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Modeling a Class of Stochastic Porous Media

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Abstract—This note extends E. H. Lloyd's model of pore structure in random fibre networks to a large class of stochastic fibre networks containing the random model as a special case. The key to the generalization is the substitution of a family of gamma distributions for the negative exponential family used for intercrossing distances on fibres. This allows closed expressions to be obtained for the variance and mean of the equivalent pore size distributions in a planar array of line elements representing fibres. The analytical details have been made available in a *Mathematica* notebook, via the World Wide Web. The result has application in modeling the forming of nonwoven textiles and paper from fibre suspensions, and in modeling their void structures and transmission of fluids.

Keywords—Stochastic, Porous media, Fibre network, Pore statistics.

1. INTRODUCTION

In a planar network of random lines, the mean number of sides per polygon is four, so Corte and Lloyd [1,2] estimated the pore size distribution as the product of two negative exponential distributions, which are known to give a good approximation to the intercrossing lengths in a random network [3]. Here we repeat this analysis using the gamma distribution as a generalisation of intercrossing length distributions for more general stochastic fibre networks; then the negative exponential distribution is a special case. We used the computer algebra package *Mathematica* for the calculus and graphics; the code for our calculations is available from the authors or via the World Wide Web [4].

In applications to nonwoven textiles and paper, a random arrangement of fibres is a common target structure. However, in commercial production, such a degree of dispersion is hard to achieve because of a tendency for the fibres to clump together, or 'flocculate.' So the random case becomes an 'upper bound on uniformity.' The new models provide a family of commercially realizable structures.

2. PORE SIZE DISTRIBUTIONS

The gamma distribution has a probability density function given by

$$f(x) = \frac{b^k}{\Gamma(k)} x^{k-1} e^{-bx}, \tag{1}$$

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with mean $\bar{x} = k/b$ and variance $\mathrm{Var}(x) = k/b^2$. The negative exponential distribution is a gamma distribution with $k = 1$. Thus, k and b are the parameters which represent the departure from the random case, for which $k = 1$ and $b_{\mathrm{rand}} = 1/\bar{x}$. For clumped or 'flocculated' stochastic fibre networks, we expect $k/b^2 > 1/{b_{\mathrm{rand}}}^2$ and to increase with increasing fibre clumping. For disperse stochastic fibre networks, we expect $b^2 \gg k$ such that $k/b^2 < 1/{b_{\mathrm{rand}}}^2$ and to decrease with increasing uniformity.

We consider the product of two independent identical gamma distributions $f(x)$ and $f(y)$ such that $xy = a$, where a is the area of a rectangular pore. The probability density of a will be given by

$$p(a) = \int_0^\infty \frac{1}{x} f(x) f\left(\frac{a}{x}\right) dx. \tag{2}$$

Evaluation of the integral in equation (2) gives us

$$p(a) = \frac{2a^{k-1}\, b^{2k}\, K_0(z)}{\Gamma(k)^2}, \qquad \text{where } z = 2b\sqrt{a}, \tag{3}$$

and $K_0(z)$ is the zeroth-order modified Bessel function of the second kind. The distribution given by equation (3) has mean $\bar{a} = k^2/b^2$ and variance $\mathrm{Var}(a) = k^2(1+2k)/b^4$.

Following Corte and Lloyd, we define an equivalent pore radius r which is given by $a = \pi r^2$. The probability of finding an equivalent pore radius $r_1 \le r \le r_2$ is given by

$$\int_{\pi {r_1}^2}^{\pi {r_2}^2} p(a)\, da = \int_{r_1}^{r_2} p\left(\pi r^2\right) 2\pi r\, dr. \tag{4}$$

So the probability density function for equivalent pore radii is

$$q(r) = 2\pi r\, p\left(\pi r^2\right), \tag{5}$$

which gives us

$$q(r) = \frac{4b^{2k}\, \pi^k\, r^{2k-1}\, K_0(z)}{\Gamma(k)^2}, \qquad \text{where } z = 2br\sqrt{\pi}, \tag{6}$$

and $\int_0^\infty q(r)\, dr = 1$. The mean and variance of $q(r)$ are given by

$$\bar{r} = \frac{\Gamma(k+1/2)^2}{b\sqrt{\pi}\,\Gamma(k)^2} \tag{7}$$

and

$$\mathrm{Var}(r) = \frac{k^2\,\Gamma(k)^4 - \Gamma(k+1/2)^4}{b^2\pi\Gamma(k)^4} = \bar{r}^2\left(\frac{k^2\,\Gamma(k)^4}{\Gamma(k+1/2)^4} - 1\right). \tag{8}$$

For a random network, $k = 1$ and the distribution of pore radii has mean $\bar{r} = \sqrt{\pi}/4b$ and variance $\mathrm{Var}(r) = (1/b^2)(1/\pi - \pi/16)$ in agreement with Corte and Lloyd [1,2]. Equation (8) gives the generalization.

Corte and Lloyd used a multiplanar model of paper with layers of capillaries of distributed radii. Then, for a fluid flow proportional to r^κ in a capillary of radius r, the mean effective radius averaged over m layers is

$$\overline{r_{\mathrm{eff}}}(m,\kappa) = \left(\frac{m}{\sum_{i=1}^m r_i^{-\kappa}}\right)^{1/\kappa}. \tag{9}$$

Typical flow regimes that may be used are: laminar or Poiseuille $\kappa = 4$; molecular or Knudsen $\kappa = 3$; turbulent $\kappa = 2$; capillary $\kappa = 1/2$. The same procedure may be used for the new family of pore radii distributions.

The derivation of pore size distribution for random networks allows a relatively simple determination of the relationship between the mean pore size and the standard deviation. This relationship is linear in the theoretical case, and approximately linear for experimental measurements [1,2]. A property of the gamma distribution, and of the new distribution $q(r)$, is that a given value of the mean may be associated with an infinite number of variances. Some plots are provided in [4], together with results of comparisons with measured pore size distributions in paper, which will be reported in detail elsewhere.

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