Dimers and Imaginary Geometry

Nathanaël Berestycki, University of Vienna

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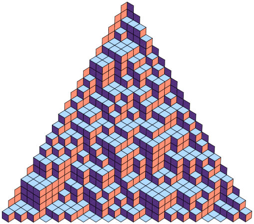
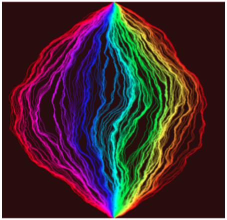




Outline

1) Introduction to the *dimer model*

- Definition. Notion of height function. Boundary conditions.
- Statement of Kenyon's theorem; Kasteleyn theory.

2) *Imaginary geometry* approach

- Temperley's bijection;
- GFF / SLE coupling.
- Convergence of winding

3) *Near-critical* (massive) dimer model

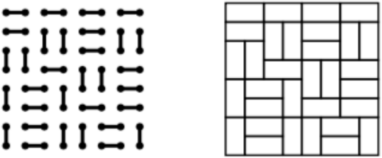
- Definition, non Gaussian scaling limit
- Connection with massive SLE.
- Exact discrete Girsanov theorem on triangular lattice

1) The dimer model

The dimer model

Let G be a finite, planar, bipartite graph.

A *dimer cover* (or *perfect matching*): a set of edges (=dimers), such that each vertex is incident to exactly one dimer.



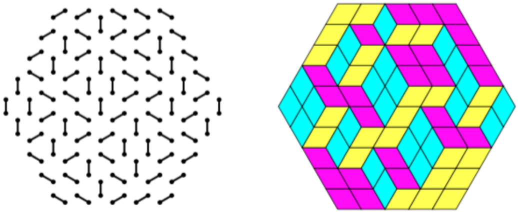
The *dimer model* with edge weights w_e:

$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{e \in \mathbf{m}} w_e .$$

Typically $w_e \equiv 1$ ($\to$ *critical*!)

The dimer model as a random surface

Honeycomb lattice: *lozenge tiling* or a stack of cubes



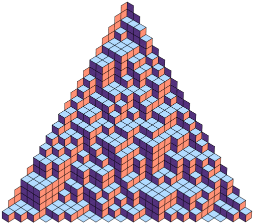
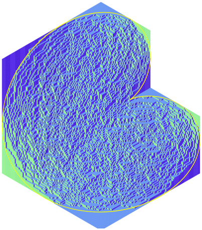
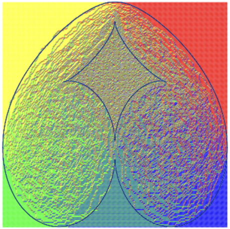
©Kenyon

Height function

Introduced by Thurston. Hence view as a random surface.

Note: depends on the choice of a reference frame.

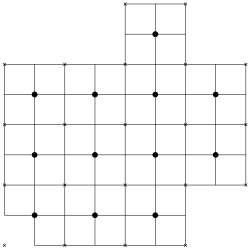
Large scale behaviour?

The effect of boundary conditions is, however, not entirely trivial and will be discussed in more detail in a subsequent paper.

P. W. Kasteleyn, 1961

Temperleyan boundary conditions



Divide the vertices into black and white.
Divide further into $B_0 = \bullet, B_1 = \times$
(and W_0, W_1).

Temperleyan: all corners are $B_1 = \times$, and one corner is removed.

Scaling limit of height function

Theorem (Kenyon '99)

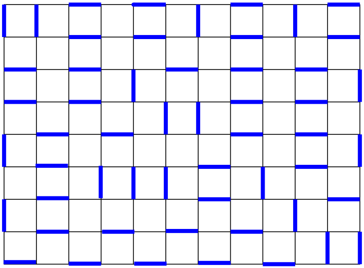
Let $\mathcal{D} \subset \mathbb{C}$ bounded domain, $\mathcal{D}^\delta = \mathcal{D} \cap \delta\mathbb{Z}^2$ with *Temperleyan* boundary conditions. Let h^δ be the associated height function. Then,

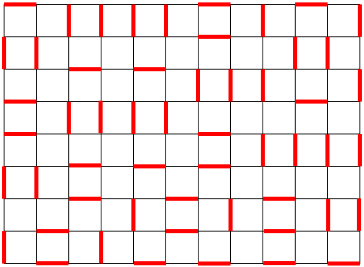
$$h^\delta - \mathbb{E}(h^\delta) \to \frac{1}{\sqrt{\pi}} h_{\mathcal{D}}^{\mathrm{GFF}} \qquad \text{as} \qquad \delta \to 0,$$

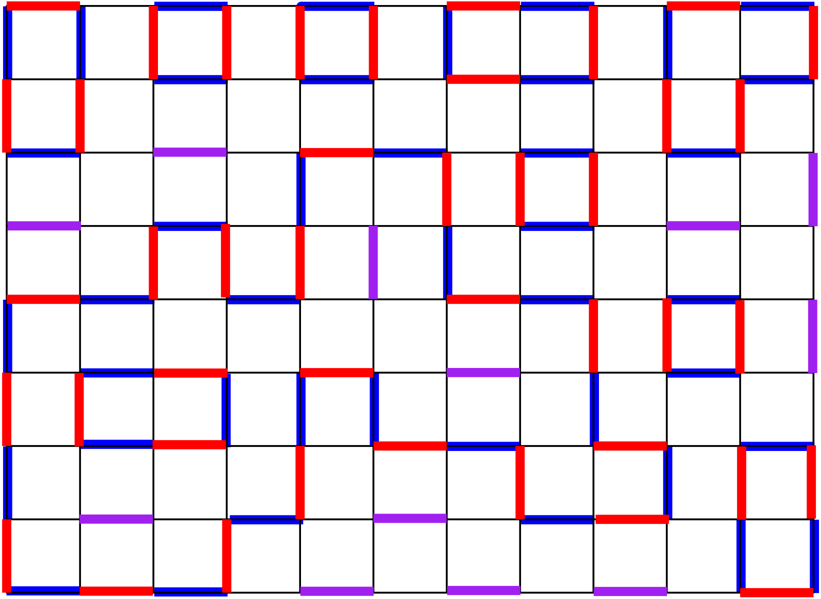
in distribution.

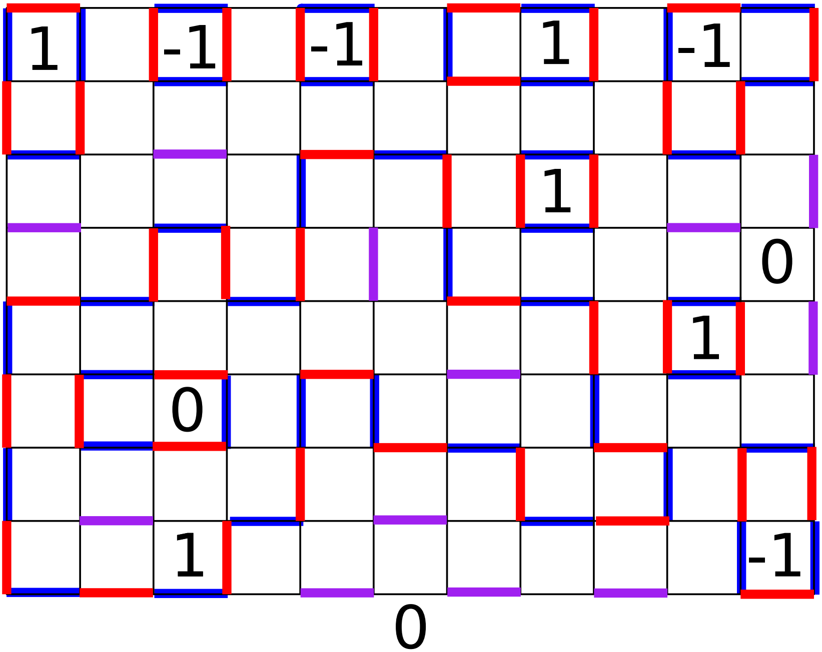
Main ingredients of the proof:

- Kasteleyn theory (exact solvability): dimer correlations are given by determinants of inverse *Kasteleyn matrix*,
- Asymptotic computation of inverse Kasteleyn matrix (discrete holomorphic + boundary conditions)
- Computation of moments





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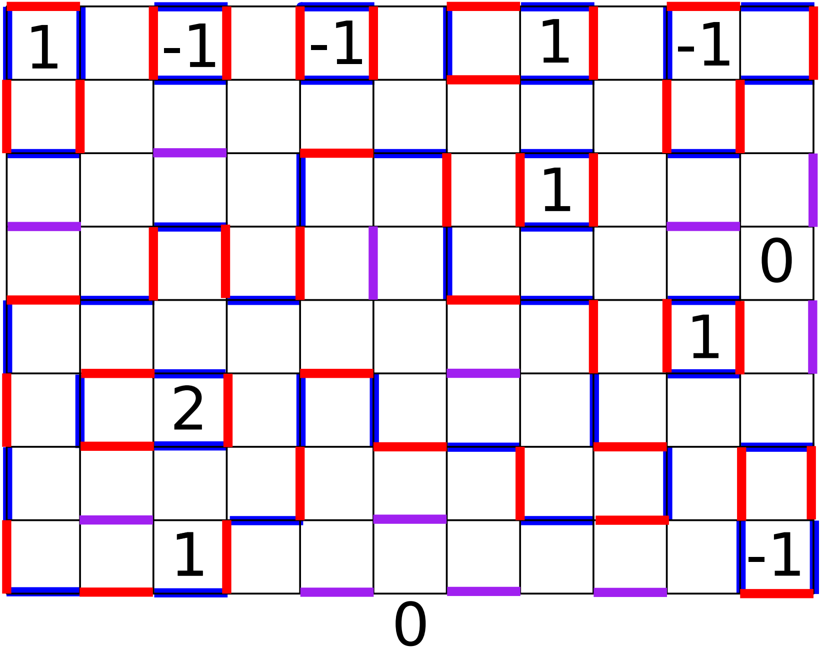

1
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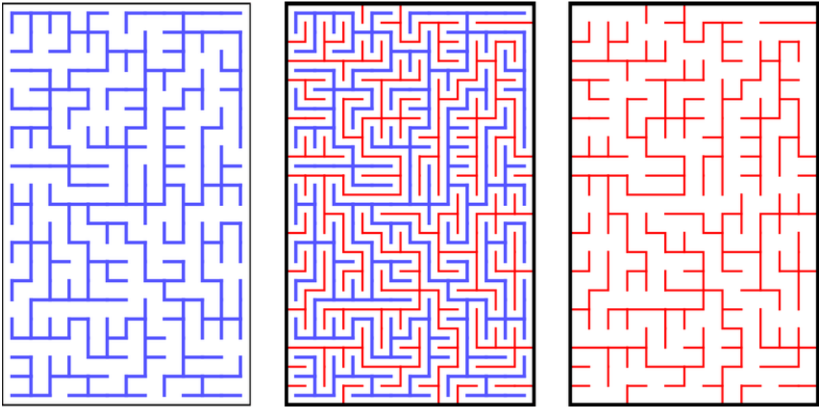
2) Imaginary Geometry

Uniform Spanning Tree

A spanning tree on a graph is a subset of edges covering each vertex and without cycles.



free boundary; self-dual trees ; wired boundary

Law

Orientation

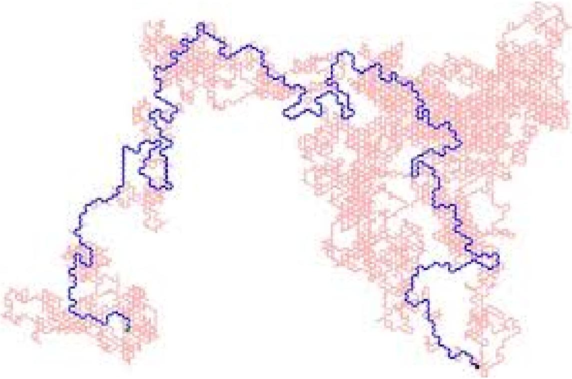
Trees can be oriented towards a designated root ρ.
For wired UST, $\rho = \partial V$.

For a weighted graph $G = (V, E)$, ∂ a fixed vertex (the boundary).

$$\mathbb{P}(T = t) = \frac{1}{Z} \prod_{e \in t} w_e$$

for every tree. Z is the partition function.

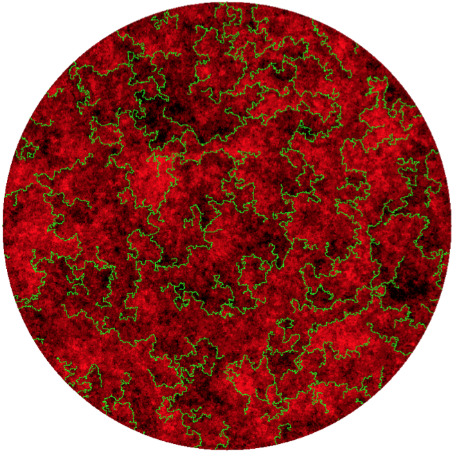
Wilson's algorithm 1.



Loop-Erased Random Walk (LERW)

Wilson's algorithm 2.

Build wired UST by adding LERW iteratively.



The tree is naturally oriented towards the boundary.

Enumeration

For a weighted graph $G = (V, E)$, ∂ a fixed vertex (the boundary).

$$\mathbb{P}(T = t) = \frac{1}{Z} \prod_{e \in t} w_e$$

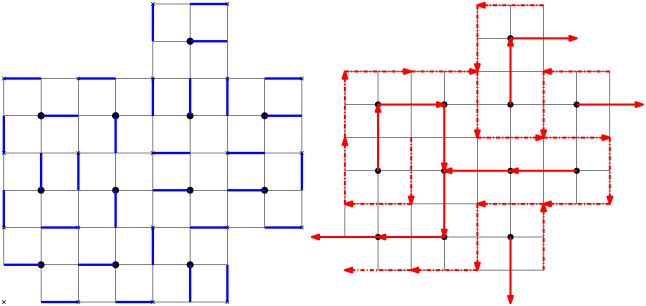
for every tree. Z is the partition function.

Fact:

Matrix tree theorem: $Z = \det(D)$ where D is the discrete Laplacian of random walk killed at ∂.

Temperley's bijection

Bijection between dimers on $G \subset \mathbb{Z}^2$ and dual spanning trees on B_0, B_1-lattices.



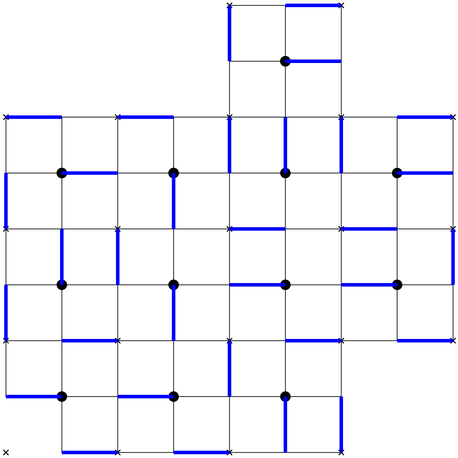
If G has **Temperleyan boundary conditions**, the B_0-tree is wired and the B_1-tree is free.

Amazing feature

Let $h =$ height function.
Then $h(f) - h(f') =$ total **winding** of branch connecting f and f' !

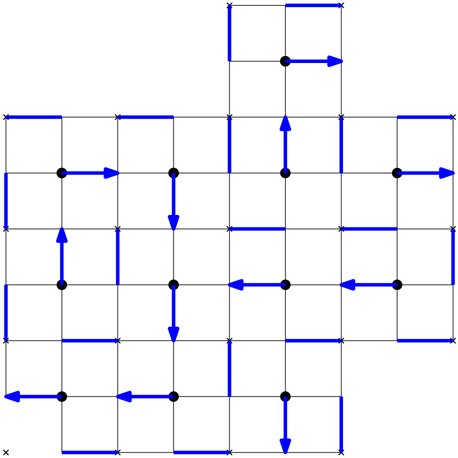
Temperley's bijection 1

Dimers on $\mathbb{Z}^2 \cap D$, Temperleyan boundary conditions.



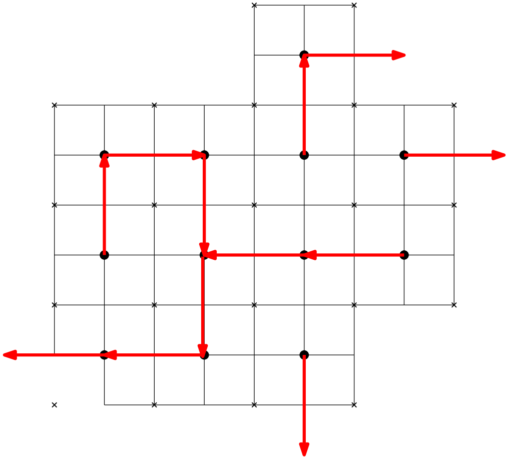
Temperley's bijection 2

Orient dimers black $\rightarrow$ white (just $B_0 = \bullet$ for now)



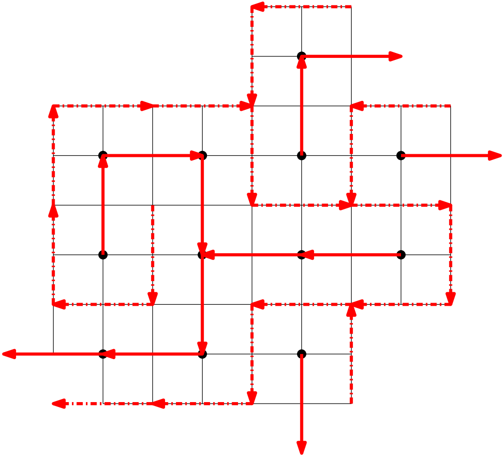
Temperley's bijection 3

Double each oriented dimer to get spanning tree on B_0 lattice (wired boundary conditions).



Temperley's bijection 4

On B_1 lattice, get dual (free boundary conditions) spanning tree.



Remarks

- The bijection is local.
- Temperleyan boundary conditions $\Rightarrow$ wired/free boundary conditions for trees.
- If $w_e \equiv 1$ then $(\mathcal{T}, \mathcal{T}^\dagger)$ uniform.
- More generally, in weighted setup, if w_e are weights on the dimer graph,

$$\mathbb{P}((\mathcal{T}, \mathcal{T}^\dagger) = (\mathbf{t}, \mathbf{t}^\dagger)) \propto \prod_{e \in \mathbf{t}} w_{e/2} \prod_{e^\dagger \in \mathbf{t}^\dagger} w_{e^\dagger/2}$$

If $w_{e^\dagger} \equiv 1$, we can just sample $\mathcal{T}$ from Wilson's algorithm; this determines $\mathcal{T}^\dagger$ by duality and then a dimer configuration by Temperley's bijection.

Winding in UST

Question

How much do branches wind in a wired UST?

For a face f, let γ_f be a path from f to ∂D which follows the UST.
Let $h^{\#\delta}(f)$ = total winding of γ_f (= height function).

Theorem (B.–Laslier–Ray (2020))

Let $G^{\#\delta}$ be a sequence of graphs embedded in $\mathbb{R}^2$. Assume $(\star)$. Let D be simply connected with ∂D locally connected.

$$h^{\#\delta} - \mathbb{E}(h^{\#\delta}) \xrightarrow[\delta\to 0]{} \frac{1}{\chi} h_{\mathrm{GFF}},$$

the Gaussian free field (Dirichlet boundary conditions); $\chi = 1/\sqrt{2}$.

Note: $\mathbb{E}(h^{\#\delta})$ itself is **not** universal, only fluctuations!

Robustness

We recover Kenyon's result, + much more:

- Balanced random environments ! (BLR 2020)
- General domains with **purely liquid** phases (Laslier 2022)
- Riemann Surfaces (BLR 2021, 2022)
- Near-critical (massive) cases (B. Haunschmid-Sibitz 2022). See tomorrow.

Still requires nice boundary conditions so that Temperley's bijection applies.

Scaling limit of Uniform Spanning Tree

Theorem (Lawler, Schramm, Werner ’03, Schramm ’00)

$D \subset \mathbb{C}$ simply connected

- *Uniform spanning tree on $D \cap \delta\mathbb{Z}^2 \to$ “A continuum tree” (continuum uniform spanning tree).*
- *Branches of the continuum tree are (radial) SLE_2 curves.*

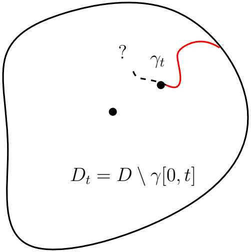
The continuum tree can be obtained by performing Wilson’s algorithm in the continuum.

Universality

Yadin–Yehudayoff 2010: assuming convergence of SRW to BM, LERW converges to SLE_2.

Schramm–Loewner–Evolution (SLE)

Oded Schramm 1961–2008



Familly of curves in (D, a, o) with $a \in \partial D$, $o \in D$.

Domain Markov property

Given $\gamma[0, t]$, law of future?
$\gamma[t, \infty)$= curve (D_t, γ_t, o).

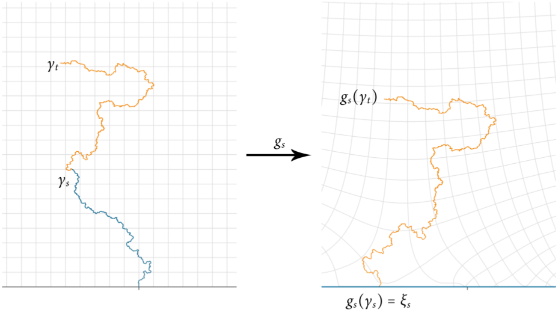
Theorem (Schramm)

Suppose γ satisfies Domain Markov + Conformal invariance. Then γ is SLE$_\kappa$ for some $\kappa \geq 0$.

Schramm–Loewner Evolution (SLE)

g_t = conformal map which removes $\gamma(0,t]$.

$$\frac{\partial g_t(z)}{\partial t} = g_t(z)\frac{\xi_t + g_t(z)}{\xi_t - g_t(z)}, \quad z \in \mathbb{D} \setminus \gamma[0,t]$$



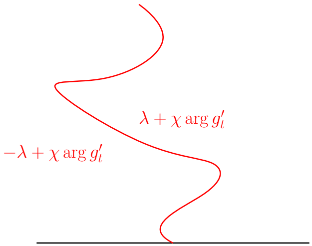
$\xi_= \exp(i\sqrt{\kappa}B_t)$ = driving function.

Imaginary Geometry

Dubédat, Miller–Sheffield: "flow lines of GFF/χ are SLE_κ curves", provided:

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

Meaning: coupling (h, η), $h = \mathrm{GFF}$, $\eta = \mathrm{SLE}_\kappa$:



Take-home message

"Values" of h/χ along curve record "winding" of SLE_κ (in sense of $\arg g'$).

Imaginary geometry

Recall:

Take-home message

"Values" of h/χ along curve record "winding" of SLE_κ (in sense of $\arg g'$).

$$\begin{array}{ccc} \text{UST} & \xrightarrow[\text{Temperley's bijection}]{\text{winding}} & h^{\#\delta} \\ {\scriptstyle\text{Lawler, Schramm, Werner}}\downarrow & & \downarrow ? \\ \text{Continuum UST} & \xrightarrow[\text{imaginary geometry}]{\text{winding?}} & (1/\chi)h_{\mathrm{GFF}} \end{array}$$

Two parts:

Part I: winding of continuum UST
Part II: making the diagramme commute

Intrinsic vs. topological winding of a path

Let $\gamma : [0, 1] \to \mathbb{C}$ smooth, *simple* curve. Let

$$W(\gamma, z) = \text{topological winding around } z$$

and let

$$W_{\text{int}}(\gamma) = \text{ intrinsic winding of } \gamma \ = \int_0^1 \arg \gamma'(s) ds$$
$$= \frac{\pi}{2}(\text{ \# left turns - \# right turns in discrete}).$$

Lemma

$W_{int}(\gamma) = W(\gamma, a) + W(\gamma, b)$ where $a = \gamma(0), b = \gamma(1)$.

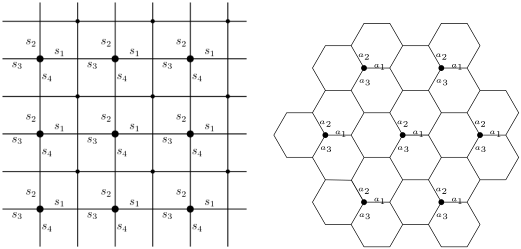
As a result this is well defined even for non-smooth paths, and is essentially continuous!

3) Near-critical dimer model

Makarov–Smirnov (2009):

The key property of SLE is its conformal invariance, which is expected in 2D lattice models only at criticality, and the question naturally arises: Can SLE success be replicated for off-critical models? In most off-critical cases to obtain a non-trivial scaling limit one has to adjust some parameter [...], sending it at an appropriate speed to the critical value. Such limits lead to massive field theories...,

Biperiodic setup

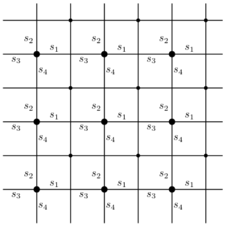


Choose $s_i = 1 + c_i\delta$, where δ = mesh size. *Gasesous/Liquid boundary...*

Massive Laplacian

Let K = Kasteleyn matrix, $D = KK^*$. Then D is (essentially) a *massive Laplacian*:

$$D(b,b) = -\sum_{i=1}^{4} s_i^2$$



but

$$\sum_{b'} D(b,b') = 2s_2s_4 + 2s_1s_3 < |D(b,b)|$$

by AM-GM.

Describes a *massive walk* (fixed killing probability).

Natural guess:

Scaling limit = Massive GFF?

$$\mathbb{E}[h(x)h(y)] = \int_0^\infty e^{-m^2 t} p_t(x,y)\,dt$$

Negative answer

Unfortunately this guess is wrong.

Theorem (Chhita, 2012)

Limiting moments of height function can be computed; no Wick rule so non Gaussian !

New results for near-critical dimers

With Levi Haunschmid-Sibitz (2022) we prove:

- Exact connection with Makarov and Smirnov's *massive* SLE_2 (and with massive Laplacian).
- Existence and universality of *scaling limit* of height function in Temperleyan domains
- Conformal *covariance* of scaling limit

Scaling limit of Temperleyan tree

Consider off-critical dimer model on square with $s_i = 1 + c_i\delta$.
Let $\mathcal{T}$ = Temperleyan B_0-tree.

$$\mathbb{P}(\mathcal{T} = \mathbf{t}) \propto \prod_{v \in B_0} s_v(\mathbf{t})$$

where $s_v(\mathbf{t}) \in \{s_1, \ldots, s_4\}$ depending on the direction of the unique outgoing edge from v in $\mathbf{t}$.

Wilson's algorithm

The branch connecting z to ∂D is LERW for the random walk on B_0 with jump probabilities $(s_i)_{i=1}^4$.

The random walk itself converges to BM with drift Δ,

$$\Delta = \frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3)$$

But what is the scaling limit of LERW?

Connection with massive SLE_2

Suppose

$$c_1 + c_3 = c_2 + c_4 = 0$$

Theorem 1 (B.–Haunschmid)

Let $z \in \Omega$. Let γ^δ = path in Temperleyan tree to $\partial\Omega$, Y_δ = endpoint. Then conditionally on $Y_\delta = y_\delta$,

$$\gamma^\delta \to \mathrm{mSLE}_2,$$

where mass $m = \|\Delta\|$.

Massive SLE_2

Consider random walk killed with probability $m^2\delta^2$ at each step.

Condition to leave Ω without dying. What is scaling limit of LERW?

Theorem (Makarov–Smirnov (2009), Chelkak-Wan (2019))

massive LERW converges to "massive SLE_2"

Described by Loewner's equation with driving function:

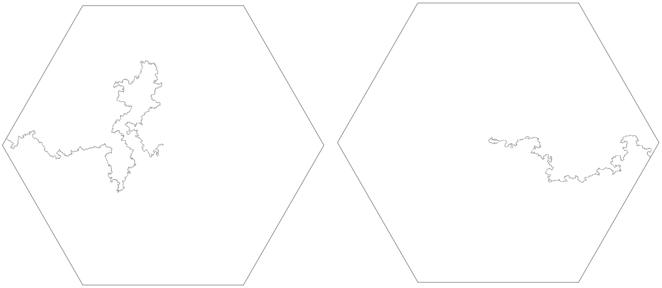
$$d\xi_t = \sqrt{2} dB_t + 2\lambda_t dt;$$

with

$$\lambda_t = \frac{\partial}{\partial w} \log \frac{P^{(m)}_{\Omega_t}(z,w)}{P^{(0)}_{\Omega_t}(z,w)}\Bigg|_{w=\gamma(t)}$$

[$m = 0$: *Lawler–Schramm–Werner 2002*]

Additional remarks



- Unconditional convergence also holds, then global Radon–Nikodym derivative:

$$\frac{d\mathbb{P}}{d\,\mathrm{mSLE}_2}(\gamma) = \exp(2\langle Y - z, \Delta\rangle)$$

where $Y =$ exit point.

- Exact same statement for hexagonal lattice $a_i = 1 + c_i\delta$,

$$\Delta = \frac{1}{3}(c_1 + c_2\tau + c_3\tau^2).$$

Convergence of height function

Corollary (B.–Haunschmid)

The Temperleyan tree $\mathcal{T}_\delta$ has a scaling limit (in Schramm topology); the limit law depends only on Δ and so is the same for hexagonal and square lattice cases.

Proof: Wilson's algorithm.

Corollary (B.–Haunschmid)

The height function of near-critical dimers in Temperleyan domains converge to the same scaling limit.

Proof: "*imaginary geometry approach*" by B.–Laslier–Ray (2020–2022).

Conformal covariance

Conformal covariance:

Image under conformal map preserved, up to power α of derivative of conformal map.

($\alpha = 0$ means conformal invariance.)

This requires allowing for general **vector field** $\Delta : \Omega \to \mathbb{R}^2 \equiv \mathbb{C}$.

Generalised near-critical dimers

At each point $z \in B_0$, assign weights $s_i = 1 + c_i\delta$, with $c_1 + c_3 = 0, c_2 + c_4 = 0$,

$$\frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3) = \Delta$$

Any drift vector can be encoded in this way.

Conformal covariance

Theorem 3. (B.–Haunschmid)

The loop-erased random walk has a scaling limit.
Hence the height function has a scaling limit, call it $h^{(\Delta);\Omega}$.

Theorem 4. (B.–Haunschmid)

Let $\phi : \tilde{\Omega} \to \Omega$ be a conformal map (with bounded derivative). In law,

$$h^{(\Delta);\Omega} \circ \phi = h^{(\tilde{\Delta});\tilde{\Omega}}$$

where at a point $w \in \tilde{\Omega}$,

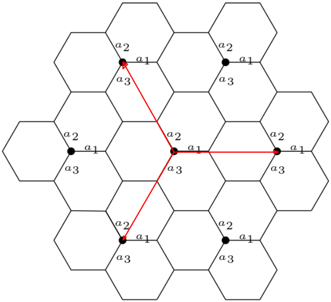
$$\tilde{\Delta}(w) = \overline{\phi'(w)} \cdot \Delta(\phi(w)).$$

Discrete Girsanov on triangular lattice $\mathbb{T}$.

Directed triangular lattice $\mathbb{T}$

if $\tau = e^{2i\pi/3}$,

$$\mathbb{Q}(x, x+\tau^{k-1}) = \frac{e^{\alpha_k}}{a}.$$



Define $\beta(v) > 0$ by

$$\exp(-\beta(v)^2) = (a/3)^{-3} \prod_{k=1}^{3} e^{\alpha_k},$$

well defined by AM-GM.

Discrete Girsanov on triangular lattice $\mathbb{T}$.

Define a vector $\alpha(v)$ at every vertex v in the graph,

$$\alpha = \alpha_1 + \alpha_2\tau + \alpha_3\tau^2,$$

Lemma

Fix any lattice path $\gamma = (x_0, \ldots, x_n)$ on $\mathbb{T}$.

$$\frac{\mathbb{Q}}{\mathbb{P}}(\gamma) = \exp(M_n - \tfrac{1}{2}V_n)$$

where $M_n = \frac{2}{3}\sum_{s=0}^{n-1}\langle\alpha(x_s), \mathrm{d}x_s\rangle$; and $V_n = \frac{2}{3}\sum_{s=0}^{n-1}\beta(x_s)^2$.

Discrete analogue of

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^t \Delta(X_s)\cdot \mathrm{d}X_s - \frac{1}{2}\int_0^t \|\Delta(X_s)\|^2 \mathrm{d}s\right).$$

Corollary (constant drift case)

$\mathbb{Q}_x(\cdot|x_n = y)$ is the same as a massive walk conditioned to survive up to time n and $X_n = y$.

Proof.

At each v, write $n_i = n_i(v) =$ number of times path goes in direction $1, \tau, \tau^2$.

$$\mathbb{Q}_x(\gamma) = \prod_v \prod_{i=1}^{3} (\frac{e^{\alpha_i}}{a})^{n_i}$$

$$= 3^{-n} \prod_v \left[((a/3)^{-3} \prod_{i=1}^{3} (e^{\alpha_i})^{\frac{n_1+n_2+n_3}{3}} \prod_{i=1}^{3} (e^{\alpha_i})^{n_i - \frac{n_1+n_2+n_3}{3}} \right]$$

$$=3^{-n} \prod_v e^{-\beta(v)^2 \frac{n_1+n_2+n_3}{3}} \exp\left(\sum_{i=1}^{3} \alpha_i (n_i - \frac{n_1+n_2+n_3}{3}) \right)$$

$$=3^{-n} e^{-\frac{1}{2}V_n} \exp\left(\sum_v \alpha_1 (\frac{2n_1-n_2-n_3}{3} + \alpha_2 (\frac{2n_2-n_1-n_3}{3}) + \alpha_3 (\frac{2n_3-n_1-n_2}{3}) \right)$$

$$=3^{-n} e^{-\frac{1}{2}V_n} \exp\left(\frac{2}{3} \sum_v \langle \alpha_1 + \alpha_2\tau + \alpha_3\tau^2, n_1 + n_2\tau + n_3\tau^2 \rangle \right)$$

$$=3^{-n} e^{-\frac{1}{2}V_n} \exp\left(\frac{2}{3} \sum_{s=0}^{n-1} \langle \alpha(x_s), \mathrm{d}x_s \rangle \right).$$

Open Problems

- Balanced condition $c_1 + c_3 = c_2 + c_4 = 0$. Is this necessary? (cf. "Loop-Erased BM")
- Is $h^{(\Delta);\Omega}$ absolutely continuous with respect to GFF?
- Coleman correspondence: massive free fermions $\leftrightarrow$ Sine-Gordon. Is there a connection?
- Bosonisation and Ising model?
- Near-critical theory for isoradial graphs?