\gamma \in \mathbb{R}$, we define

$$\Phi_\gamma(h) := \{z \in \mathbb{D} : \lim_{r \to 0} \frac{H_{I(z,r)-1}(z)}{-\log r} = \frac{\gamma}{\sqrt{2\pi}}\}.$$

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Theorem (Aru, P., Powell, 2022)

Let h be a GFF in $\mathbb{D}$ with Dirichlet boundary conditions. Then, with probability one, $\mathcal{T}_\gamma(h) = \Phi_\gamma(h)$ for every $\gamma \in [-2, 2] \setminus \{0\}$.

Key ideas of the proof

Recalling the definition of $\Phi_\gamma(h)$

$$\Phi_\gamma(h) = \{z \in \mathbb{D} : \lim_{r \to 0} \frac{H_{I(z,r)-1}(z)}{-\log r} = \frac{\gamma}{\sqrt{2\pi}}\},$$

we can see that, roughly speaking, for $\gamma \in [-2, 2] \setminus \{0\}$, the points in $\Phi_\gamma(h)$ should be points in $\mathbb{D} \setminus \Gamma$, where Γ is the nested CLE_4 in $\mathbb{D}$ coupled to h as its level lines. So the question becomes:

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Theorem (Aru, P., Powell, 2022)

Let Γ be a nested CLE_4 in $\mathbb{D}$. Then the complement of Γ, i.e., the complement in $\mathbb{D}$ of the union of all loops in Γ, is almost surely totally disconnected.

Further results

- The set of thick points of a weighted nesting CLE_κ field with $\kappa \in (8/3, 4]$ and distribution μ having 0 mean and finite second moment is almost surely totally disconnected.

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- Application to a certain class of local sets of the GFF.
- Application to supercritical LQG metrics.