Pólya urns with growing initial compositions

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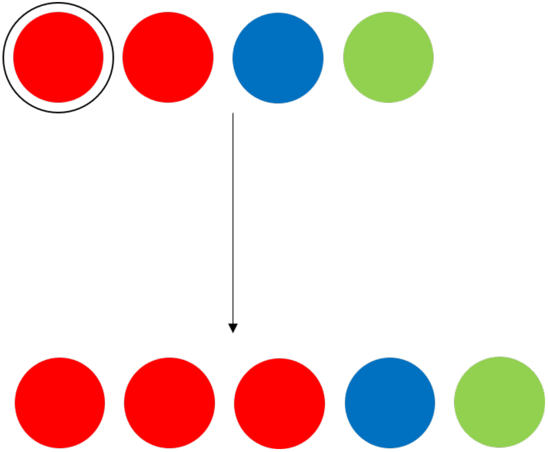
UK Easter Probability Meeting 2023

March 29, 2023

Introduction to Pólya urns



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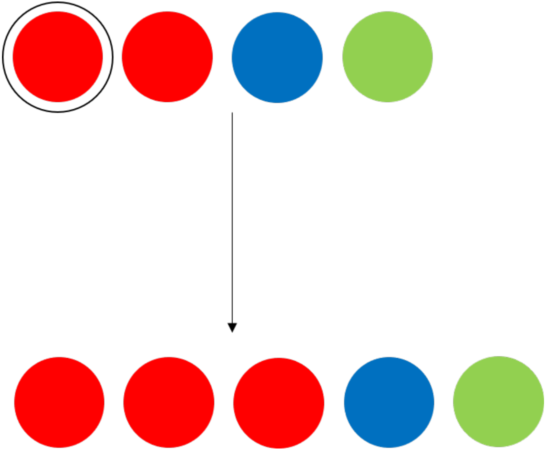

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Introduction to Pólya urns

A d-colour generalized Pólya urn process $(\boldsymbol{U}(n))_{n\geq 0}$ is a Markov process which depends on two parameters:

- The initial composition $\boldsymbol{U}(0) \in \mathbb{Z}^d_{\geq 0}$.
- The non-negative integer-valued replacement matrix R.

The process evolves from step n to $n+1$ as follows:

1. Select a ball u.a.r. from the urn.
2. If a ball of colour i is chosen, place the ball back into the urn along with R_{ij} balls of colour j.

Asymptotic behaviour: Two canonical cases

A typical question when studying Pólya urns is how the urn behaves as the number of draws n tends to infinity.

- What does the colour composition $\boldsymbol{U}(n)/\sum_{i=1}^{d} U(n)_i$ converge to?
- What are the size, scale, and shape of the urns fluctuations around its asymptotic colour composition?

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This question can be answered for most replacement matrices by solving the following two canonical cases:

Identity replacement matrix: Take $R = SI$, $S \in \mathbb{Z}_{\geq 1}$.

Irreducible replacement matrix: For all $1 \leq i, j \leq d$, if $\boldsymbol{U}(0) = \boldsymbol{e}_i$ there exists $n(i,j) \geq 0$, such that $U(n(i,j))_j > 0$ has positive probability, e.g.

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Growing initial composition urns

Let $(\boldsymbol{U}_n)_{n\geq 1}$ be a sequence of urns with identical replacement matrix, and set $N(n) := \sum_{i=1}^{d} U_n(0)_i$ to be the number of initial balls in the urn.

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Three regimes appear:

- **Initial Ball Dominant** - $n = o(N)$.
- **Transitional Regime** - $n/N \to 1$ (w.l.o.g.).
- **Time Step Dominant** - $N = o(n)$.

Asymptotic behaviour: Irreducible replacement matrix

(Athreya & Karlin, 1968) Let $\boldsymbol{U}$ have an irreducible replacement matrix R. Let λ_1 denote the Perron-Frobenius eigenvalue of R. Then, a.s.

$$\lim_{n\to\infty} \frac{\boldsymbol{U}(n)}{\sum_{i=1}^{d} U(n)_i} = \boldsymbol{v_1},$$

where $\boldsymbol{v}_1$ is the left eigenvector of λ_1.

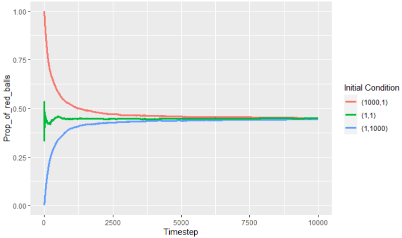

Figure: Simulations of a two colour urn with $R = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $\boldsymbol{v_1} \approx (0.45, 0.55)$.

Asymptotic behaviour: Irreducible replacement matrix

(Janson, 2004) Assume R is diagonalizable with real eigenvalues. Let λ_1 be the Perron-Frobenius eigenvalue and λ_2 the second largest eigenvalue.

Fluctuations of the irreducible urn around $\boldsymbol{v}_1$ as $n \to \infty$:

- Small urns: If $\lambda_2 < \lambda_1/2$, then the fluctuations converge to an Ornstein-Uhlenbeck process of size $n^{1/2}$ and scale nt.
- Critical urns: If $\lambda_2 = \lambda_1/2$, then the fluctuations converge to a Brownian motion of size $n^{1/2} \log(n)^{1/2}$ and scale n^t.
- Large urns: If $\lambda_2 > \lambda_1/2$, then the fluctuations converge to a non-Gaussian random variable of size n^{λ_2/λ_1} and scale n^t that depends on $\boldsymbol{U}(0)$.

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Let $(\boldsymbol{U}_n)_{n\geq 1}$ be a sequence of urns with identical irreducible replacement matrix, and set $N(n) := \sum_{i=1}^{d} U_n(0)_i$ to be the number of initial balls in the urn.

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We assume that:

- **Fixed initial colour composition** - There is a vector $\boldsymbol{\mu}$ such that

$$\boldsymbol{\mu}_n := N^{-1}\boldsymbol{U}_n(0) = \boldsymbol{\mu}, \;\; n \geq 1.$$

- **Balanced replacement matrix** - We add $S \geq 1$ balls to the urn at each time step a.s.

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Sub-urn representation

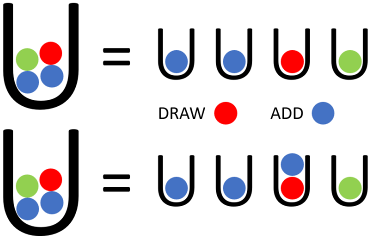
Let $\boldsymbol{U}$ be an urn with N initial balls. We have

$$\boldsymbol{U}(n) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{d} \sum_{j=1}^{U(0)_i} \boldsymbol{U}_{ij}(D_{ij}(n)), \quad n \geq 0.$$

- $\boldsymbol{U}_{ij}$ - Pólya urns with initial conditions $\boldsymbol{e}_i$ and identical replacement matrix to $\boldsymbol{U}$.
- $D_{ij}(n)$ - The number of times the urn $\boldsymbol{U}_{ij}$ has been drawn from by time step n of $\boldsymbol{U}$.

$$\boldsymbol{U}(0) = ({\color{blue}2}, {\color{red}1}, {\color{green}1})$$

$$R = \begin{pmatrix} {\color{blue}0} & {\color{blue}0} & {\color{blue}1} \\ {\color{red}1} & {\color{red}0} & {\color{red}0} \\ {\color{green}0} & {\color{green}1} & {\color{green}0} \end{pmatrix}$$



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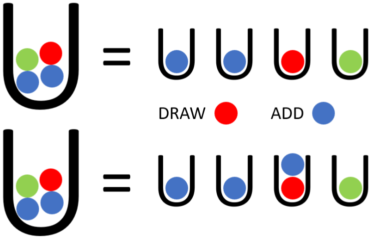

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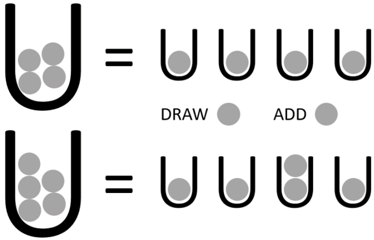

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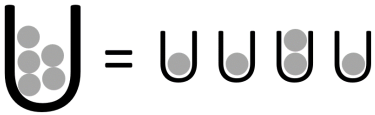

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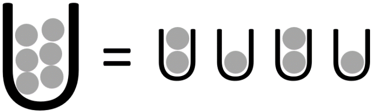

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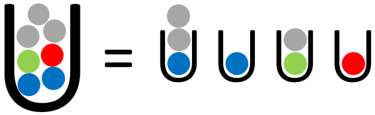

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Initial ball dominant regime $n = o(N)$

Let the sequence $(\boldsymbol{U}_n)_{n\geq 1}$ have a balanced irreducible replacement matrix and initial colour composition $\boldsymbol{\mu}$. Let $\ell_1(n,t) = \log(1 + Snt/N)$.

Initial Ball Dominant: As $n \to \infty$

$$n^{-1/2}(\boldsymbol{U}_n(\lfloor nt \rfloor) - N\mathrm{e}^{R'S^{-1}\ell_1(n,t)}\boldsymbol{\mu}) \xrightarrow{\mathrm{d}} \boldsymbol{W}_1(t) \text{ in } D[0,\infty),$$

$$N^{-1}\boldsymbol{U}_n(\lfloor nt \rfloor) \xrightarrow{\mathrm{p}} \boldsymbol{\mu} \text{ in } D[0,\infty).$$

- **Asymptotic colour composition $\boldsymbol{\mu}$** - Initial colour composition dominates the asymptotic colour composition.
- **Brownian fluctuations $\boldsymbol{W}_1$** - The draws of the urn are close to a random walk with jump probabilities given by $\boldsymbol{\mu}$ and jumps given by R.

Transitional regime $n \sim N$

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$$n^{-1/2}(\boldsymbol{U}_n(\lfloor nt \rfloor) - N\mathrm{e}^{R'S^{-1}\ell_1(n,t)}\boldsymbol{\mu}) \xrightarrow{\mathrm{d}} \boldsymbol{W_2}(t) \text{ in } D[0,\infty),$$
$$N^{-1}\boldsymbol{U}_n(\lfloor nt \rfloor) \xrightarrow{\mathrm{p}} \mathrm{e}^{R'S^{-1}\log(1+St)}\boldsymbol{\mu} \text{ in } D[0,\infty).$$

- **Asymptotic colour composition** $\mathrm{e}^{R'S^{-1}\log(1+St)}\boldsymbol{\mu}$ **(normalized)** - Can be seen as the "expected" composition of an urn with initial colour composition $\boldsymbol{\mu}$ number of initial balls 1 and number of draws t.
- **Gaussian fluctuations $\boldsymbol{W_2}$** - The shape of the fluctuations depend on the fluctuations (variance) of the urn with a finite number of draws.

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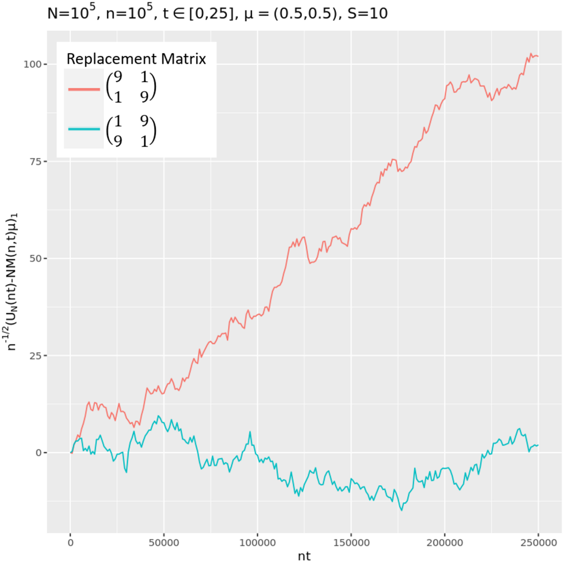

Time step dominant regime $N = o(n)$

Time Step Dominant: Assume R is diagonalizable with real eigenvalues. S is the Perron-Frobenius eigenvalue, and let λ_2 be the second largest eigenvalue. Then, as $n \to \infty$

$$\frac{\boldsymbol{U}_n(\lfloor N(n/N)^t \rfloor)}{SN(n/N)^t} \xrightarrow{\mathrm{p}} \boldsymbol{v}_1 \text{ in } D(0, \infty).$$

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Small Urns: Let $\ell_1(n,t) = \log(1 + Snt/N)$. If $\lambda_2 < S/2$, then as $n \to \infty$

$$n^{-1/2}(\boldsymbol{U}_n(\lfloor nt \rfloor) - N\mathrm{e}^{R'S^{-1}\ell_1(n,t)}\boldsymbol{\mu}) \xrightarrow{\mathrm{d}} t^{1/2}\boldsymbol{W}_s(\log(t)) \text{ in } D(0,\infty).$$

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Large Urns: If $\lambda_2 > S/2$, then as $n \to \infty$

$$\frac{\boldsymbol{U}_n(\lfloor N(n/N)^t \rfloor) - N\mathrm{e}^{R'S^{-1}\ell_2(n,t)}\boldsymbol{\mu}}{N^{1/2}(n/N)^{\lambda_2 t/S}} \xrightarrow{\mathrm{d}} \boldsymbol{V}_\ell \text{ in } D(0,\infty).$$

Thank you!