Tail asymptotics for exponential functionals of subordinators and extinction times of self-similar fragmentation processes

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Exponential functional of subordinators

$(\xi_r, r \geq 0)$: subordinator, i.e. an increasing Lévy process, with no drift and no killing

Distribution characterized by its Laplace exponent

$$\phi(x) = \int_0^\infty (1 - \exp(-xu))\pi(\mathrm{d}u), x \geq 0$$

where $\int_0^\infty (1 \wedge u)\pi(\mathrm{d}u) < \infty$ (then: $\mathbb{E}[\exp(-x\xi_r)] = \exp(-r\phi(x)), \forall r, x \geq 0$)

Exponential functional of ξ:

$$I = \int_0^\infty \exp(-\xi_r)\mathrm{d}r$$

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Such random variables:

- are involved in the description of various processes ranging from the analysis of algorithms to coagulation or fragmentation processes
- correspond to the extinction times of non-increasing, non-negative self-similar Markov processes (by Lamperti's transformation)

Survey: Bertoin-Yor 05 + many papers since then.

Moments and density

ξ: subordinator of Laplace exponent $\phi(x) = \int_0^\infty (1 - \exp(-xu))\pi(\mathrm{d}u)$

$I = \int_0^\infty \exp(-\xi_r)\mathrm{d}r$

Proposition (Carmona-Petit-Yor 97)

For all $k \in \mathbb{N}$,

$$\mathbb{E}[I^k] = \frac{k!}{\phi(1)\dots\phi(k)}.$$

$\Rightarrow$ I has exponential moments $\quad (\mathbb{E}\left[e^{aI}\right] < \infty$ for some $a > 0)$

Proposition (Carmona-Petit-Yor 97, Pardo-Rivero-Schaik 13)

I has a density on $\mathbb{R}_+^*$, denoted by k, which satisfies

$$k(x) = \int_0^\infty \left(\int_x^{xe^v} k(y)\mathrm{d}y \right) \pi(\mathrm{d}v), \quad x > 0$$

On the logarithm of the tail of I

Let ψ be the inverse of $x \mapsto x/\phi(x)$: $\frac{\psi(x)}{\phi(\psi(x))} = x$, for $x > x_\psi$

Theorem (Rivero 03)

If ϕ is regularly varying at ∞ with index $\gamma \in [0, 1)$, then:

$$\ln \mathbb{P}(I > t) \underset{t \to \infty}{\sim} -(1-\gamma)\psi(t).$$

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Main hypothesis in the following (not restrictive at all!):

$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} < 1 \qquad \textbf{(H)}$$

Theorem (H.-Rivero 12)

Assume (**H**). Then

$$\ln \mathbb{P}(I > t) \underset{t\to\infty}{\sim} -\int_{x_\psi}^{t} \frac{\psi(r)}{r} \mathrm{d}r.$$

In some special cases when π is finite, Maulik-Zwart 06 ***remove the logarithm***; e.g. when $\pi(0,u) = bu + o(u^{1+\delta})$ as $u \downarrow 0$ for some $b \geq 0, \delta > 0$, then

$$\mathbb{P}(I > t) \underset{t\to\infty}{\sim} \mathrm{c}\, t^{\frac{b}{|\pi|}} \exp(-|\pi| t).$$

On the tail of I

k: density of I, ψ defined by $\frac{\psi(x)}{\phi(\psi(x))} = x$

Theorem 1 (H. 22; Minchev-Savov 22)

Assume (**H**). Then there exists $\mathrm{c} \in (0, \infty)$ such that

$$\mathbb{P}(I > t) \; = \; \mathrm{c} \; \frac{t(\psi'(t))^{1/2}}{\psi(t)} \; \exp\left(-\int_{x_\psi+1}^{t} \frac{\psi(r)}{r}\mathrm{d}r\right)\left(1 + O\left(\frac{1}{\psi(t)}\right)\right)$$

and

$$k(t) \; = \; \mathrm{c} \; (\psi'(t))^{1/2} \; \exp\left(-\int_{x_\psi+1}^{t} \frac{\psi(r)}{r}\mathrm{d}r\right)\left(1 + O\left(\frac{1}{\psi(t)}\right)\right).$$

Remarks. Different methods of proof:

- Minchev-Savov 22 obtain an explicit expression of c and estimates of the derivatives of k.
- H.22 obtains the first order $O\left(\frac{1}{\psi(t)}\right)$ and possibility to get further orders by iterating the proof.

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Ex.1: Stable subordinators $\quad \phi(x) = x^{\alpha}, 0 < \alpha < 1,$ then

$$\mathbb{P}(I > t) \underset{t\to\infty}{\propto} t^{-\frac{\alpha}{2(1-\alpha)}} \exp\left((-(1-\alpha)t^{\frac{1}{1-\alpha}}\right)\left(1 + O\left(t^{-\frac{1}{1-\alpha}}\right)\right).$$

Ex.2: Gamma subordinator $\quad \pi(\mathrm{d}u) = u^{-1}e^{-u}\mathrm{d}u,$ then $\phi(x) = \ln(1+x)$ and

$$\mathbb{P}(I > t) \underset{t\to\infty}{\propto} \exp\left(-t\ln(t) - t\ln(\ln(t)) + t + O\left(\frac{t\ln(\ln(t))}{\ln(t)}\right)\right)$$

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Ex.3: π probability such that $\pi(0, u) = \sum_{i=1}^{p} c_i u^{\gamma_i} + O(u^{1+\varepsilon})$ as $u \downarrow 0$ where $1/2 < \gamma_1 < \ldots < \gamma_{p-1} < \gamma_p = 1$, $\varepsilon > 0$, then:

$$\mathbb{P}(I > t) \underset{t\to\infty}{\propto} t^{c_p} \exp\left(-t + \sum_{i=1}^{p-1} \frac{c_i \Gamma(1+\gamma_i)}{1-\gamma_i} t^{1-\gamma_i}\right)$$

Idea of proof: compare two integral equations

On the one hand, using that $\frac{\psi(x)}{\phi(\psi(x))} = x, x > x_\psi$, we have:

$$\frac{\psi(x)}{x} = \phi(\psi(x)) = \int_0^\infty \left(1 - \exp\left(-\frac{\psi(x)}{x} xv\right)\right) \pi(\mathrm{d}v), \quad \forall x > x_\psi. \tag{1}$$

On the other hand, setting $f(x) := -\ln \mathbb{P}(I > x)$ and using the equation satisfied by k, we get:

$$f'(x) = \int_0^\infty \left(1 - \exp\left(-\int_x^{xe^v} f'(u)\mathrm{d}u\right)\right) \pi(\mathrm{d}v), \quad \forall x > 0. \tag{2}$$

Rk.: $\int_x^{xe^v} f'(u)\mathrm{d}u \approx xvf'(x)$ when $v \downarrow 0$

Idea of proof: compare two integral equations

Equations (1) and (2) are "close". More precisely we have:

Proposition 1 (H.22)

Assume (**H**). Then as $x \to \infty$

$$f'(x) = \frac{\psi(x)}{x} + \frac{\psi'(x)}{\psi(x)} - \frac{1}{x} - \frac{\psi''(x)}{2\psi'(x)} + O\left(\frac{\psi'(x)}{(\psi(x))^2}\right)$$

Which immediately implies the estimates on $\mathbb{P}(I > t)$ and $k(t)$ as $t \to \infty$.

Application to the tails of extinction times of self-similar fragmentations

Self-similar fragmentations

Describe the evolution of masses of particles that split repeatedly as time goes on:

- each particle is characterized by a mass $m \in (0,1]$
- each particle of mass m splits in particles of masses $(ms_k)_{k\in\mathbb{N}}$, where $(s_k)_{k\in\mathbb{N}} \in \mathcal{S}^{\downarrow} := \left\{(s_i)_{i\geq 1} : s_1 \geq s_2 \geq s_3 \ldots ; \sum_{i=1}^{\infty} s_i = 1\right\}$ at rate

$$m^{\alpha}\nu(\mathrm{d}\mathbf{s})$$

where $\alpha \in \mathbb{R}$ and ν is a measure on $\mathcal{S}^{\downarrow}$ such that $\int_{\mathcal{S}^{\downarrow}}(1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$

- different particles evolve independently (branching property)
- the process starts from a unique particle, of mass 1

First ref.: Kolmogorov 41, Filippov 61, Brennan and Durrett 86-87, Bertoin 01-02

Many studies on those models since 2000+.

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- different particles evolve independently (branching property)
- the process starts from a unique particle, of mass 1

Hypotheses: $\alpha < 0$ and $\nu(\mathcal{S}^{\downarrow}) > 0 \Rightarrow$ very small objects split very quickly!

Let ζ be the first time at which the entire initial mass is reduced to an amount of 0-mass particles.

Proposition (Filippov 61, McGrady & Ziff 87, Bertoin 02)

The extinction time ζ is finite almost surely.

Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

The parameters $\alpha < 0$ and ν are fixed; ζ: corresponding extinction time.

Two functions: we let for x large enough

$$\phi(x) = \int_{\mathcal{S}^{\downarrow}} (1 - s_1^{x+1}) \nu(\mathrm{d}\mathbf{s}) \quad \text{and} \quad \psi : \quad \frac{\psi(x)}{\phi(\psi(x))} = x$$

Main hypothesis:

$$\limsup_{x \to \infty} \frac{\phi'(x) x}{\phi(x)} < 1 \tag{H}$$

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$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} < 1 \qquad (\mathbf{H})$$

Theorem 2 (H. 22)

Assume (**H**). Then

$$\mathbb{P}(\zeta > t) \asymp \left(\frac{\psi(|\alpha|t)}{t}\right)^{\frac{1}{|\alpha|}-1} (\psi'(|\alpha|t))^{\frac{1}{2}} \exp\left(-\int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r}\mathrm{d}r\right)$$

where for positive functions f, g, $f(t) \asymp g(t)$ means there exists $a, b > 0$ such that $a \cdot g(t) \leq f(t) \leq b \cdot g(t)$ for t large enough.

Examples with finite splitting rates

Here $\psi(x) \underset{x\to\infty}{\sim} \nu(\mathcal{S}^{\downarrow})x$, hence $\int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r}\mathrm{d}r = \nu(\mathcal{S}^{\downarrow})t + o(t)$.

Ex.1: Fragmentations into k identical pieces. A fragment of size m splits into k fragments of same sizes m/k. For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \underset{t\to\infty}{\sim} \mathrm{c}\exp(-t)$$

for some $\mathrm{c} \in (0,\infty)$.

Ex.2: Uniform fragmentation. A fragment of size m splits into two fragments of sizes $mU, m(1-U)$, where U is uniform on $[0,1]$. For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \asymp t^{\frac{2}{|\alpha|}} \exp(-t).$$

Examples with finite splitting rates

Ex.3: Beta fragmentations. A fragment of size m splits into two fragments of sizes $mB, m(1-B)$, where $B \sim \mathrm{Beta}(a,b)$, $b \geq a > 0$ (density on $(0,1)$ proportional to $x^{a-1}(1-x)^{b-1}$). For all indices of self-similarity $\alpha < 0$:

$$\mathbb{P}(\zeta > t) \asymp \begin{cases} \exp(-t) & \text{if } b \geq a > 1 \\ t^{\frac{1}{|\alpha|}} \exp(-t) & \text{if } b > a = 1 \\ t^{\frac{2}{|\alpha|}} \exp(-t) & \text{if } b = a = 1 \\ \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a}\right) & \text{if } b > 1 > a > 1/2 \\ t^{\frac{1}{|\alpha|}} \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a}\right) & \text{if } 1 = b \geq a > 1/2 \\ \exp\left(-t + \frac{\Gamma(a)}{(1-a)|\alpha|^a} t^{1-a} + \frac{\Gamma(b)}{(1-b)|\alpha|^b} t^{1-b}\right) & \text{if } 1 > b \geq a > 1/2. \end{cases}$$

If a (and possibly b) is smaller than 1/2, there will be additional terms.

Examples with infinite splitting rates

Ex.4: Aldous' beta-splitting models. Those are scaling limits of discrete models introduced by Aldous 96 to interpolate between some phylogenetic trees.

Parametrized by $\beta \in (-2,-1)$; binary splitting ($\nu(s_1+s_2<1)=0$) and

$$\nu(s_1 \in \mathrm{d}u) = \frac{-\beta-1}{\Gamma(2+\beta)}(u(1-u))^{\beta}, u \in (1/2,1) \quad \text{and} \quad \alpha = 1+\beta.$$

Then for $\beta \in (-2,-3/2]$:

$$\mathbb{P}(\zeta > t) \asymp t^{\frac{-2\beta-1}{2(\beta+2)}} \exp\left(-a_\beta t^{\frac{1}{\beta+2}} + b_\beta t\right)$$

where $a_\beta = (-\beta-1)^{\frac{-\beta-1}{\beta+2}}(\beta+2)$ and $b_\beta = \frac{(2\beta+3)\Gamma(\beta+2)}{(\beta+2)\Gamma(2\beta+4)}$.

For $\beta \in (-3/2,1)$: additional power terms in the exponential.

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For $\beta \in (-3/2,1)$: additional power terms in the exponential.

Ex.5: Applications to random rooted real trees. Gives/retrieves some precise estimates for the tails of heights of some random real trees. E.g. the Brownian tree, the stable Lévy trees, Ford's phylogenetic trees, etc.

Outline of the proof of Theorem 2

An intermediate tool: the extinction time of a typical point, denoted by I

Proposition (Bertoin 02)

$$I = \int_0^\infty \exp(\alpha \xi_t) \mathrm{d}t$$

where ξ is a subordinator with Laplace exponent $\bar{\phi}(x) = \int_{\mathcal{S}} (1 - \sum_i s_i^{x+1}) \nu(\mathrm{d}\mathbf{s})$.

Rk.: $\bar{\phi}(x) = \phi(x) + O(2^{-x})$ as $x \to \infty$.

Connections between the tails of ζ and I?

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Proposition 2 (H. 22)

Assume (**H**). Then,

$$\mathbb{P}(\zeta > t) \asymp \left(\frac{\psi(|\alpha| t)}{t} \right)^{\frac{1}{|\alpha|}} \cdot \mathbb{P}(I > t)$$

Some hints to prove Proposition 2

Step 1. Connections with moments of typical fragments: U_1, U_2 i.i.d uniform on $(0,1)$, $\Lambda_{(i)}(t)$: mass of the fragment containing U_i at time t, $i = 1, 2$

Proposition

There exists $\mathrm{c} \in (0, \infty)$ such that for all t large enough

$$\frac{\mathbb{E}\left[\Lambda_{(1)}(t)\right]^2}{\mathbb{E}\left[\Lambda_{(1)}(t)\Lambda_{(2)}(t)\right]} \leq \mathbb{P}(\zeta > t) \leq \mathrm{c}\left(\frac{\psi(|\alpha| t)}{t}\right)^{\frac{2}{|\alpha|}} \mathbb{E}\left[\Lambda_{(1)}(t)\right]$$

Idea: Introduce $S(t) := \sum_{i \geq 1} (F_i(t))^2$ and use the first and second moments methods.

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Step 2. Asymptotics of moments of 1 and 2 typical fragments.

Proposition

For all $a > 0$ there exists a constant $\mathrm{c} \in (0, \infty)$ such that

$$\mathbb{E}\left[\Lambda_{(1)}^a(t)\right] \underset{t\to\infty}{\sim} \mathrm{c}\left(\frac{t}{\psi(|\alpha| t)}\right)^{\frac{a}{|\alpha|}} \mathbb{P}(I > t)$$

Proposition

For all $a, b > 0$,

$$\mathbb{E}\left[\Lambda_{(1)}^a(t)\Lambda_{(2)}^b(t)\right] \asymp \left(\frac{t}{\psi(|\alpha| t)}\right)^{\frac{a+b+1}{|\alpha|}} \mathbb{P}(I > t).$$

A conjecture

Let $F(t)$ be the decreasing sequence of masses of particles present at time t, $\forall t \geq 0$.

Yaglom limit of the process F conditioned on non-extinction?

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Yaglom limit of the process F conditioned on non-extinction?

Probably:

$F(t) \left(\frac{\psi(|\alpha| t)}{t} \right)^{\frac{1}{|\alpha|}} \mid \zeta > t$ converges in distribution in ℓ_1 to a non-trivial limit.

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