Rates on Yaglom's theorem for Galton-Watson processes in varying environment

UK Easter Probability Meeting

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Galton-Watson processes in varying environment:

- Branching processes where the individuals reproduce independently of each other.
- All individuals in the same generation have the same offspring distribution but these distributions **vary among generations**.
- Critical regime: Yaglom's theorem

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Main question:

What is the rate of convergence of the Yaglom's limit with respect to the Wasserstein distance d_W?

$$d_W\left(\left\{ \frac{Z_n}{a_n} \;\middle|\; Z_n > 0 \right\}, \mathbf{e} \right) \leq C f(n)$$

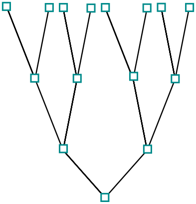
Consider a collection $Q = \{q_n, n \geq 1\}$ of probability measures supported on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

A **Galton-Watson process** $Z = \{Z_n : n \geq 0\}$ **in the environment** Q is a Markov chain defined recursively as follows:

$$Z_0 = 1 \quad \text{and} \quad Z_n = \sum_{i=1}^{Z_{n-1}} \chi_i^{(n)}, \qquad n \geq 1,$$

where $\{\chi_i^{(n)} : i, n \geq 1\}$ is a sequence of independent random variables satisfying

$$\mathbb{P}(\chi_i^{(n)} = k) = q_n(k), \qquad k \in \mathbb{N}_0,\ i, n \geq 1.$$



$\chi_i^{(4)} \sim q_4$

$\chi_i^{(3)} \sim q_3$

$\chi_i^{(2)} \sim q_2$

$\chi_i^{(1)} \sim q_1$

$\chi_i^{(n)}$ is the offspring of the i-th individual in the $(n-1)$-th generation.

For every $n \geq 1$, denote by f_n the **generating function** associated with the reproduction law q_n, i.e.

$$f_n(s) := \sum_{k=0}^{\infty} s^k q_n(k), \qquad 0 \leq s \leq 1,\ n \geq 1.$$

By a recursive application of the branching property, we deduce

$$\mathbb{E}\left[s^{Z_n}\right] = f_1 \circ \cdots \circ f_n(s), \qquad 0 \leq s \leq 1,\ n \geq 1,$$

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Let $\mu_0 := 1$ and for any $n \geq 1$,

$$\mu_n := f_1'(1) \cdots f_n'(1), \quad \nu_n := \frac{f_n''(1)}{f_n'(1)^2} \quad \text{and} \quad \rho_{0,n} := \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}.$$

The **mean** and the **normalized second factorial moment** of Z_n satisfy

$$\mathbb{E}\left[Z_n\right] = \mu_n, \qquad \text{and} \qquad \frac{\mathbb{E}\left[Z_n(Z_n - 1)\right]}{\mathbb{E}\left[Z_n^2\right]} = \rho_{0,n}.$$

Hypothesis in the environment: There exists $c > 0$ such that

$$f_n'''(1) \leq c f_n''(1)(1 + f_n'(1)), \quad \text{for any } n \geq 1. \qquad \textbf{(A)}$$

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Critical regime:

$$\rho_{0,n} \to \infty \qquad \text{and} \qquad \rho_{0,n}\mu_n \to \infty \quad \text{as} \quad n \to \infty.$$

In other words,

$$\mathbb{P}(Z_n > 0) \to 0 \quad \text{and} \quad \mathbb{E}\left[Z_n \mid Z_n > 0\right] \to \infty, \quad \text{as} \quad n \to \infty.$$

Question: Is there a sequence $\{a_n, n \geq 1\}$ such that Z_n/a_n conditioned on $\{Z_n > 0\}$ converges to a random variable?

The mean of the conditioned process is growing at rate $2(\mu_n \rho_{0,n})^{-1}$, i.e.

$$\lim_{n\to\infty} \frac{2}{\mu_n \rho_{0,n}} \mathbb{E}[Z_n \mid Z_n > 0] = 1.$$

Theorem 1 (Kersting (2020), C.-T. and Palau (2021))

Let $\{Z_n : n \geq 0\}$ *be a* ***critical*** *GWVE that satisfies condition* $(\mathbf{A})$*. Then*

$$\left\{ \frac{2Z_n}{\mu_n \rho_{0,n}} \;\middle|\; Z_n > 0 \right\} \xrightarrow{(d)} \boldsymbol{e}, \qquad as \;\; n \to \infty,$$

where $\boldsymbol{e}$ *is a* ***standard exponential random variable****.*

In the **constant environment** case and in the critical regime, we have

$$\mu_n = 1 \quad \text{and} \quad \rho_{0,n} = \sigma^2 n, \qquad \text{for any} \quad n \geq 1$$

where $\sigma^2 := \mathrm{Var}(Z_1)$.

In **constant environment** ($\mu_n = 1,\ \rho_{0,n} = \sigma^2 n$), Peköz and Röllin (2011) showed

$$d_W\left(\left\{\frac{2Z_n}{\sigma^2 n}\ \middle|\ Z_n > 0\right\}, \mathbf{e}\right) \le C\frac{\log(n)}{n}.$$

In **varying environment** ($\rho_{0,n} \to \infty$ and $\mu_n \rho_{0,n} \to \infty$ as $n \to \infty$), we pursuit a bound of the form

$$d_W\left(\left\{\frac{2Z_n}{\mu_n \rho_{0,n}}\ \middle|\ Z_n > 0\right\}, \mathbf{e}\right) \le C(\psi(\rho_{0,n}\mu_n) + \phi(\rho_{0,n})),$$

where ψ, ϕ continuously vanishing at infinity.

Trade-off between sharp and simple bounds valid for a **reduced family of Q** and semi-explicit bounds valid for **larger families of Q**.

Theorem 2 (C-T, Jaramillo, Palau (2023+))

*Let $\{Z_n : n \geq 0\}$ be a **critical** GWVE that satisfies condition $(\mathbf{A})$. Assume that there exists $0 < a \leq A < \infty$ such that $a \leq f_n'(1) \leq A$ for all $n \geq 1$. If*

$$\sum_{k=2}^{\infty} \log(\mu_k \rho_{0,k}) \left| \frac{\nu_{k-1}}{\mu_{k-2}} - \frac{\nu_k}{\mu_{k-1}} \right| < \infty.$$

Then, the following bound holds

$$d_W \left(\left\{ \frac{2Z_n}{\mu_n \rho_{0,n}} \;\middle|\; Z_n > 0 \right\}, \boldsymbol{e} \right) \leq C \left(\frac{\log(\mu_n \rho_{0,n})}{\mu_n \rho_{0,n}} + \frac{1}{\rho_{0,n}} \right),$$

where $C > 0$ is a constant independent of n.

In **constant environment**,

$$d_W \left(\left\{ \frac{2Z_n}{\sigma^2 n} \;\middle|\; Z_n > 0 \right\}, \mathbf{e} \right) \leq C \left(\frac{\log(\sigma^2 n)}{\sigma^2 n} + \frac{1}{\sigma^2 n} \right) \leq \tilde{C} \frac{\log(n)}{n}.$$

Theorem 3 (C-T, Jaramillo, Palau (2023+))

Let $\{Z_n : n \geq 0\}$ be a ***critical*** *GWVE that satisfies condition* $(\mathbf{A})$*. Then, the following bound holds*

$$d_W\left(\left\{\frac{2Z_n}{\mu_n\rho_{0,n}} \;\middle|\; Z_n > 0\right\}, \boldsymbol{e}\right) \leq C\left(\frac{\log(\mu_n\rho_{0,n})}{\mu_n\rho_{0,n}} + \frac{r_n}{\rho_{0,n}}\right),$$

where $C > 0$ is a constant independent of n and

$$r_n := \sum_{j=1}^{n-1}\left(\frac{\nu_j}{\mu_{j-1}}\right)^2 \frac{1}{(\rho_{0,n} - \rho_{0,j})}(1 + f_j'(1)) + \frac{\nu_n}{\mu_{n-1}}(1 + f_n'(1)).$$

Examples:

- **Poisson distributions with μ_n increasing linearly:** Assume that $\{q_n : n \geq 1\}$ are Poisson(λ_n) such that

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_n = \frac{n-1}{n}, \qquad n > 1.$$

It follows that

$$\mu_n = n \qquad \text{and} \qquad \rho_{0,n} \sim \log(n) \qquad \text{as} \quad n \to \infty.$$

From Theorem 2, we have

$$d_W\left(\left\{\frac{2Z_n}{\mu_n \rho_{0,n}} \;\middle|\; Z_n > 0\right\}, \mathbf{e}\right) \leq C\left(\frac{\log(n\log(n))}{n\log(n)} + \frac{1}{\log(n)}\right).$$

- **Poisson distributions with μ_n decreasing exponentially:** Assume that $\{q_n : n \geq 1\}$ are Poisson(λ_n) such that

$$\lambda_1 = e^{-1} \qquad \text{and} \qquad \lambda_n = \frac{\exp(-\sqrt{n})}{\exp(-\sqrt{n-1})} \qquad n > 1.$$

It follows that

$$\mu_n = \exp(-\sqrt{n}) \quad \text{and} \quad \rho_{0,n} \sim 2\sqrt{n}\exp(\sqrt{n}) \quad \text{as} \quad n \to \infty.$$

From Theorem 2, we have

$$d_W\left(\left\{\frac{2Z_n}{\mu_n \rho_{0,n}} \;\middle|\; Z_n > 0\right\}, \mathbf{e}\right) \leq C\frac{\log(\sqrt{n})}{\sqrt{n}}.$$

- **Linear fractional distributions:** For each $n \geq 1$, there exists $p_n \in (0,1)$ and $a_n \in (0,1]$, such that

$$q_n(0) = 1 - a_n \quad \text{and} \quad q_n(k) = a_n p_n (1-p_n)^{k-1}, \quad k \geq 1.$$

The random variable Z_n conditioned on $\{Z_n > 0\}$ has a geometric distribution, i.e.

$$\mathbb{P}(Z_n = k \mid Z_n > 0) = \widehat{p}_n (1 - \widehat{p}_n)^k, \qquad \text{where} \qquad \widehat{p}_n = \frac{2}{2 + \mu_n \rho_{0,n}}.$$

Assume that μ_n and $\rho_{0,n}$ satisfy the criticality conditions. Then

$$d_W\left(\left\{ \frac{2Z_n}{\mu_n \rho_{0,n}} \;\middle|\; Z_n > 0 \right\}, \mathbf{e} \right) \leq C \frac{1}{\mu_n \rho_{0,n}}.$$

Thank you for your attention!

- G. Kersting. *A unifying approach to branching processes in a varying environment.* J. Appl. Probab. 57.1, 2019.
- E. A. Peköz and A. Rölin. *New rates for exponential approximation and the theorems of Rényi and Yaglom.* Ann. Probab., 39(2):587-608, 2011.
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