Limit theorems for Gibbs functionals: Stein's method meets disagreement percolation

Moritz Otto (Aarhus)

based on joint work with Christian Hirsch (Aarhus), Günter Last (Karlsruhe) and Anne Marie Svane (Aalborg)

UK Easter Probability Meeting
March 2023

Gibbs point processes

- $(\mathbb{X}, \mathcal{X})$ Borel space with σ-finite measure λ
- ξ Gibbs point process with Papangelou intensity (PI) κ if

$$\mathbb{E}\left[\int f(x,\xi)\,\xi(dx)\right] = \int \mathbb{E}\left[f(x,\xi+\delta_x)\kappa(x,\xi)\right]\,\lambda(dx), \quad f \geq 0.$$

- Hamiltonian $H\colon \mathsf{N} \times \mathsf{N} \to (-\infty, \infty]$ (based on κ) defined by

$$H(\mu,\psi) := \begin{cases} 0, & \text{if } \mu(\mathbb{X}) = 0, \\ -\log \kappa_m(x_1,\dots,x_m,\psi), & \text{if } \mu = \delta_{x_1} + \cdots + \delta_{x_m}, \\ \infty, & \text{if } \mu(\mathbb{X}) = \infty, \end{cases}$$

where we define recursively

$$\kappa_{m+1}(x_1,\dots,x_{m+1},\mu) \\ := \kappa(x_{m+1}, \mu + \delta_{x_1} + \cdots + \delta_{x_m})\,\kappa_m(x_1,\dots,x_m,\mu), \quad m \geq 1.$$

Assumptions on the PI

Assume

$$\kappa(x, \mu) \leq \alpha \text{ for some } \alpha > 0 \qquad \text{(DOM)},$$
$$\kappa(x, \mu) = \kappa(x, C(x, \mu)) \qquad \text{(LOC)},$$

where

$$C(x, \mu) := \sum_{y \in \mu} 1\{y \neq x\} \, 1\{x \text{ and } y \text{ are connected via } \mu\} \, \delta_y$$

is the connected component of x in μ wrt some symmetric relation $\sim$ on $\mathbb{X}$.

Examples: Strauss process, Area interaction model, Widom-Rowlinson model, Continuum random cluster model

Disagreement coupling

Let

- λ diffuse and σ-finite
- $W \in \mathcal{X}$ with $\lambda(W) < \infty$
- $\psi \in \mathsf{N}_{W^c}$ boundary condition

Put

$$\kappa_{W,\psi}(x,\mu) := 1\{x \in W\}\kappa(x, \mu \cup \psi).$$

Theorem. (Last–O. 22)
We find Gibbs processes ξ, ξ' on W with PI $\kappa_{W,\psi}$, $\kappa_{W,\psi'}$ such that

- Every point in $\xi \Delta \xi'$ is connected via $\xi \cup \xi'$ to $\psi \cup \psi'$.
- There is a Poisson process η with intensity measure $\alpha\lambda(\cdot \cap W)$ such that

$$\mathrm{supp}(\xi \cup \xi') \subset \mathrm{supp}(\eta) \quad \text{a.s.}$$

Poisson process approximation via Stein's method

Assume that:

- Γ finite point process with intensity measure K
- For K-a.a. $x \in \mathbb{X}$, let $\Gamma_x \stackrel{d}{=} \Gamma$ and let Γ^x be a reduced Palm version of Γ at x
- ζ finite Poisson process with intensity measure L

Then

- (Barbour–Brown 92):

$$\mathsf{d}_{\mathsf{TV}}(\Gamma, \zeta) \leq d_{TV}(\mathsf{K}, \mathsf{L}) + \int \mathbb{E}(\Gamma_x \Delta \Gamma^x)(\mathbb{X})\, \mathsf{K}(\mathrm{d}x).$$

- (Bobrowski–Schulte–Yogeshwaran 22):

$$\mathsf{d}_{\mathsf{KR}}(\Gamma, \zeta) \leq d_{TV}(\mathsf{K}, \mathsf{L}) + 2 \int \mathbb{E}(\Gamma_x \Delta \Gamma^x)(\mathbb{X})\, \mathsf{K}(\mathrm{d}x).$$

Poisson approximation of Gibbs functionals

Let $\mathbb{X} := \mathbb{R}^d \times \mathbb{Y}$ and $\lambda := \lambda^d \otimes \mathbb{Q}$. For compact $W \subset \mathbb{R}^d$ consider

$$\Gamma := \sum_{(x,r)\in\xi\cap W\times\mathbb{Y}} g(x,r,\xi)\,\delta_{(x,r)},$$

where $g\colon \mathbb{X} \times \mathsf{N} \to \{0,1\}$ satisfies

$$g(x,r,\mu) = g(x,r,\mu \cap R_x), \quad (x,r,\mu) \in \mathbb{R}^d \times \mathbb{Y} \times \mathsf{N},$$

with $R_x := (x+R) \times \mathbb{Y}$ for some Borel set $R \subset \mathbb{R}^d$.

Lemma. The reduced Palm process $\xi^{x,r,\Gamma}$ of ξ at (x,r) wrt Γ is a Gibbs process with PI

$$\kappa^{x,r}(y,s,\mu) := \kappa(y,s,\mu+\delta_{(x,r)})\frac{g(x,r,\mu+\delta_{(y,s)})}{g(x,r,\mu)}.$$

Theorem. (Last–O. 22) Let $R \subset S$, $S_x := (x+S) \times \mathbb{Y}$ and let ζ be a finite Poisson process on $\mathbb{X}$. Then

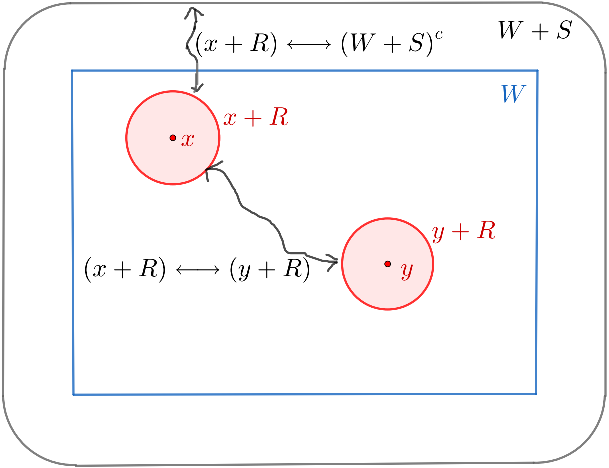
$$\mathrm{d}_{\mathsf{KR}}(\Gamma, \zeta) \leq d_{\mathsf{TV}}(\mathbb{E}[\Gamma], \mathbb{E}[\zeta]) + T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 =& 2 \iint_{W \times W} \mathbb{E}[g(x,r,\xi)\kappa(x,r,\xi)]\, \mathbb{E}[g(y,s,\xi)\kappa(y,s,\xi)] \\ & \times \mathbb{1}\{S_x \cap S_y \neq \emptyset\}\, \mathrm{d}(x,y)\, \mathbb{Q}^2(\mathrm{d}(r,s)), \\ T_2 =& 2 \iint_{W \times W} \mathbb{E}[g(x,r,\xi+\delta_{(y,s)}) g(y,s,\xi+\delta_{(x,r)}) \kappa_2((x,r),(y,s),\xi)] \\ & \times \mathbb{1}\{S_x \cap S_y \neq \emptyset\}\, \mathrm{d}(x,y)\, \mathbb{Q}^2(\mathrm{d}(r,s)), \\ T_3 =& 2\alpha^2 \int_{W \times W} \mathbb{1}\{S_x \cap S_y = \emptyset\}\, \mathbb{P}(R_x \stackrel{\eta}{\longleftrightarrow} (W+S)^c \cup R_y)\, \mathrm{d}(x,y), \end{aligned}$$

where η is a Poisson process on $\mathbb{X}$ with intensity measure $\alpha\lambda_d \otimes \mathbb{Q}$.

Interpretation of $\mathbb{P}(R_x \overset{\eta}{\longleftrightarrow} (W+S)^c \cup R_y)$



Normal approximation of Gibbs functionals

Now assume

- $\mathbb{Y} := [0, r_0]$ for some $r_0 > 0$
- $(x, r) \sim (y, s) \quad \Leftrightarrow \quad \|x - y\| \leq r + s$
- $\kappa(x, \mu) \leq \alpha < \alpha_c(r_0)$ critical intensity for Poisson Boolean percolation with radius r_0
- $g : \mathbb{X} \times \mathsf{N} \to \mathbb{R}$ translation invariant
- $|W_n| \to \infty$ as $n \to \infty$

Consider

$$\Gamma := \sum_{(x,r)\in\xi} g(x, r, \xi)\delta_{(x,r)} \qquad \text{and} \qquad H_n := \Gamma(W_n \times \mathbb{Y}).$$

Aim: Show that $\frac{H_n - \mathbb{E}H_n}{\sqrt{\mathrm{Var}(H_n)}} \to \mathcal{N}(0, 1)$ under conditions on g and ξ.

Normal approximation of Gibbs functionals

Chen–Röllin–Xia (2021): $d_K\big(\frac{H_n-\mathbb{E}H_n}{\sqrt{\mathrm{Var}(H_n)}},\mathcal{N}(0,1)\big)$ can be bounded using a coupling of Γ with Palm versions Γ^x, $x \in \mathbb{X}$.

Difficulty: Disagreement coupling uses different version of Γ for each $x \in \mathbb{X}$!

Theorem. (Hirsch–O.–Svane 23) Assume that g together with ξ satisfy conditions on moments, variance lower bounds and exponential stabilization. Then

$$\mathrm{d_K}\left(\frac{H_n-\mathbb{E}H_n}{\sqrt{\mathrm{Var}(H_n)}},\mathcal{N}(0,1)\right) \le O\left(\frac{\log|W_n|^a}{\sqrt{|W_n|}}\right),$$

where $\mathrm{d_K}(X,Y) := \sup_{u\in\mathbb{R}} |\mathbb{P}(X \le u) - \mathbb{P}(Y \le u)|$ Kolmogorov distance.

Open and related problems

- What happens beyond the critical threshold/at criticality?
- Disagreement coupling for Gibbs processes not satisfying (LOC)?

G. Last and MO (2022+). Disagreement coupling of Gibbs processes with an application to Poisson approximation. *To appear in Ann. Appl. Probab.*

C. Hirsch, MO and A. M. Svane (2023+). Normal approximation for Gibbs processes via disagreement couplings. *In preparation.*

Thank you!