The Determination of the Variance of Local Grammage in Random Fibre Networks by the Line Method

W.W. SAMPSON

An approximate technique known as the line method, used to determine the variance of local grammage in random fibre networks, was presented by Corte and Dodson in 1969. In their original article, and in several subsequent publications, the derived equations have been published incorrectly. Here, these equations have been rederived and are stated correctly. The use of the line method to account for the influence of fibre length distribution is also presented. Comparison is made with the established analytic determination of the fractional between zones variance derived by Dodson. Agreement between the techniques is excellent.

INTRODUCTION

The distribution of mass in paper is often compared with that determined analytically for random fibre networks given a knowledge of fibre width, coarseness and length. The variance of local grammage in random fibre networks is determined via the use of numerical integration to calculate the fractional between-zones variance. This property of random networks weights the variance observed at points, allowing the variance for finite zones to be calculated, and was derived by Dodson [1, 2] and is described in detail in [3]. The wavelength power spectrum for random networks is also used widely and was derived by Haglund et al. [4] and its use is discussed in [5]. Both techniques involve numerical integration of expressions including fibre dimensions and inspection zone size as parameters. The structures of real sheets are influenced, of course, by highly stochastic processes such as fibre orientation, flocculation and preferential drainage effects; for a recent review discussing these and the use of random network models as reference structures see [6].

An alternative approach to determining the variance of local grammage in random fibre networks was devised originally by E.H. Lloyd and reported by Corte and Dodson [7] using a technique known as the line method. The technique yields expressions that allow determination of the mass distribution without knowledge of fibre width and without further integration. Introducing a parameter \( K \) to account for autocorrelation, Corte and Dodson [7] observed that, for square inspection zones of side \( x \), the variance of local grammage in a random network of mean grammage \( \bar{\beta} \) formed from fibres with length \( \lambda \) and coarseness \( \delta \) is given by

\[
\sigma_x^2(\bar{\beta}) = \frac{\lambda \delta}{x^2} \bar{\beta} K
\]

(1)

where \( 0 \leq K \leq 1 \) and is a function of fibre geometry and the inspection zone size, \( x \). It follows directly that the coefficient of variation of local grammage \( CV_x(\bar{\beta}) \) is given by

\[
CV_x(\bar{\beta}) = \frac{1}{\bar{\beta}} \sqrt{\frac{\lambda \delta}{x^2} K}
\]

(2)

Derivation of expressions for \( K \) involves trigonometric determination of the overlapping distance of random lines of length \( \lambda \) and angle \( \theta \) with centres occurring within square zones of side \( x \). Three different functions determine \( K \), each applying to a given range of \( \lambda \) as a multiple of \( x \). In the original article [7], the equations for \( K \) to be applied over the range \( x \leq \lambda \leq \sqrt{2}x \) was published incorrectly. The equations have been published several times in different forms and the expression for the range \( x \leq \lambda \leq \sqrt{2}x \) is incorrect in every publication seen by this author (see, e.g. [8, pp. 207–208]; [9–11], and a widely circulated translation of [7]). It should be noted that these errors are not simply restatements of the original incorrect expression but each yields a different and incorrect value of \( K \) on application. The full derivation was not given in [7] but brief instructions to determine the relevant integrals were. Accordingly, these integrals have been recalculated and the correct equations for \( K \) are presented here; the parameter has been tabulated also for the typically applicable range of fibre length and inspection zone sizes.

CORRECTED EQUATIONS

Calculation of \( K \), following the instructions given in [7] for determining the integrals yields, on simplification, Eq. (3). Note that...
Equation (3)

\[
K = \frac{1}{\pi} \left( \frac{x}{\lambda} \right) \log \left( \frac{x}{\lambda} \right) - \log \left( 1 - \frac{1}{2} \frac{x^2}{\lambda^2} \right) + \arcsin \left( \frac{x}{\lambda} \right) \left( \frac{x^2}{3\lambda^2} \right) - \frac{1}{6\pi} \frac{\lambda^2}{x^2} + \frac{4}{3\pi} \frac{\lambda}{x} - \frac{1}{3\pi} \frac{x^2}{\lambda^2} - \frac{2}{\pi} - 1
\]

for \( \lambda \leq x \)

\[
K = \frac{1}{\pi} \left( \frac{x}{\lambda} \right) \left( \frac{1}{3} \left( 1 - \frac{x}{\sqrt{2}} \right) + \log \left( 1 + \frac{x}{\sqrt{2}} \right) \right) - \frac{x^2}{\lambda^2}
\]

for \( \lambda \geq \sqrt{2}x \)

These differ from those given in [7,8,10,11] only in the second range, i.e. for \( x \leq \lambda \leq \sqrt{2}x \).

There is no doubt that Eq. (3) was derived correctly by Corte and Dodson [7] since it has been plotted correctly as a function of \( \lambda \) and \( x \) at least twice [8,11]. The function is plotted in Fig. 1 against \( \lambda x \); for \( \lambda x \) less than 0.1, the value of \( K \) continues to become asymptotic to 1.

Tables I and II give the value of \( K \) to three decimal places for the range of \( \lambda x \) and \( x/\lambda \) typically used in the analysis of paper.

**COMPARISON WITH FRACTIONAL BETWEEN-ZONES VARIANCE**

The variance of local grammage in a random fibre network is given by Dodson [1–3] as

\[
\sigma_i^2(\beta) = \sigma^2(\beta) \bar{\beta}
\]

(4)

\[
\sigma_i^2(\beta) = \frac{\delta \beta}{\omega} \bar{\beta}
\]

(5)

where \( \sigma_i^2(\beta) \) is the variance at points, \( \omega \) is the fibre width, and \( \bar{\beta} \) is the fractional between-zones variance. Expressions for the determination of \( \bar{\beta} \) were determined analytically and are given in [1–3]; the parameter is tabulated in [3,12] for fibres of width \( \omega = 20 \mu m \) over a range of fibre lengths and inspection zone sizes. It turns out that \( \bar{\beta} \) is very nearly proportional to fibre width over the typical range of application for paper.

Comparison of Eq. (1) with Eq. (5) yields

\[
K = \frac{x^2}{\lambda} \frac{\bar{\beta}}{\omega}
\]

(6)

We observe, therefore, that \( K \) is a weighted form of the fractional between-zones variance \( \bar{\beta} \), the weighting being given by the ratio of the area of a square inspection zone to that of a rectangular fibre.

The line method is compared with the analytic formula in Fig. 2 for fibres of width 30 \( \mu m \). Agreement between the two methods is excellent; the greatest error occurs for small \( x \) and is less than 0.25% for the data plotted. Note that the line method makes implicit use of the proportionality between \( \bar{\beta} \) and \( \omega \) discussed above. Corte and Dodson pointed out that the importance of the agreement between the two techniques is that it allows the fractional between-zones variance to be expressed in a form independent of fibre width such that

\[
\bar{\beta}_i = \frac{\bar{\beta}}{\omega} = K \frac{\lambda}{x^2}
\]

(7)

where \( \lambda_i \) and, since the variance contributions of each length fraction are additive, the variance of local grammage is given by

\[
\sigma_i^2(\bar{\beta}) = \frac{\delta \beta}{x^2} \lambda K
\]

\[
\sigma_i^2(\bar{\beta}) = \frac{\delta \beta}{x^2} \sum m_i \lambda_i K_i
\]

(8)

where \( m_i \) is the mass fraction of fibres with class centre \( \lambda_i \); the treatment assumes uniform fibre width. Note that Eq. (8) is previously unstated and extends the use of the line method.

Equation (2) may be rewritten as

\[
CV_i(\bar{\beta}) = \frac{\sqrt{m_i} \sqrt{K}}{\bar{\beta}}
\]

(9)

where \( m_i \) is the mass of a fibre. Thus the coefficient of variation of local grammage for...
networks formed at a given mean grammage from a given furnish is determined by the quotient \( \frac{K}{x} \); this is plotted against \( \frac{1}{x} \) in Fig. 3. The effect of doubling fibre length is to reduce \( \frac{K}{x} \) by up to about 25% depending on the inspection zone size; thus we expect a plot of \( CV_x(\frac{1}{x}) \) against \( mf_{ib} \) to be approximately linear for random networks of fibres within the range of dimensions typically found in paper.

**ACKNOWLEDGEMENTS**

The author wishes to thank Professor Richard Kerekes of the University of British Columbia, Vancouver, BC, Canada and Professor Kit Dodson of UMIST, Manchester, UK for bringing the line method to his attention.

**REFERENCES**