The Effect of Fibre Length Distribution on Suspension Crowding

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Expressions are developed to extend the concept of fibre crowding, developed by Kerekes et al., to account for the distribution of fibre lengths which are often lognormal. The theory shows that the mean crowding number for fibres with a distribution of lengths is higher than that for fibres of uniform length equal to the mean by a factor dependent only on the coefficient of variation of fibre length. For a lognormal distribution of fibre lengths, the distribution of crowding numbers itself resembles a lognormal distribution and this is consistent with measurements of floc size in the literature. The theory allows also determination of the fraction of fibres in suspension with a crowding number greater or less than a given value and provides a new basis for determination of the critical concentration of a given fibre type. The results are of use in studies of flocculation and paper forming dynamics.

On élabore une nouvelle expression afin d'élargir le concept du nombre de fibres mis au point par Kerekes et ses collaborateurs afin de tenir compte de la distribution des longueurs de fibres qui, bien souvent, est log-normale. Selon la théorie, le nombre moyen de fibres de longueurs inégales est supérieur au nombre de fibres de longueur uniforme égale à la moyenne et, ceci, par un facteur qui ne dépend que du coefficient de variation des longueurs de fibres. Dans le cas d'une distribution log-normale des longueurs de fibres, la répartition elle-même des nombres s'apparente à une distribution log-normale ; ce phénomène est donc conforme aux calculs de la taille du floculat indiquée dans la documentation. Cette théorie permet également de calculer la fraction de fibres dans la suspension avec un nombre de fibres supérieur ou inférieur à une valeur donnée, et fournit une nouvelle base pour le calcul de la concentration critique d'un type de fibre donné. Les résultats peuvent être utilisés dans des études sur la dynamique de la floculation et de la formation du papier.

INTRODUCTION

The concept of characterizing the flocculation propensity of a fibre suspension by a crowding factor was developed by Kerekes et al. [1,2] from the work of Mason [3]. The crowding factor, or crowding number, is defined as the expected number of fibres in a sphere of diameter one mean fibre length and is given by

$$n_c = \frac{2}{3}A^2C_v \tag{1}$$

$$= \frac{\pi}{6}\frac{\bar{\lambda}^2}{\delta}C_m \tag{2}$$

$$\approx \frac{\bar{\lambda}^2}{2\delta}C_m \tag{3}$$

where

A = fibre aspect ratio, defined as the ratio of mean fibre length to fibre diameter
C_v = dimensionless volumetric fibre consistency
$\bar{\lambda}$ = mean fibre length (m)
δ = fibre coarseness ($\mathrm{kg \cdot m^{-1}}$)
C_m = mass consistency ($\mathrm{kg \cdot m^{-3}}$)

The crowding number has been shown to be useful in characterizing flocculated suspensions and correlations have been found with the formation of papers formed on commercial and pilot Fourdrinier machines [4,5]. Where pulps of different geometries have been used in floc-

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culation studies, the crowding number can be used to provide the appropriate reference for comparison as in the recent work of Karema et al. [6] and Kellomäki et al. [7]. The concept of fibre crowding was recently extended by Dodson [8] to derive expressions for the expected number of contacts per fibre; this result has itself been developed to allow the variance of density in a 3D random fibre network to be expressed in terms of fibre geometry [9].

Papermaking fibres typically have a distribution of lengths and often these are well described by the lognormal distribution [10–12]; this implies that there exists also a distribution of fibre masses. We expect, therefore, that the number of fibres within the sphere of influence of a given fibre in suspension will exhibit a distribution dependent on the fibre length and mass distributions. Here we address this issue by deriving the probability density function for the local crowding number of fibres with a lognormal distribution of lengths.

THEORY

Typically, the mass consistency of a fibre suspension is determined more readily than its volumetric consistency. Also, automated fibre length analyzers typically give mean fibre coarseness as a standard output, making Eq. (2) the more easily applied of the two expressions for crowding number. Accordingly, this is the expression we shall develop first; equivalent expressions in terms of fibre diameter and volumetric consistency are given in the sequel.

The crowding number, n_c as given by Eq. (2) may be written as

$$n_c = \frac{C_m}{\overline{m}_f}\overline{V}_{sp} \tag{4}$$

where $\overline{m}_f = \overline{\lambda}\delta$ is the mean mass of a fibre and $\overline{V}_{sp}$ is the volume of a sphere of diameter one mean fibre length. If fibre lengths have a lognormal distribution, then the probability density of fibre length is given by

$$f(\lambda) = \frac{1}{\sqrt{2\pi}\lambda\sigma}\,\mathrm{e}^{-\frac{(\log(\lambda)-\mu)^2}{2\sigma^2}} \tag{5}$$

and has mean

$$\overline{\lambda} = \mathrm{e}^{\frac{\sigma^2}{2}+\mu}$$

variance,

$$\mathrm{Var}(\lambda) = (\mathrm{e}^{\sigma^2}-1)\mathrm{e}^{2\mu+\sigma^2}$$

and coefficient of variation,

$CV(\lambda) = (\mathrm{e}^{\sigma^2}-1)^{1/2}$. Parameters μ and σ may be expressed in terms of the mean and coefficient of variation of fibre length, which are readily determined:

$$\mu = \frac{1}{2}\log\left(\frac{\overline{\lambda}^2}{1+CV^2(\lambda)}\right) \tag{6}$$

$$\sigma = \sqrt{\log\left(1+CV^2(\lambda)\right)} \tag{7}$$

Crowding Number Distribution

The volume of a sphere of diameter one fibre length is $\pi\lambda^3/6$, so we have

$$\lambda(V_{sp}) = \left(\frac{6V_{sp}}{\pi}\right)^{\frac{1}{3}} \tag{8}$$

and the probability density of V_{sp} is therefore

$$g(V_{sp}) = f\left(\lambda(V_{sp})\right)\left|\frac{\mathrm{d}\lambda(V_{sp})}{\mathrm{d}V_{sp}}\right| \tag{9}$$

$$= \frac{1}{3\sqrt{2\pi}V_{sp}\sigma}\,\mathrm{e}^{-\frac{\left(\log\left(\frac{6V_{sp}}{\pi}\right)-3\mu\right)^2}{18\sigma^2}} \tag{10}$$

Letting

$$y = \frac{C_m}{m_f} = \frac{C_m}{\lambda\delta}$$

we have

$$\lambda(y) = \frac{C_m}{y\delta} \tag{11}$$

and the probability density of y is given by

$$h(y) = f\left(\lambda(y)\right)\left|\frac{\mathrm{d}\lambda(y)}{\mathrm{d}y}\right| \tag{12}$$

$$= \frac{1}{\sqrt{2\pi}y\sigma}\,\mathrm{e}^{-\frac{\left(\log\left(\frac{C_m}{y\delta}\right)-\mu\right)^2}{2\sigma^2}} \tag{13}$$

For a distribution of fibre lengths, we shall denote the crowding number n_c^*; the probability density function for n_c^* is given by

$$p(n_c^*) = \int_0^\infty \frac{1}{V_{sp}}g(V_{sp})\,h\left(\frac{n_c^*}{V_{sp}}\right).\mathrm{d}V_{sp} \tag{14}$$

Substitution of Eqs. (10) and (13) in (14) and evaluating the integral yields, on simplification,

$$p(n_c^*) = \frac{1}{2n_c^*\sqrt{5\pi}\sigma}\,\mathrm{e}^{-\frac{\left(2\mu+\log\left(\frac{\pi C_m}{6\delta n_c^*}\right)\right)^2}{20\sigma^2}} \tag{15}$$

Assuming cylindrical fibres of diameter d, the derivation of $p(n_c^*)$ using volume consistency as a basis follows that given above with $\delta \mapsto \frac{\pi d^2}{4}$ and $C_m \mapsto C_v$ such that

$$p(n_c^*) = \frac{1}{2n_c^*\sqrt{5\pi}\sigma}\,\mathrm{e}^{-\frac{\left(2\mu+\log\left(\frac{2C_v}{3d^2 n_c^*}\right)\right)^2}{20\sigma^2}} \tag{16}$$

Equations (15) and (16) are equivalent and $\int_0^\infty p(n_c^*)\cdot\mathrm{d}n_c^* = 1$

The mean, variance and coefficient of variation of the crowding number are given by,

$$\overline{n_c^*} = \frac{\pi C_m}{6\delta}\mathrm{e}^{2\mu+5\sigma^2} = \frac{2C_v}{3d^2}\mathrm{e}^{2\mu+5\sigma^2} \tag{17}$$

$$= n_c\mathrm{e}^{4\sigma^2} \tag{18}$$

$$= n_c\left(1+CV^2(\lambda)\right)^4 \tag{19}$$

$$Var(\overline{n_c^*}) = \overline{n_c^*}^2\left(\mathrm{e}^{10\sigma^2}-1\right) \tag{20}$$

$$= \overline{n_c^*}^2\left(\left(1+CV^2(\lambda)\right)^{10}-1\right) \tag{21}$$

$$CV(n_c^*) = \left(\left(1+CV^2(\lambda)\right)^{10}-1\right)^{\frac{1}{2}} \tag{22}$$

We observe from Eq. (19) that, for a given mean fibre length, the effect of a distribution of fibre lengths is to increase the crowding number from that given by Eq. (2) by a factor dependent only on the coefficient of variation of fibre length. This is illustrated in Fig. 1; a coefficient of variation of fibre length of 44% doubles the crowding number and a coefficient of variation of fibre length of 88% increases the crowding number by a factor of 10.

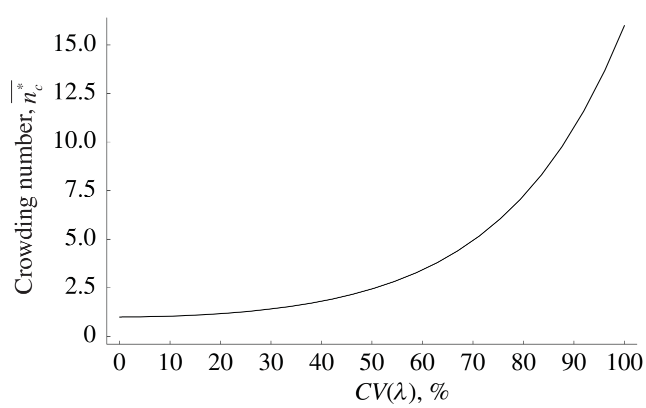

Fig. 1. The mean crowding number, $\overline{n_c^*}$ for mixed length fibres in units of the crowding number n_c for uniform fibres, at the same consistency, as a function of $CV(\lambda)$. For typical furnishes, the effective mean crowding number is more than double the nominal value calculated from the mean fibre length.

Calculations

The fibre length distributions for a TMP and a chemical softwood pulp are shown in Fig. 2. The points represent the length weighted distribution of fibre lengths as measured on a Kajaani FS-200; the lines are the result of a least squares fit of the cumulative distribution function for the lognormal distribution to the cumulative data. The cumulative plots are shown on the left and the relative frequency plots for the same data on the right. Table I provides a summary of the data as determined directly from the Kajaani and by the least squares fit. Although the fit gives a slightly higher estimate of the coefficient of variation of fibre length than that from the Kajaani, it illustrates that the use of the lognormal distribution is appropriate for these pulps. In the calculations which follow, we shall use the data from the Kajaani to characterize the pulps.

For the two pulps described here, the crowding number calculated using Eq. (2) at a typical headbox consistency of 0.75% would be 32 for the TMP and 132 for the chemical pulp; so we would expect the chemical pulp fibres to be interacting with, on average, about 4 times as many fibres in suspension than the TMP fibres. Considering the fibre length distribution and applying Eq. (19), we have, at the same consistency, a mean crowding number of 152 for the TMP and 295 for the chemical pulp. Thus, the chemical pulp fibres are, on average, interacting with about twice as many fibres than the TMP fibres; also, accounting for the length distribution increases the mean crowding number for the TMP by about five times and approximately doubles that of the chemical pulp.

Equation (19) should be applied also in the selection of consistencies for comparison of pulps in flocculation experiments. A crowding number of 60 was identified by Kerekes and Schell [2] as the lower limit where coherent flocs may form in suspension. Using the original theory, this is equivalent to a consistency, C_m of 1.42 and 0.34% for the TMP and the chemical pulp, respectively. However, considering the length distribution and using Eq. (19) indicates that a mean crowding number of 60 occurs at mass consistencies of 0.29 and 0.15% for the TMP and the chemical pulp, respectively. Plots of the probability density function for n_c^* for these two pulps are shown in Fig. 3. In Fig. 3A, the consistencies used are those required to give $\overline{n_c^*} = 60$, while in Fig. 3B the consistencies used are such that Eq. (2) gives $n_c = 60$.

TABLE I
SUMMARY OF FIBRE LENGTH AND COARSENESS DATA FOR PULPS ILLUSTRATED IN FIG. 2.

	Data from Kajaani				Data from fit		
	$\delta \times 10^7$ (kg · m^{-1})	$\bar{\lambda}$ (mm)	$Var(\lambda)$ (mm²)	$CV(\lambda)$ –	$\bar{\lambda}$ (mm)	$Var(\lambda)$ (mm²)	$CV(\lambda)$ –
Chemical	1.45	2.21	1.081	0.471	2.21	1.146	0.485
TMP	2.22	1.34	0.858	0.693	1.40	1.298	0.810

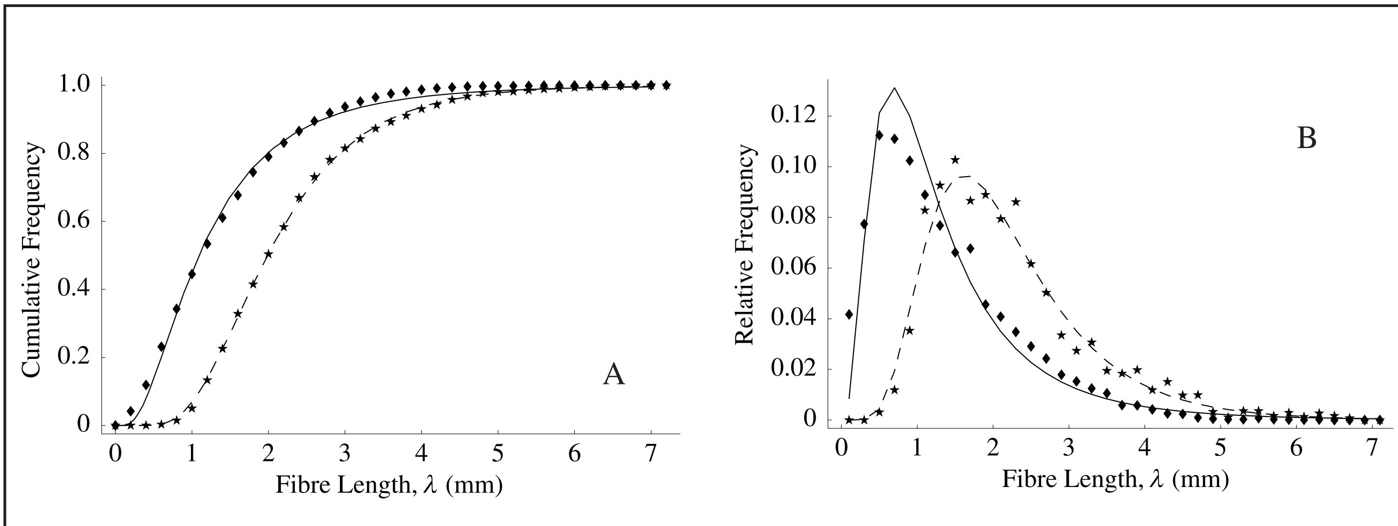

Fig. 2. Length weighted fibre length distributions for two pulps (chemical softwood – stars and broken line; TMP – diamonds and solid line).

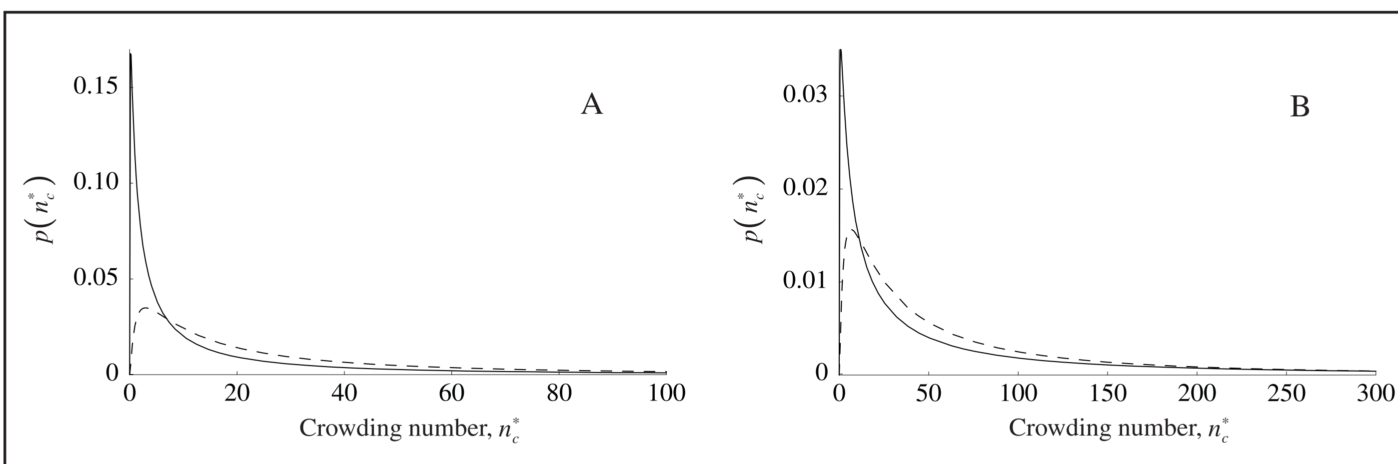

Fig. 3. Plots of probability density functions for crowding number. A) Mean crowding number by Eq. (19), $\overline{n_c^*}$ = 60; B) crowding number by original theory, n_c = 60 (chemical softwood – broken line; TMP – solid line).

We observe that, in both plots, the distribution for the chemical pulp has a longer tail. A high value of n_c^* may occur in flocs of any size, but it is reasonable to expect that high values of n_c^* will be associated with larger flocs. Thus, from the plots in Fig. 3 we may expect the chemical pulp to have more large flocs than the TMP at the same mean crowding number. We note also that the form of Eqs. (15) and (16) is similar to that of the probability density function of the lognormal distribution and the plots show curves with a positive skew and a long tail. The result is consistent with the observation of Farnood et al. [13] that the variance statistics of random structures of disks, or flocs, with a lognormal distribution of diameters, closely resemble those of paper. Also, floc size distributions measured by Karema et al. [6], Kellomäki et al. [7], and by Hourani [14,15] have a lognormal shape.

The use of the new equations in the design of flocculation experiments should allow more precise determination of the relative influence of mechanical and chemical forces and, for example, the influence of fibre surface chemistry.

Free Fibre Fraction

When the mechanism of flocculation is elastic fibre bending, as discussed by Kerekes et al. [1], a minimum of three contacts per fibre are required. Fibres in contact with less than three others in the suspension therefore may not be considered flocculated and will form the 'free fibre fraction' in the suspension. The theory allows us to determine what percentage of fibres in suspension are expected to have a crowding number greater or less than a given value. The cumulative distribution function for n_c is given by

$$q(n_c^*) = \frac{1}{2}\left(1 - \mathrm{Erf}\left[\frac{2\mu + \log\left(\frac{C_m \pi}{6\delta n_c^*}\right)}{2\sqrt{5}\sigma}\right]\right) \tag{23}$$

$$= \frac{1}{2}\left(1 - \mathrm{Erf}\left[\frac{2\mu + \log\left(\frac{2C_v}{3d^2 n_c^*}\right)}{2\sqrt{5}\sigma}\right]\right) \tag{24}$$

where Erf[x] is the error function.

If adjacent fibre contacts occur on alternate sides of a given fibre, this gives the criterion that we require, $n_c > 4$; if two adjacent contacts occur on the same side of a given fibre, we require $n_c > 5$. For the pulps described above, we have seen that a mean crowding number, $\overline{n_c^*}$ of 60 occurs at mass consistencies of 0.29 and 0.15% for the TMP and the chemical pulp, respectively. From Eq. (23) we find that, at these consistencies, approximately 35% of the TMP fibres and approximately 11% of the chemical pulp fibres have a crowding number of 4 or less and may be expected to behave essentially as a fluid.

Critical Concentration

The critical concentration of a fibre suspension, as defined by Mason [3], is the concentration where fibres can be expected to be independent in suspension and occurs at $n_c = 1$. Accounting for the fibre length distribution, $\overline{n_c^*} = 1$ occurs at

$$C_{m,crit} = \frac{6\delta}{\pi\bar{\lambda}^2}\frac{1}{\left(1 + CV^2(\lambda)\right)^4} \tag{25}$$

$$C_{v,crit} = \frac{3}{2A^2}\frac{1}{\left(1 + CV^2(\lambda)\right)^4} \tag{26}$$

For the TMP, this corresponds to a mass consistency of 0.0049% compared to 0.0236% when the original theory is used; for the chemical pulp, this corresponds to a mass consistency of 0.0025% compared to 0.0057% neglecting the distribution of fibre lengths. Also, we calculate that, at these consistencies, around 4.5% of the fibres for both pulps have a crowding number of 4 or more. In the British Standard Sheet Former, C_m = 0.171 kg · m^{-3}; at this consistency we have, for the TMP, a mean crowding number of 3.5 and about 15% of the fibres have a crowding number of 4 or more, and for the chemical pulp we have a mean crowding number of 6.7 and about 27% of the fibres have a crowding number of 4 or more. Using Eq. (2), the TMP has a crowding number of 0.7 and the chemical pulp has a crowding number of 3.

DISCUSSION

The theory presented here allows the effect of fibre length distribution on the crowding number of fibres in suspension to be determined when fibres have a lognormal distribution of lengths. Typically, the coefficient of variation of fibre length is between 40 and 80% and this range results in an increase of mean crowding number of between 1.8 and 7 times that calculated on the basis of the mean fibre length; the consequences for the critical consistency at which the mean crowding number is 1 are the same. Knowing the distribution of crowding numbers in a suspension allows determination of the fraction of fibres which experience a crowding number greater or less than a given value; this in turn allows determination of the 'free fibre fraction' in suspension. It is interesting to note the result of Dodson [12], who found that the influence of a distribution of fibre lengths on the formation of sheets was to reduce the coefficient of variation of local grammage by a few percent when compared to that of fibres of uniform length and the same mean. Our results suggest that the influence of a fibre length distrbution is to increase the nonuniformity of a suspension. The discrepancy is likely to arise from the different bases used to quantify variability. For real sheets, preferential drainage effects during sheet forming, as discussed by Sampson et al. [16] and Norman et al. [17], will contribute to the observed nonuniformity.

No account is taken in the theory of the fines fraction in a furnish, which often is undetected in automated fibre length analyzers. However, the results of Beghello [18] indicate that floc size in suspension is insensitive to the presence of fines and fillers. It would be desirable also to derive expressions for the probability density of crowding number, and hence its mean and variance, in the case of normally distributed fibre lengths which are common in blended and recycled furnishes. Our attempts at this derivation have proved unsuccessful to date; expressions for the probability density of $\overline{V_{sp}}$ and C_m/m_f have been determined, and the former is a gamma distribution. However, attempts at the integral to yield the probability density function of their product have not been successful.

CONCLUSIONS

Expressions have been derived for the probability density function, and hence the mean and variance, of crowding number for fibre suspensions with a lognormal distribution of fibre lengths. A distribution of lengths increases the mean crowding number compared to that of a suspension of uniform length equal to the mean. Also, the distribution of crowding numbers in such a suspension has a positive skew and resembles a lognormal distribution. The theory allows calculation of the fibre fraction exhibiting a crowding number greater or less than a given value and hence allows determination of the fraction of the suspension which may be considered flocculated.

NOMENCLATURE

A	Fibre aspect ratio
C_m	Mass consistency, kg · m^{-3}
$C_{m,\,crit}$	Critical mass consistency, kg · m^{-3}
C_v	Volume consistency
$C_{v,\,crit}$	Critical volume consistency
d	Fibre diameter, m
δ	Fibre coarseness, kg · m^{-1}
λ	Fibre length, m
$\bar{\lambda}$	Mean fibre length, m
m_f	Mass of a fibre, kg
$\overline{m_f}$	Mean mass of a fibre, kg
μ	Parameter characterizing lognormal distribution
n_c	Crowding number for uniform fibre lengths
n_c^*	Crowding number for a fibre length distribution
$\overline{n_c^*}$	Mean crowding number for a fibre length distribution
σ	Parameter characterizing lognormal distribution
V_{sp}	Volume of a sphere of diameter λ, m^3
$\overline{V_{sp}}$	Mean volume of a sphere of diameter λ, m^3
y	Ratio of consistency to fibre mass, m^{-3}

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