The Determination of the Variance of Local Grammage in Random Fibre Networks by the Line Method

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An approximate technique known as the line method, used to determine the variance of local grammage in random fibre networks, was presented by Corte and Dodson in 1969. In their original article, and in several subsequent publications, the derived equations have been published incorrectly. Here, these equations have been rederived following Corte and Dodson and are stated correctly. The use of the line method to account for the influence of fibre length distribution is also presented. Comparison is made with the established analytic determination of the fractional between zones variance derived by Dodson. Agreement between the techniques is excellent.

Une méthode approximative employée pour déterminer la variation du grammage local dans des réseaux aléatoires de fibres a été présentée par Corte et Dodson en 1969. Lors de la première publication de cet article et de plusieurs autres par la suite, il y avait une erreur dans les équations dérivées. Nous avons ici dérivé à nouveau ces équations selon Corte et Dodson et elles sont indiquées de la manière appropriée. Nous indiquons aussi comment utiliser la méthode approximative en tenant compte de l'influence de la répartition des fibres selon leur longueur. Nous comparons la détermination analytique de la variance fractionnaire entre les zones dérivée établie par Dodson. La concordance entre les méthodes est excellente.

INTRODUCTION

The distribution of mass in paper is often compared with that determined analytically for random fibre networks given a knowledge of fibre width, coarseness and length. The variance of local grammage in random fibre networks is determined via the use of numerical integration to calculate the fractional between-zones variance. This property of random networks weights the variance observed at points, allowing the variance for finite zones to be calculated, and was derived by Dodson [1,2] and is described in detail in [3]. The wavelength power spectrum for random networks is also used widely and was derived by Haglund et al. [4] and its use is discussed in [5]. Both techniques involve numerical integration of expressions including fibre dimensions and inspection zone size as parameters. The structures of real sheets are influenced, of course, by highly stochastic processes such as fibre orientation, flocculation and preferential drainage effects; for a recent review discussing these and the use of random network models as reference structures see [6].

An alternative approach to determining the variance of local grammage in random fibre networks was devised originally by E.H. Lloyd and reported by Corte and Dodson [7] using a technique known as the line method. The technique yields expressions that allow determination of the mass distribution without knowledge of fibre width and without further integration. Introducing a parameter K to account for autocorrelation, Corte and Dodson [7] observed that, for square inspection zones of side x, the variance of local grammage in a random network of mean grammage $\bar{\beta}$ formed from fibres with length λ and coarseness δ is given by

$$\sigma_x^2(\tilde{\beta}) = \frac{\lambda\delta}{x^2}\bar{\beta}K \qquad (1)$$

where $0 \le K \le 1$ and is a function of fibre geometry and the inspection zone size, x. It follows directly that the coefficient of variation of local grammage $CV_x(\tilde{\beta})$ is given by

$$CV_x(\tilde{\beta}) = \frac{1}{x}\sqrt{\frac{\lambda\delta}{\bar{\beta}}K} \qquad (2)$$

Derivation of expressions for K involves trigonometric determination of the overlapping distance of random lines of length λ and angle θ with centres occurring within square zones of side x. Three different functions determine K, each applying to a given range of λ as a multiple of x. In the original article [7], the equation for K to be applied over the range $x \le \lambda \le \sqrt{2}x$ was published incorrectly. The equations have been published several times in different forms and the expression for the range $x \le \lambda \le \sqrt{2}x$ is incorrect in every publication seen by this author (see, e.g. [8, pp. 207–208]; [9–11], and a widely circulated translation of [7]). It should be noted that these errors are not simply restatements of the original incorrect expression but each yields a different and incorrect value of K on application. The full derivation was not given in [7] but brief instructions to determine the relevant integrals were. Accordingly, these integrals have been recalculated and the correct equations for K are presented here; the parameter has been tabulated also for the typically applicable range of fibre length and inspection zone sizes.

CORRECTED EQUATIONS

Calculation of K, following the instructions given in [7] for determining the integrals yields, on simplification, Eq. (3). Note that

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Equation (3)

$$K\begin{cases} = 1 - \dfrac{4}{3\pi}\dfrac{\lambda}{x} + \dfrac{1}{6\pi}\dfrac{\lambda^2}{x^2} & \text{for } \lambda \le x \\[2ex] = \dfrac{4}{\pi}\left(\dfrac{x}{\lambda}\left(\log\left(\dfrac{x}{\lambda}\right) - \log\left(1 - \sqrt{1 - \dfrac{x^2}{\lambda^2}}\right)\right) + \arcsin\left(\dfrac{x}{\lambda}\right) + \dfrac{\left(\lambda^2 - x^2\right)^{\frac{3}{2}}}{3x\lambda^2}\right) - \dfrac{1}{6\pi}\dfrac{\lambda^2}{x^2} + \dfrac{4}{3\pi}\dfrac{x}{\lambda} - \dfrac{1}{3\pi}\dfrac{x^2}{\lambda^2} - \dfrac{2}{\pi} - 1 & \text{for } x \le \lambda \le \sqrt{2}x \\[2ex] = \dfrac{1}{\pi}\left(4\dfrac{x}{\lambda}\left(\dfrac{1}{3}\left(1 - \sqrt{2}\right) + \log\left(1 + \sqrt{2}\right)\right) - \dfrac{x^2}{\lambda^2}\right) & \text{for } \lambda \ge \sqrt{2}x \end{cases}$$

these differ from those given in [7,8,10,11] only in the second range, i.e. for $x \le \lambda \le \sqrt{2}x$.

There is no doubt that Eq. (3) was derived correctly by Corte and Dodson [7] since it has been plotted correctly as a function of λ and x at least twice [8,11]. The function is plotted in Fig. 1 against λ/x; for λ/x less than 0.1, the value of K continues to become asymptotic to 1. Tables I and II give the value of K to three decimal places for the range of λ/x and x/λ typically used in the analysis of paper.

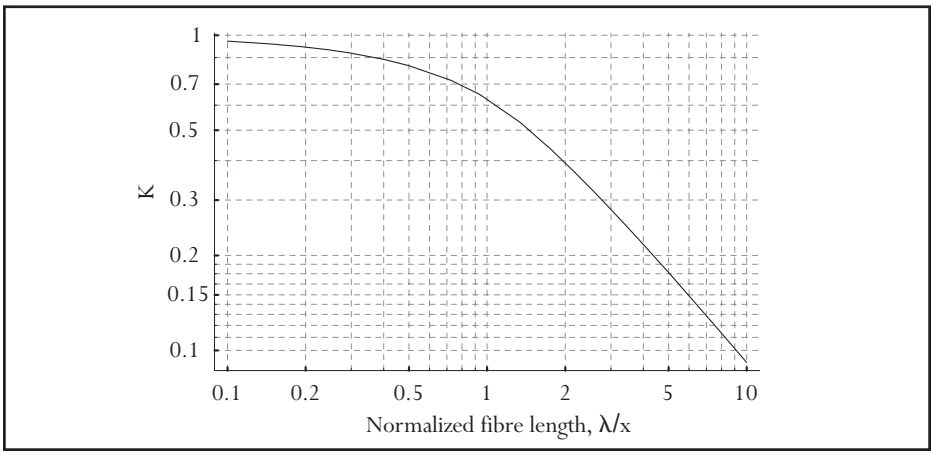

Fig. 1. Parameter *K* as a function of fibre length and inspection zone size.

COMPARISON WITH FRACTIONAL BETWEEN-ZONES VARIANCE

The variance of local grammage in a random fibre network is given by Dodson [1–3] as

$$\sigma_x^2(\tilde{\beta}) = \sigma^2(\beta)\tilde{\rho} \tag{4}$$

$$= \frac{\delta}{\omega}\bar{\beta}\tilde{\rho} \tag{5}$$

where $\sigma^2(\beta)$ is the variance at points, ω is the fibre width, and $\tilde{\rho}$ is the fractional between-zones variance. Expressions for the determination of $\tilde{\rho}$ were determined analytically and are given in [1–3]; the parameter is tabulated in [3,12] for fibres of width $\omega = 20$ µm over a range of fibre lengths and inspection zone sizes. It turns out that $\tilde{\rho}$ is very nearly proportional to fibre width over the typical range of application for paper.

Comparison of Eq. (1) with Eq. (5) yields

$$K = \frac{x^2}{\lambda}\frac{\tilde{\rho}}{\omega} \tag{6}$$

We observe, therefore, that K is a weighted form of the fractional between-zones variance $\tilde{\rho}$, the weighting being given by the ratio of the area of a square inspection zone to that of a rectangular fibre.

The line method is compared with the analytic formula in Fig. 2 for fibres of width 30 µm. Agreement between the two methods is excellent; the greatest error occurs for small x and λ and is less than 0.25% for the data plotted. Note that the line method makes implicit use of the proportionality between $\tilde{\rho}$ and ω discussed above. Corte and Dodson pointed out that the importance of the agreement between the two techniques is that it allows the fractional between-zones variance to be expressed in a form independent of fibre width such that

$$\tilde{\rho}^* = \frac{\tilde{\rho}}{\omega} = K\frac{\lambda}{x^2} \tag{7}$$

Where there is a distribution of lengths, K_i should be determined for each fibre length class λ_i and, since the variance contributions of each length fraction are additive, the variance of local grammage is given by

$$\sigma_x^2(\bar{\beta}) = \frac{\delta}{x^2}\bar{\beta}\lambda K$$

$$= \frac{\delta}{x^2}\bar{\beta}\sum_i^n m_i\lambda_i K_i \tag{8}$$

where m_i is the mass fraction of fibres with class centre λ_i; the treatment assumes uniform fibre width. Note that Eq. (8) is previously unstated and extends the use of the line method.

Equation (2) may be rewritten as

$$CV_x(\tilde{\beta}) = \sqrt{\frac{m_{fib}}{\bar{\beta}}}\frac{\sqrt{K}}{x} \tag{9}$$

where m_{fib} is the mass of a fibre. Thus the coefficient of variation of local grammage for

TABLE I
PARAMETER *K* AS GIVEN BY EQ. (3)
FOR λ/x BETWEEN 0.01 AND 100

λ/x	K	λ/x	K	λ/x	K	λ/x	K
0.01	0.996	0.1	0.958	1	0.629	10	0.992
0.02	0.992	0.2	0.917	2	0.394	20	0.047
0.03	0.987	0.3	0.877	3	0.280	30	0.031
0.04	0.983	0.4	0.839	4	0.217	40	0.024
0.05	0.979	0.5	0.801	5	0.177	50	0.019
0.06	0.975	0.6	0.764	6	0.149	60	0.016
0.07	0.971	0.7	0.729	7	0.129	70	0.014
0.08	0.966	0.8	0.694	8	0.113	80	0.012
0.09	0.962	0.9	0.661	9	0.101	90	0.011
						100	0.009

TABLE II
PARAMETER *K* AS GIVEN BY EQ. (3)
FOR x/λ BETWEEN 0.01 AND 100

x/λ	K	x/λ	K	x/λ	K	x/λ	K
0.01	0.009	0.1	0.092	1	0.629	10	0.958
0.02	0.019	0.2	0.177	2	0.801	20	0.979
0.03	0.028	0.3	0.255	3	0.864	30	0.986
0.04	0.037	0.4	0.328	4	0.897	40	0.989
0.05	0.047	0.5	0.394	5	0.917	50	0.992
0.06	0.056	0.6	0.453	6	0.931	60	0.993
0.07	0.065	0.7	0.507	7	0.940	70	0.994
0.08	0.074	0.8	0.553	8	0.948	80	0.995
0.09	0.083	0.9	0.594	9	0.953	90	0.995
						100	0.996

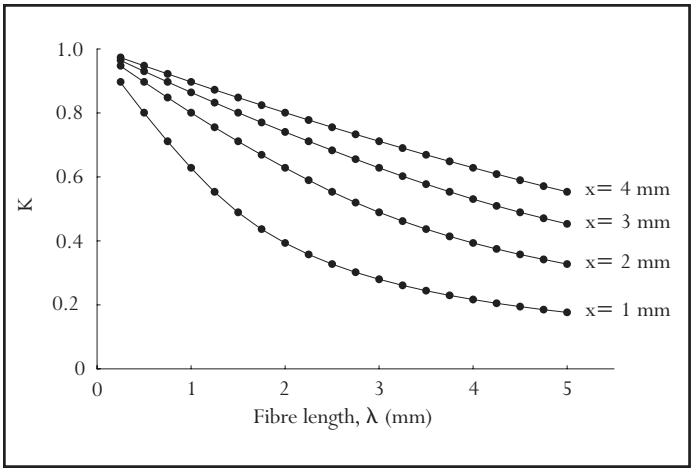

Fig. 2. Comparison of *K* determined by Eqs. (3) and (6). Points calculated using Eq. (3), lines calculated using Eq. (6); all curves calculated for ω = 30 μm. Agreement between the two methods is excellent.

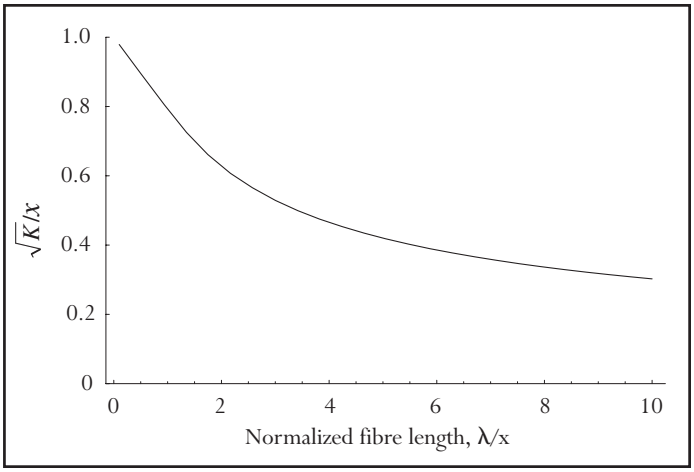

Fig. 3. Coefficient $\sqrt{K}/x$ as a function of fibre length and inspection zone size.

networks formed at a given mean grammage from a given furnish is determined by the quotient $\sqrt{K}/x$; this is plotted against λ/x in Fig. 3. The effect of doubling fibre length is to reduce $\sqrt{K}/x$ by up to about 25% depending on the inspection zone size; thus we expect a plot of $CV_x(\tilde{\beta})$ against $\sqrt{m_{fib}}$ to be approximately linear for random networks of fibres within the range of dimensions typically found in paper.

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