1 Introduction

Many systems and processes are classified as being stochastic in nature. The classification applies to those processes and systems that are influenced by random processes and are therefore subject to a degree of uncertainty in their characterisation.

Perhaps the simplest examples of stochastic systems are the tossing of a coin and the rolling of a die. When the coin is tossed a sufficient number of times, we expect the outcome to be 'heads' 50% of the time, and 'tails' 50% of the time. The system is stochastic because each time the coin is tossed, we know only the possible number of outcomes, i.e. 2, and their probability of occurring, i.e. 0.5. Similarly, we expect the probability of throwing say, a 4, using an unbiased die to be $\frac{1}{6}$.

Consider instead a cannonball being shot from a cannon. If we know the mass of the cannonball, the energy imparted to it, the angle of its initial trajectory, and the geography of our environment, then we may readily compute the precise location at which the ball will make its first impact. We know that there may be small errors in our calculation because of, e.g. wind speed, the viscosity of the air, or the acceleration due to gravity at our given location on the planet\(^1\), but expect these to be negligible and recognise that given this information we could account for it in our calculations. Such processes are classified as deterministic – given sufficient information about the inputs to the process we are able to predict the outcome.

Clearly there is a direct contrast between stochastic and deterministic processes. The difference between them is essentially one of uncertainty. If we knew enough about the position, orientation in three-dimensions, and velocity of the die and the relevant elastic moduli and coefficients of friction of the die and the surface onto which we are rolling, then we could solve the appropriate equations of motion and compute its precise position at rest and hence predict the number that will be rolled. This is a difficult enough problem to solve if all these variables were known and typically we expect that at least the first three will be unknown. Accordingly we have uncertainty in our system. Despite this uncertainty, we may be confident that the probability of rolling any number is $\frac{1}{6}$. Thus, whilst we cannot predict the outcome, we know what

\(^1\)We assume that our experiment is to be conducted on Earth.
all the possible outcomes are, i.e. 1, 2, 3, 4, 5 and 6, and the probability of their occurrence.

In this course we will consider the theory of stochastic processes in 1, 2 and 3-dimensions. Stochastic processes arise in a number of interesting contexts such as the motion of gases, the evolution of bubbles of different sizes in bread and the clustering of traffic on the motorway. The system of interest to us in this module is the structure of stochastic fibrous networks. Probably the most familiar of these is paper, though most of us have also encountered such structures as mats of, e.g. glass fibres used in composites or as laboratory filters. Paper is a convenient material to study, because its use in society is widespread and therefore it is not normally difficult to obtain a sample for observation. The stochastic nature of paper structure is immediately apparent on holding a sheet of unprinted paper up to the light. The passage of light through the sheet is non-uniform and some regions appear darker than others. This difference in optical uniformity is highly stochastic in its nature and the sheet appears ‘cloudy’. Often, a periodic mark is also detectable when viewing a sheet in this way. This usually comes from the woven textile on which the network is formed which is periodic and therefore is deterministic also, so is not a component of the stochastic variability in the sheet.

From time-to-time through the module we will use paper as an example, but throughout we should bear in mind that the structures we are studying have a broad range of application in materials processing. Before considering network structures we must first consider the statistical background that we shall use to develop theoretical models. We begin by considering the simplest systems and proceed to consider the interaction between stochastic variables.
2 Statistics of events

We consider first the application of statistics to the description of systems where the events within that system or the outcomes of it are discrete. This means that each possible event or outcome has a definite probability of occurrence. In the introduction to this module we considered two such processes; the rolling of an unbiased die and the tossing of a coin. If we assume that the probability of the die coming to rest on one of its edges is infinitesimal, then we may state that the probability of each event is \( \frac{1}{6} \). Similarly, we know that it is not possible to throw the die and have the uppermost face show, for example, \( 4 \frac{1}{2} \) spots. So the outcome of rolling the die is a discrete random variable.

Consider now the distribution of the weights of eggs produced by free-range hens. The probability that an egg weighs precisely 60 g is very small; as is the probability that it weighs precisely 59.9 g or 60.000001 g. It is much easier, and certainly more meaningful, to state the probability that eggs from these hens weigh between say 55 and 65 g or between 45 and 55 g, etc. Clearly, the weights of the eggs differs from the rolling of a die in that we do not have discrete outcomes; the weights of the eggs are therefore classified as a continuous random variable.

2.1 Characterising statistics

Given data from a system, e.g. the outcomes, \( x_i \) of \( n \) rolls of a die or the weights of \( n \) eggs, we may use statistics to characterise the population. The main statistics that we use are familiar to you and are defined below.

**Mean** The mean value of the data is given by the sum of all the data divided by the number of observations. For data \( x_1, x_2 \ldots x_n \) we denote the mean \( \bar{x} \) (pronounced “x bar”) and this is given by,

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i
\]

The mean is often termed the *expectation* or, in every-day language, the *average*.

**Mode** The mode is the value within our data that occurs with the greatest frequency. For discrete data, this is found by inspection; for continuous data the mode is estimated from a histogram of the data as the mid point of the tallest column.

**Median** The median is occasionally used instead of the mean for the characterisation of data that has a histogram that is not symmetric about the mean; such data is described as skewed. The median is found by sorting the data by magnitude and selecting the middle observation such that half the observations are numerically greater than the median and half are numerically smaller.
**Variance** The variance of our data is the mean square difference from the mean, *i.e.* it is the expected value of \((x_i - \bar{x})^2\). It is denoted \(\sigma^2(x)\) and is given by,

\[
\sigma^2(x) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}
\]

**Standard Deviation** The standard deviation is the square root of the variance and is denoted \(\sigma(x)\). It is often preferred to the variance as it has the same units as the original data.

\[
\sigma(x) = \sqrt{\sigma^2(x)} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}
\]

**Coefficient of Variation** The coefficient of variation is the standard deviation relative to the mean. We denote it \(CV(x)\) and it is given by,

\[
CV(x) = \frac{\sigma(x)}{\bar{x}}
\]

Note that the coefficient of variation is dimensionless and may often be reported as a percentage.

The mean, mode and median are described as measures of location; the variance, standard deviation and coefficient of variation are described as measures of spread.

**Example calculation**

Two unbiased dice are rolled 20 times and the numbers on the uppermost faces recorded. The outcomes of the experiment are recorded in the following graphic:

![Die Rolls](image)

The values obtained by summing the numbers shown on each pair of dice are our \(x_i\) so we have,

\[x_i = \{9, 5, 8, 10, 7, 6, 10, 8, 6, 7, 8, 4, 6, 8, 3, 8, 7, 9\}\]

The mean is given by adding the \(x_i\) on all the dice and dividing by \(n = 20\):

\[
\bar{x} = \frac{9 + 5 + 8 + 10 + 7 + 6 + 10 + 8 + 6 + 7 + 8 + 4 + 6 + 8 + 3 + 8 + 7 + 9}{20}
\]

\[
= \frac{142}{20}
\]

\[
= 7.1
\]
Note that whilst the mean for this set of data is 7.1, this is not a possible outcome.

To find the median and the mode, it is convenient to sort our \( x_i \):

\[
x_i = \{3, 4, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 10, 10\}.
\]

The median is the middle observation in our sorted list, so is 7; the mode is the most frequently occurring value, so is 8.

The variance is given by:

\[
\sigma^2(x) = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}
\]

so we have,

\[
\begin{align*}
x_i - \bar{x} &= \{1.9, -2.1, 0.9, 2.9, -0.1, -1.1, 2.9, 0.9, -1.1, -0.1, -0.1, 0.9, -3.1, \\
&\quad -1.1, 0.9, -4.1, 0.9, -0.1, 1.9\} \\
(x_i - \bar{x})^2 &= \{3.61, 4.41, 0.81, 8.41, 0.01, 1.21, 8.41, 0.81, 1.21, 0.01, 1.21, 0.01, 0.81, \\
&\quad 9.61, 1.21, 0.81, 16.81, 0.81, 0.01, 3.61\}
\end{align*}
\]

and hence,

\[
\begin{align*}
\sigma^2(x) &= \frac{63.8}{20} \\
&= 3.19
\end{align*}
\]

Note that it is often more convenient to compute the variance as,

\[
\sigma^2(x) = \sum_{i=1}^{n} \frac{x_i^2}{n} - \bar{x}^2
\]

\[
= 53.6 - 50.41
\]

\[
= 3.19
\]

The standard deviation is

\[
\sigma(x) = \sqrt{\sigma^2(x)} = \sqrt{3.19} = 1.786 \quad (3 \text{ dec. pl.})
\]

and the coefficient of variation is

\[
CV(x) = \frac{\sigma(x)}{\bar{x}} = \frac{1.786}{7.1} = 0.252 \quad (3 \text{ dec. pl.})
\]

It is worth noting that for small samples such as that used in this example, it is often appropriate to use the denominator \((n-1)\) rather than \(n\) to calculate the variance, and hence the standard deviation and coefficient of variation. This arises because each of the \(n\) values in the sample can be compared with \(n-1\) other values and \(\sum (x_i - \bar{x}) = 0\); accordingly, we lose one ‘degree of freedom’ in our system. Usually, we have sufficient data that the difference between \(n-1\) and \(n\) is small and therefore the effect on our calculation of the variance is small; we will not consider this further here.
2.2 Functions describing distributions

It is often more satisfactory to calculate the properties of distributed random variables without carrying out experiments. As we proceed through this course, we shall see that often it is much easier to perform calculations than to carry out experiments and indeed that there are certain properties of paper that are difficult to measure, yet can be calculated. Since we are concerned with stochastic systems, most of our variables have distributions and if we are to model the influence of these distributed variables on given property of the system, then we require functions that describe the distributions. The distributions of discrete random variables are described by probability functions; those of continuous random variables are described by probability density functions. Below we give some examples of probability functions and probability density functions. We will use several of these through the course and will introduce each as they occur.

Examples of probability functions

The probability of occurrence of a random variable $X$ where $X$ is a positive integer with equal probability of all $X$ between 1 and $X_{\text{max}}$ is given by the probability function for the discrete uniform distribution,

$$f(X) = \frac{1}{X_{\text{max}}} \quad \text{for } X = 0, 1, 2, \ldots$$

(1)

So, for a single unbiased die, we have $X_{\text{max}} = 6$ and the probability, $f(X)$ of $X = 1, 2, \ldots, 6$ is $\frac{1}{6}$. We can use the probability function to determine the expected value of $X$ using

$$\bar{X} = \sum_{X=1}^{X_{\text{max}}} X f(X)$$

(2)

$$= \frac{1 + X_{\text{max}}}{2}$$

(3)

The expected value of $X^2$ is given by

$$\bar{X^2} = \sum_{X=1}^{X_{\text{max}}} X^2 f(X)$$

(4)

$$= \frac{1}{6}(1 + X_{\text{max}})(1 + 2X_{\text{max}})$$

(5)

So we can compute the variance as

$$\sigma^2(X) = \bar{X^2} - \bar{X}^2$$

(6)

$$= \frac{1}{12}(X_{\text{max}}^2 - 1)$$

(7)

and calculation of the standard deviation and coefficient of variation of $X$ follow directly.
So, for an unbiased die, we have,

\[
\bar{X} = \frac{3}{2}
\]

\[
\sigma^2(X) = \frac{35}{12} \approx 2.9
\]

\[
\sigma(X) = \sqrt{\frac{35}{12}} \approx 1.7
\]

\[
CV(X) = \sqrt{\frac{5}{21}} \approx 0.49
\]

Consider now a random variable \( Y = X_1 + X_2 \) where \( 1 \leq X_1, X_2 \leq X_{\text{max}} \) and \( X_1 \) and \( X_2 \) are independent random variables with probability function given by the discrete uniform distribution. Thus, \( Y \) is a random variable taking values between \( 2 \) and \( 2X_{\text{max}} \). The probability function for \( Y \) is given by,

\[
f(Y) = \frac{X_{\text{max}} - |X_{\text{max}} + 1 - Y|}{X_{\text{max}}^2}
\]  

(8)

**Exercise**

By considering all the possible outcomes that can be obtained by rolling pairs of 4, 5 and 6 sided dice, satisfy yourself that Equation (8) is correct. Plot a histogram of the probabilities of all \( Y \) that can be obtained using a pair of 6-sided dice.

So for our two-dice experiment, we have

\[
f(Y) = \frac{6 - |7 - Y|}{36}
\]

\[
\sum_{Y=2}^{12} Y f(Y) = 7
\]

\[
\sigma^2(Y) = \sum_{Y=2}^{12} Y^2 f(Y) - \left( \frac{35}{6} \right) \approx 5.83
\]

\[
\sigma(Y) = \sqrt{\frac{35}{6}} \approx 2.41
\]

\[
CV(Y) = \sqrt{\frac{5}{42}} \approx 0.34
\]

It is illustrative to examine again the characterising statistics determined earlier for our data for 20 rolls of a pair of dice. The mean obtained for those data is close to that determined from the probability function. The variance and hence the other measures of spread are noticeably lower. This arises because we considered only 20
If we repeated the experiment again, then we would obtain a different estimate of the mean, the variance, etc. If we then combined the data from the two experiments, then we would obtain a better estimate of the mean, the variance, etc. What is important here is that knowing the probability function allows us to determine precisely what the mean and variance, etc. of our random variable are; the experimental approach allows us only to estimate these properties.

**Examples of probability density functions**

A widely used probability distribution is the Normal or Gaussian distribution. We will examine the reasons for this later in the course, but for now it is sufficient to note that the random variables studied in many experiments are found to have distributions that are well approximated by the Normal distribution.

The Normal distribution has probability density given by,

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

where \(x\) is our normally distributed random variable and the distribution has mean \(\bar{x} = \mu\) and variance \(\sigma^2(x) = \sigma^2\). This is convenient, since the probability density function (pdf) is defined in terms of the mean and variance of the distribution. The integral of a pdf over the domain of the variable must be equal to 1; this signifies that we consider all possible probabilities. The Normal distribution applies to \(-\infty \leq x \leq \infty\) so

\[
\int_{-\infty}^{\infty} f(x).dx = 1 ,
\]

and the probability that \(x\) lies in the range, \(x_1 \leq x \leq x_2\) is given by

\[
P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f(x).dx
\]

The fraction of the distribution with value less than \(x\) is called the cumulative distribution function and is given by

\[
\int_{-\infty}^{x} f(x).dx = \frac{1}{2} \left(1 + \text{Erf} \left( \frac{x-\mu}{\sqrt{2}\sigma} \right) \right)
\]

where \(\text{Erf}(\zeta)\) is the Error Function; values of this function are available in statistical tables and mathematical software.

The mean value of \(x\) is given by

\[
\bar{x} = \int_{-\infty}^{\infty} x f(x).dx = \mu .
\]

The mean value of \(x^2\) is given by

\[
\overline{x^2} = \int_{-\infty}^{\infty} x^2 f(x).dx = \mu^2 + \sigma^2 ,
\]
so the variance is given by

\[ \sigma^2(x) = \bar{x}^2 - \bar{x}^2 \]
\[ = \sigma^2. \]  \hspace{1cm} (12)
\[ = \sigma^2. \]  \hspace{1cm} (13)

The lognormal distribution has probability density given by,

\[ g(y) = \frac{1}{\sqrt{2\pi} y \sigma} e^{-\frac{(\log(y) - \mu)^2}{2\sigma^2}} \]  \hspace{1cm} (14)

where \( y \) is our lognormally distributed random variable and \( 0 \leq y \leq \infty \). We calculate the mean and variance in the same way, such that

\[ \bar{y} = \int_{0}^{\infty} y g(y).dy = e^{\mu + \frac{\sigma^2}{2}}. \]  \hspace{1cm} (15)

and

\[ \sigma^2(y) = \bar{y}^2 - \bar{y}^2 \]
\[ = \int_{0}^{\infty} y^2 g(y).dy - \bar{y}^2 \]
\[ = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1). \]  \hspace{1cm} (16)
3 Random fibre networks

We have already identified some examples of stochastic fibre networks. Over the next few lectures, we shall consider a particular class of stochastic fibre network: the random fibre network. A random fibre network is defined as one where,

- the fibres are deposited independently of one another,
- the fibres have an equal probability of landing at all points in the network, i.e. the fibre centres are randomly distributed over the area,
- the fibres have an equal probability of making all possible angles with any arbitrarily chosen, fixed axis, i.e. they have a uniform random orientation.

The precise difference between random and stochastic systems and processes is subtle and we shall illustrate it by reference to paper. For a system to be random, then the events must be independent; for our system, the events are the locations of the centres of fibres. The structure of paper is influenced by the dynamics of the forming section; for example, increasing shear and decreasing the forming consistency both increase the mobility of fibres in the suspension and hence improve the uniformity of the sheet. Thus, through changes in the forming conditions, we are able to influence the locations of fibres relative to each other and therefore they are not deposited independently. In fact, fibres usually exist in suspension as loose clumps or flocs; also so again, they are not deposited independently. Finally, we know that in machine-made papers, fibres are preferentially oriented in the direction of forming; as such they do not have an equal probability of making all possible angles with any arbitrarily chosen fixed axis. We will term such effects ‘departures from randomness’; they are certainly not deterministic, and we expect our structure still to be heavily influenced by random processes.

We now develop our definition of a random fibre network by applying the appropriate mathematics to derive expressions that describe important features of the structure of random fibre networks and their distributions. Unless otherwise stated, we consider fibres to be rectangles of length, \( \lambda \) and width, \( \omega \) with \( \lambda > \omega \); as before, we denote fibre coarseness, \( \delta \) where coarseness is the mass per unit length of the fibres. In the coursework for this module, you will be required to generate a graphical representation of a low grammage random fibre network. It should look something like the network shown in Figure 1. The caption for Figure 1 gives details of the fibre and network properties and the images shown are of one network at different magnifications; the centre point of each image is the same.

For now, there is one particular feature of the networks that you should note, that is that random networks are not uniform – it is immediately apparent that there are regions with dense and sparse coverage. So, random processes are ‘clumpy’ with clusters of events and corresponding regions with few events. The clustered nature of random events is easily appreciated when listening to the clicks of a Geiger counter that has been placed near a radioactive source. The clicks come in clusters with non-uniform intervals between them. We see the same feature in our random fibre network.
3.1 The Poisson distribution

The standard model for a random process of events with mean $\mu$ is a Poisson distribution with mean $\mu$. The probability that precisely $k$ events will occur is

$$P(k) = \begin{cases} \frac{e^{-\mu} \mu^k}{k!} & \text{for } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{k=0}^{\infty} P(k) = 1.$$ 

So Equation (19) is a probability function and yields a probability between zero and one for all positive integer values of $k$; the second line of the equation tells us that for non-integer or negative values of $k$, the function yields a probability of zero.

An important characteristic of the Poisson distribution is that the variance is equal to the mean; thus, $\bar{k} = \sigma^2(k) = \mu$. Note that whilst $k$ must take positive integer values, $\mu$ need only be real and positive.

3.2 Coverage

Consider a random two-dimensional network of fibres, as described above, formed by a Poisson process for fibre centres with mean density $\mu$ fibre centres per unit area. The number of fibres covering a point is a Poisson variable $c$ which we shall call the coverage.

The mass of a fibre is $\lambda \delta$ so the expected number of fibre centres per unit area in a network of mean grammage, $\bar{\beta}$ is

$$\mu = \frac{\bar{\beta}}{\lambda \delta}.$$ 


We have defined our network as one where the fibre centres are positioned independently and with equal likelihood of each location. Thus we use the Poisson distribution to describe the occurrence of fibres covering points in the network. Accordingly, the probability that a point will be covered by exactly \( k \) fibres is given by

\[
P(c = k) = \frac{e^{-\bar{c}} \bar{c}^k}{k!}
\]

We may relate the expected coverage of points, \( \bar{c} \), to the expected number of fibre centres per unit area through the projected area of a fibre, \( \lambda \omega \),

\[
\bar{c} = \mu \lambda \omega .
\]

More often, it is convenient to determine \( \bar{c} \) from the mean grammage of the network \( \bar{\beta} \) and the mean grammage of a fibre \( \frac{\delta}{\omega} \), using,

\[
\bar{c} = \frac{\bar{\beta} \omega}{\delta} .
\]

From our knowledge of fibre geometries and network properties, we can immediately make some statements about the structure of the network. For example, the grammage of a wood pulp fibre is around 5 \( g m^{-2} \) and that of newsprint is around 45 \( g m^{-2} \) so the mean coverage, \( \bar{c} = 9 \). Thus, from Equation (21) the probability that a point is covered by \( k \) fibres is

\[
P(c = k) = \frac{e^{-9} \cdot 9^k}{k!}
\]

The probability of a point being covered by no fibres, \( i.e. \) the probability of a pinhole is

\[
P(0) = e^{-9} ,
\]

\[
= 0.00012341 .
\]

The probability that a point is covered by exactly 9 fibres is

\[
P(9) = \frac{e^{-9} \cdot 9^9}{9!}
\]

\[
= 0.132 \hspace{1cm} (3 \hspace{1cm} \text{dec. pl.}).
\]

and the probability that a point is covered by more than 9 fibres is

\[
P(c > 9) = 1 - (P(0) + P(1) + P(2) + \ldots + P(8) + P(9)) ,
\]

\[
= 0.413 \hspace{1cm} (3 \hspace{1cm} \text{dec. pl.}).
\]

This is important. When we manufacture fibre networks in an commercial environment, we specify a grammage; for the newsprint in our example this was 45 \( g m^{-2} \). Assuming a fibre grammage, we have estimated that this corresponds to a coverage of 9. Now,
the treatment above tells us that for a random network of fibres, only about 13% of the network has this coverage, i.e. 87% is either heavier or lighter than our target grammage.

Plots of the Poisson distribution for mean coverage of points by 4 and 9 fibres are shown in Figure 2. As the mean value of a Poisson distributed variable increases, the distribution becomes less skew and is well approximated by a Normal distribution for means greater than about 20; though we note that the Normal distribution is a probability density function.

**Exercise**

A random network of fibres has grammage $15 g m^{-2}$; this is approximately the grammage of a single ply of bathroom tissue. The fibres in the network have coarseness $1.6 \times 10^{-7} kg m^{-1}$ and width $23 \mu m$.

1. calculate the expected coverage of the network,
2. calculate the probability of a pinhole occurring in the network,
3. plot a histogram of the probability function for coverage.

### 3.3 Two-dimensional networks

Typically, industrially formed networks have sufficiently high coverage that there will often be significant vertical separation between fibres lying over a given point in the plane. This separation may be filled with other fibres, or may be voids generated where fibres bridge gaps between adjacent fibres. Whichever is the case, this complicates the mathematics describing the structural properties of the network. Much of
the earliest work describing the structure of random fibre networks therefore considered two-dimensional networks. Qualitatively these can be considered low grammage networks with sufficiently low coverage in all regions that vertical separation between fibres is negligible. Much of the earliest work describing the structure of fibre networks was carried out by two scientists, Corte and Kallmes. They defined a two-dimensional network as one where,

“...the number of fibres in the network is so small that the area covered by more than two fibres is negligible, i.e. less than 1%...”

We can easily calculate what this limiting coverage is. The area covered by more than two fibres is \(1 - (P(0) + P(1) + P(2))\), where \(P(0), P(1), P(2)\) are the probabilities of coverage 0, 1 and 2 given by the Poisson distribution. So

\[
1 - (P(0) + P(1) + P(2)) = 1 - \left(e^{-\bar{c}} + \bar{c}e^{-\bar{c}} + \frac{\bar{c}^2}{2}e^{-\bar{c}}\right) \tag{24}
\]

Setting the right hand side of Equation (24) equal to 0.01 allows determination of \(\bar{c}\) using, for example, Newton’s method. This yields \(\bar{c} \approx 0.436\). So we may state that networks of mean coverage less than 0.436 have less than 1% of their area covered by more than two fibres and may therefore be considered two-dimensional.

Over the next few lectures, we shall consider some of the structural properties of two-dimensional networks and consider their relevance to the physical properties of stochastic fibrous structures.

3.3.1 Number of crossings per fibre

The strength of bonded fibrous networks is determined by the intrinsic strength of the constituent fibres and the number and strength of the bonds between them. The number of crossings per fibre is an important property therefore, since fibres may bond to each other only where they are in contact. To determine the number of crossings per fibre in a two dimensional random fibre network, we consider two fibres within an area \(x^2\) of the network; two such fibres are represented by the wide lines in Figure 3. Despite this graphical representation, for now, we assume that the fibres have no width, i.e. they are lines.

The longitudinal axes of the two wide lines in Figure 3 cross at an angle \(\phi\). The parallelogram surrounding the horizontal line in the figure is a rhombus of side \(\lambda\); for the two lines to cross, the centre of the line with orientation \(\phi\) must fall within the area of this rhombus. The area of the rhombus is \(\lambda^2 \sin(\phi)\) so the probability that these two lines cross is given by the ratio of the area of the rhombus to that of the region of interest, i.e.

\[
P_{\text{cross},\phi} = \frac{\lambda^2}{x^2} \sin(\phi) .
\]
Figure 3: Two fibres of length $\lambda$ oriented at an angle $\phi$ to each other within a square area of side $x$.

This tells us the probability that a pair of lines within an area $x^2$ will intersect with angle $\phi$. The probability that a pair of lines cross with any angle is given by

$$\int P_{\text{cross,}\phi} f(\phi) d\phi,$$

where $f(\phi)$ is the probability density function for $\phi$. Because of symmetry, we need only consider $0 \leq \phi \leq \frac{\pi}{2}$. Since $\phi$ has a uniform distribution and is a continuous random variable,

$$f(\phi) = \frac{2}{\pi}$$

such that $\int_0^{\frac{\pi}{2}} f(\phi) = 1$. Thus the probability that two lines with any angle cross is

$$P_{\text{cross}} = \frac{2}{\pi} \frac{\lambda^2}{x^2} \int_0^{\frac{\pi}{2}} \sin(\phi) d\phi,$$

$$= \frac{2}{\pi} \frac{\lambda^2}{x^2}.$$  \hspace{1cm} (25)

Equation (26) gives the probability that two lines of length $\lambda$ cross within an area $x^2$. In our two-dimensional network, we have say, $\bar{n}_{\text{fib}}$ fibres in an area $x^2$. Each fibre can cross all others except itself so the number of ways that pairs of fibres may be selected is $\bar{n}_{\text{fib}}(\bar{n}_{\text{fib}} - 1)/2$ where the divisor 2 is applied because each fibre, being one of a pair, is counted twice.

Given this information, we may now state that the expected number of crossings in an area $x^2$ is given by

$$\bar{n}_{\text{cross,x}} = P_{\text{cross}} (\bar{n}_{\text{fib}} - 1) \frac{\bar{n}_{\text{fib}}}{2}$$ \hspace{1cm} (27)
\[ \bar{n}_{\text{fib}} = \frac{1}{\pi} \frac{\lambda^2}{x^2} (\bar{n}_{\text{fib}} - 1) \bar{n}_{\text{fib}} \]  

(28)

Usually \( \bar{n}_{\text{fib}} \ll 1 \) so we have

\[ \bar{n}_{\text{cross},x} \approx \frac{1}{\pi} \frac{\lambda^2}{x^2} \bar{n}_{\text{fib}}^2 \]  

(29)

and the expected number of crossings per unit area is given by \( \bar{n}_{\text{cross},x}/x^2 \) such that,

\[ \bar{n}_{\text{cross}} \approx \frac{\lambda^2}{\pi x^4} \bar{n}_{\text{fib}}^2 \]  

(30)

The expected number of fibres of length \( \lambda \) and width \( \omega \) in a zone of area \( x^2 \), \( \bar{n}_{\text{fib}} \) is given by

\[ \bar{n}_{\text{fib}} = \frac{x^2}{\lambda \omega} \bar{c} \]  

(31)

Substitution of Equation (31) in Equation (30) and rearranging yields

\[ \bar{n}_{\text{cross}} \approx \frac{\bar{c}^2}{\pi \omega^2} \]  

(32)

So, Equation (32) gives the number of crossings per unit area and we observe that it depends only on the coverage and the width of fibres.

To determine the expected number of crossings per fibre, \( \bar{n}_{\text{cross, fib}} \) we simply divide Equation (32) by the number of fibres per unit area, \( \bar{n}_{\text{fib}}/x^2 \) and multiply by two, since each crossing makes a contact on both fibres involved. Thus,

\[ \bar{n}_{\text{cross, fib}} = 2 \frac{\bar{n}_{\text{cross}}}{{\bar{n}_{\text{fib}}}} \bar{n}_{\text{fib}} \approx 2 \frac{\bar{c}^2}{\pi \omega^2} \frac{\lambda \omega}{\bar{c}} \approx 2 \frac{\bar{c}}{\pi} \frac{\lambda}{\omega} = 2 \frac{\bar{c}}{\pi} \bar{A} \]  

(33)

where \( \bar{A} = \frac{\lambda}{\omega} \) is the aspect ratio of the fibres.

It is important to remember that Equation (33) is applicable to two-dimensional networks only. We have defined these as networks of mean coverage less that about 0.44.

**Exercise**

Determine the mean coverage required for a network of fibres of aspect ratio \( A \) to have on average one contact per fibre. Use this value of coverage to determine expressions for the fractions of the network with coverage greater than 0 and greater than 1. Denoting these fractions \( f_0 \) and \( f_1 \) respectively, satisfy yourself that the expected area of one contact is approximated by \( \bar{A} \frac{f_1}{f_0} \).
3.3.2 Mean ligament length

We have derived an expression for the expected number of contacts per fibre in a two-dimensional random fibre network. Now, if we know the number of contacts on a fibre and we know also the mean length of a fibre then we may readily determine the expected distance between fibre axes. For fibres of length $\bar{\lambda}$ this distance, $\bar{g}_a$ is,

$$\bar{g}_a = \frac{\bar{\lambda}}{\bar{n}_{\text{cross, fib}} - 1}$$  \hfill (34)

where the denominator is one crossing less than the expected number because each distance between crossings is bounded by two crossings.

The maximum mean coverage that we have identified for our two-dimensional networks is about 0.4, so assuming an aspect ratio of say 100, we expect more than 10 crossings per fibre for networks with coverage around 0.2. We may state then that typically $\bar{n}_{\text{cross, fib}} \gg 1$ such that

$$\bar{g}_a = \frac{\bar{\lambda}}{\bar{n}_{\text{cross, fib}}} \quad \text{for} \quad \bar{n}_{\text{cross, fib}} \gg 1$$  \hfill (35)

now, the number of crossings per fibre is given by Equation (33) as $2\bar{c}A/\pi$, and $A = \frac{\bar{\lambda}}{\bar{\omega}}$, so we have

$$\bar{g}_a = \frac{\pi \bar{\omega}}{2\bar{c}}$$  \hfill (36)

Refer now to Figure 4. This shows a section of a fibre with horizontal orientation crossed by two other fibres; the longitudinal axis of each fibre is represented by a
broken line and the distance between fibre axes, \( g_a \) is marked on the figure. Three other distances are marked on the figure also; these illustrate the difficulty in determining what precisely is the distance between a pair of crossings when we consider fibres of finite width. For the pair of crossings shown, the distance between the crossings, or the \textit{ligament length}, has maximum length \( g_2 \) and minimum, \( g_1 \). Of the distances marked on the figure, the best estimate of the length of the ligament between our pair of crossings is that between the edges of the crossing fibres measured along the fibre axis and denoted \( g \). We note that this distance will be determined by the angles at which our fibres cross the horizontal fibres; to a first approximation however, we expect \( g \) to be about one fibre width less than \( g_a \), \textit{i.e.} \( g \approx g_a - \omega \). Accordingly, we have

\[
\bar{g} = \frac{\pi \omega}{2 \bar{c}} - \omega ,
\]

\[
= \left( \frac{\pi}{2 \bar{c}} - 1 \right) \omega .
\]  

(37)

This is an important result since it tells us that the expected ligament length of a two-dimensional random fibre network is dependent only on the coverage of the network and the width of the fibres. Recall however that the expected coverage of a two dimensional fibre network is given by the mean grammage of the network, \( \bar{\beta} \) divided by the mean grammage of a fibre, \( \bar{\beta}_{rh} \) and that the grammage of a fibre is given by its coarseness, \( \delta \) divided by its width \( \omega \) such that

\[
\bar{c} = \frac{\bar{\beta} \omega}{\delta} ,
\]

and

\[
\bar{g} = \frac{\pi \delta}{2 \bar{\beta}} - \omega .
\]  

(38)

So the expected ligament length of a two-dimensional network of mean grammage \( \bar{\beta} \) is determined by the coarseness and width of the fibres. Increasing fibre coarseness results in an increase in the ligament length because at constant grammage there are fewer fibres per unit area and hence fewer crossings per fibre.

### 3.3.3 Ligament-length distribution

We have derived expressions for the expected, or mean, ligament length in a two-dimensional random network of fibres. Of interest also is the distribution of ligament lengths. We shall see that this is particularly important when modelling the pore size distribution of the network. It is relevant also to the way that a network behaves under load since we expect unbonded segments of the fibre length to exhibit different strain behaviour to bonded segments.

To derive expressions for the distribution of ligament lengths, we return to the Poisson distribution. The events of interest are the crossings of fibres; for a point Poisson process along a line with \( \mu \) events \textit{per unit length} the expected number of
events in an interval of length $g$ along that line is $\mu g$ and the probability that there will be precisely $\kappa$ events in that interval is:

$$P(\kappa) = \frac{e^{-\mu g} (\mu g)^{\kappa}}{\kappa!} \quad \kappa = 0, 1, 2, \ldots$$

The probability of a gap of length $g$ is the probability of zero events occurring in an interval $g$; this is

$$P(0) = e^{-\mu g} \quad g > 0$$

(39)

To describe the distribution of ligament lengths we require a probability density function (pdf) and this must integrate to 1 over the applicable range of our random variable. We proceed therefore by integrating Equation (39) over the range $0 \leq g \leq \infty$

$$\int_0^\infty P(0) \, dg = \int_0^\infty e^{-\mu g} \, dg = \frac{1}{\mu}$$

So the probability density function for $g$ is given by $e^{-\mu g}$ divided by $1/\mu$ such that

$$f(g) = \mu e^{-\mu g}$$

(40)

and $\int_0^\infty f(g) \, dg = 1$. From earlier lectures we recall that the mean value is determined by integrating the product $g f(g)$ over the applicable range of the variable such that

$$\bar{g} = \int_0^\infty g f(g) \, dg = \int_0^\infty \mu g e^{-\mu g} \, dg = \frac{1}{\mu}$$

(41)

and hence

$$\mu = \frac{1}{\bar{g}}$$

(42)

Substitution of Equation (42) in Equation (40) yields the final form of the probability density function for ligament lengths:

$$f(g) = \frac{1}{\bar{g}} e^{-\frac{g}{\bar{g}}}$$

(43)

We determine the variance of the ligament length distribution following the procedure described earlier:

$$\sigma^2(g) = \int_0^\infty g^2 f(g) \, dg - \bar{g}^2 = \frac{1}{\bar{g}^2}$$

(44)

The pdf given by Equation (43) is known as the exponential distribution or the negative exponential distribution.
Exercise
A two dimensional random fibre network has mean coverage 0.3 and is formed from fibres of width 20 µm. Determine,

1. the expected free-fibre-length,
2. the fraction of the free-fibre-length distribution that is greater than the mean,
3. the fraction of the free-fibre-length distribution that lies within ±10 % of the mean.

3.3.4 Fractional Contact Area
We term the fraction of the fibre surface that is in contact with other fibres the fractional contact area (FCA). The concept of fibre contact is itself rather subtle. If, for example, we lie two pieces of paper on top of each other, then we may consider them to be in full contact; however, at the fibre scale, the roughness of the sheet reduces the contact area significantly. The same effect applies when fibre surfaces cross each other, the ‘quality of contact’ is determined by the roughness and conformability of the surfaces and this in turn influences the area available for bonding between them.

Bearing in mind then, that the fractional contact area represents an upper limit on the available area for bonding, here we will begin by deriving expressions for the fractional contact area in a two-dimensional random fibre network. Given this, we will proceed to consider networks with more realistic mean coverage.

In a two-dimensional network, we can assume that fibres are in contact with each other in any region where they cross. Also, in a two-dimensional network, we can assume that the fraction of the network with coverage greater than 3 is negligible. Now, we want to know the fraction of the fibre surface that is in contact with other fibres. The easiest way to determine this is to determine first the fraction that is not in contact with other fibres. This fraction is the fraction of the fibre surfaces that is in the surfaces of the network. Assuming that the fraction of the network with $c > 3$ is negligible, it is

$$f = \frac{P(1) + P(2) + P(3)}{\bar{c}},$$

and

$$P(3) = 1 - P(0) - P(1) - P(2).$$

So

$$f = \frac{1 - P(0)}{\bar{c}}.$$ 

The fractional contact area of a two-dimensional network is therefore

$$\Phi_{2D} = 1 - f = 1 - \frac{1 - P(0)}{\bar{c}}.$$
The probability of coverage zero is given in terms of the mean coverage only, i.e.

\[ P(0) = e^{-\bar{c}}. \]

so we have

\[ \Phi_{2D} = 1 - \frac{1 - e^{-\bar{c}}}{\bar{c}} \]  

(49)

The probability of coverage zero is known as the fractional open area, \( \epsilon \), of the sheet, i.e. the two-dimensional analogue of porosity:

\[ \epsilon = P(0) = e^{-\bar{c}}, \]

such that

\[ \bar{c} = \log(1/\epsilon) = -\log(\epsilon) \]  

(50)

We therefore obtain a final expression for \( \Phi_{2D} \) in terms of fractional open area only:

\[ \Phi_{2D} = 1 + \frac{1 - \epsilon}{\log(\epsilon)}. \]  

(51)

**Thicker structures**

Real fibrous materials typically have greater mean coverage than permitted for two-dimensional networks. We can extend our theory for the fractional contact area of two-dimensional networks to thicker materials by modelling these as consisting of several layers. We therefore model a structure with infinite mean as being formed by superimposing several independent and identical two-dimensional structures. Consider first three layers of this structure, each with fractional open area equal to the porosity of the network, \( \epsilon \). A given fibre in the central layer may form additional contacts with the surrounding layers in those regions where its own layer has no contact on either side, or is in contact with other fibres above or below it only. The fraction of the projected solid area making contacts between layers is

\[ f_b = (1 - \epsilon)^2, \]  

(52)

and the fraction of the fibre surface that is available for additional contacts is \( (1 - \Phi_{2D}) \). Accordingly, for a network with infinite coverage formed from the superposition of an infinite number two-dimensional of layers, we have,

\[
\Phi_{\infty} &= \Phi_{2D} + f_b (1 - \Phi_{2D}) \\
&= \Phi_{2D} + (1 - \epsilon)^2 (1 - \Phi_{2D}) \\
&= 1 + \frac{1 - \epsilon}{\log(\epsilon)} - (1 - \epsilon)^2 \left( \frac{1 - \epsilon}{\log(\epsilon)} \right) \\
&= 1 + \frac{1 - \epsilon}{\log(\epsilon)} \left( 1 - (1 - \epsilon)^2 \right) \\
&= 1 + \frac{(1 - \epsilon)(2 - \epsilon)\epsilon}{\log(\epsilon)}. \]  

(53)
When
\[ \epsilon = 0 \quad \Phi_\infty = 1 \]
\[ \epsilon = 1 \quad \Phi_\infty = 0 \]
as expected.

For multiplanar structures of finite mean coverage \( \bar{c} \), we must take account of the fact that the outermost fibre surfaces are not available for contact. To a reasonable approximation, this fraction is \( 1 - 1/\bar{c} \), for \( \bar{c} \) greater than about 10. The fractional contact area of a network with mean coverage \( \bar{c} \) and porosity \( \epsilon \) is therefore
\[
\Phi_c = \left(1 - \frac{1}{\bar{c}}\right) \Phi_\infty = \left(1 - \frac{1}{\bar{c}}\right) \left(1 + \frac{(1 - \epsilon)(2 - \epsilon)\epsilon}{\log(\epsilon)}\right).
\]

### 3.3.5 Pore radius distribution

So far, we have studied the theory describing the number of crossings between fibres in two-dimensional random fibre networks, the distances between crossings and the fraction of a given fibre’s surface that is in contact with other fibres. Here we shall see that the number of crossings per fibre and the ligament length distribution couple to control the size of the voids between fibres in the network. This is an important property of the network as it influences its interactions with fluids during say an impregnation stage during manufacture of a composite; it controls also the ability of a network to capture particles when acting as a filter.

Figure 5 shows an area of a two-dimensional random fibre network. Our interest is the distribution of the size of the polygons that are generated by the crossings of fibres in the network. We begin with some qualitative statements arrived at by inspection of the figure, these being that the polygons generated in regions where there are several crossings, e.g. towards the right of the figure, are smaller than those where there are few, e.g. those towards the top left of the figure. We observe also that the boundaries of the polygons are the ligaments of the network.

We may now perform a crude semi-quantitative analysis of the polygons in Figure 5. Counting the number of sides per polygon, we observe that many of the polygons are triangles, some are four-sided, a few are five-sided and the largest, in the top left of our network, has six sides. If we were to count the sides of enough polygons in a random network of fibres, then we would find that the average number of sides per polygon is four and this means that more of the polygons are triangles than have five, six or more sides. A robust proof of this exists and is attributable to the mathematician Miles; the proof is beyond the scope of this course, but we may satisfy ourselves that the result is reasonable by noting that each crossing of a pair of fibres is associated with four polygons as shown in Figure 6 and as such, each crossing generates four ligaments.
Since ligaments are the boundaries of our polygons, and they are generated in multiples of four, then an expected number of sides per polygon of four seems reasonable. Before we are able to derive expressions for the distribution of polygon sizes in our two-dimensional random fibre network, we require one further piece of information. This we derived previously and it is that the probability density function for ligament lengths is negative exponential. As the expected number of sides per polygon is four, then we simplify the structure of the network and consider polygons to be rectangles with sides of length distributed according to the negative exponential distribution. We begin by deriving the probability density function for the areas of such rectangles.

We seek the probability density function for the areas, \( a \), of rectangles with sides \( g_x \) and \( g_y \) such that \( a = g_x \cdot g_y \) and \( g_x \) and \( g_y \) are ligament lengths with independent and identical distributions given by the negative exponential distribution with mean \( \bar{g}_x = \bar{g}_y = \bar{g} \) such that,

\[
 f(g_x) = \frac{1}{\bar{g}} e^{-\frac{g_x}{\bar{g}}}, \quad (55) \\
 f(g_y) = \frac{1}{\bar{g}} e^{-\frac{g_y}{\bar{g}}}, \quad (56)
\]

Now, the area \( a \) is given by the product of \( g_x \) and \( g_y \) and we need to account for all \( g_x \) and \( g_y \). The probability density for \( a \) is given by,

\[
 g(a) = \int_0^\infty f(g_x) f(g_y) \cdot dg_x 
\]

\( i.e. \) it is given by the integral for all \( g_x \) of the product of the probability densities of \( g_x \) and \( g_y \). To evaluate the integral in Equation (57) we need the expression for \( f(g_y) \) in
terms of \( g_x \). We note that
\[
g_y = \frac{a}{g_x}
\]
and
\[
\frac{d g_y}{d a} = \frac{1}{g_x}
\]
So the probability density for \( g_y \) may be written,
\[
f(g_y) = f \left( \frac{a}{g_x} \right) \left| \frac{d g_y}{d a} \right|
\]
\[
= f \left( \frac{a}{g_x} \right) \frac{1}{g_x}
\]
\[
= \frac{1}{g g_x} e^{-\frac{a}{g x \sigma}} .
\]

### Changing variables in probability density functions.

The steps performed between Equations (58) and (60) are those required to change the variable in the probability density function of interest to us here. The term \( \left| \frac{d g_y}{d a} \right| \) is called the Jacobian and is required such that the integral of the pdf over the applicable range is 1. The Jacobian is the modulus of the derivative of the old variable (in our case \( g_y = \frac{a}{g} \)) with respect to the new variable (in our case \( a \)).

Substitution of Equations (60) and (55) into Equation (57) yields,
\[
g(a) = \int_0^\infty \frac{1}{g} e^{-\frac{a}{g}} \frac{1}{g g_x} e^{-\frac{a}{g x \sigma}} . dg_x
\]
Evaluation of the integral yields,
\[ g(a) = \frac{2}{\bar{g}^2} K_0(\zeta) \quad \text{where} \quad \zeta = 2 \sqrt{a/\bar{g}} \] (64)

where \( K_0(\zeta) \) is the zeroth order modified Bessel function of the second kind. Such functions are known as special functions and may not be familiar to many scientists. They allow, in particular, the evaluation of certain integrals; here we are concerned more with the nature of the result and we may rely on computer packages that allow symbolic computation such as Mathematica to generate and manipulate these functions for us. In Mathematica, the zeroth order modified Bessel function of the second kind is input as \texttt{BesselK[0, x]}.

It is a good idea to check that what you have derived is indeed a pdf; for the case of pore areas that we have discussed here, the applicable range of the distribution is \( 0 \leq a \leq \infty \) and we find that \( \int_0^\infty g(a) \, da = 1 \) so we know that we have considered all probabilities and therefore that we have a pdf.

The mean pore area is given by,
\[ \bar{a} = \int_0^\infty a \, g(a) \, da \]
\[ = \bar{g}^2 , \] (65)

and the variance is given by
\[ \sigma^2(a) = \int_0^\infty a^2 \, g(a) \, da - \bar{a}^2 , \]
\[ = 4 \bar{g}^4 - \bar{g}^4 , \]
\[ = 3 \bar{g}^4 \] (66)

such that the coefficient of variation of pore areas \( CV(a) = \sqrt{3} \).

We now require one further change of variable to characterise the size of the voids in the network and seek a parameter with dimensions of length. We could, of course, simply take the square root of the pore areas though this is perhaps a little too simplistic. The preferred measure has been the radius of a circle of the same area as a pore, \( i.e. \ r = \sqrt{\frac{a}{\pi}} \) and, as we are to change the variable, we need also to determine the Jacobian. So we have
\[ \frac{a}{d} = \pi r^2 \]
\[ \frac{da}{dr} = 2 \pi r \] (67)
so

\[ g(r) = g(\pi r^2) \left| \frac{d}{dr} \right| = \frac{4 \pi r}{g^2} K_0(\zeta) \quad \text{where} \quad \zeta = 2 r \sqrt{\pi/g} \] (68)

Again we check \( \int_0^\infty g(r) = 1 \) and we obtain the mean and variance of pore radii as,

\[ \bar{r} = \frac{\sqrt{\pi}}{4} \hat{g} \] (69)

\[ \sigma^2(r) = \left( \frac{1}{\pi} - \frac{\pi}{16} \right) g^2 \] (70)

such that \( CV(r) = \frac{1}{\pi} \sqrt{\frac{16}{\pi^2}} - 1 \).

There are a few important points to note from our derivation. Firstly, the mean pore radius is a function only of the mean ligament length and, since we derived expressions for this and saw that it depended on fibre width and coverage only, we may now state that for two-dimensional random fibre networks, the mean pore size also depends on fibre width and coverage only. Secondly, we observe that the coefficient of variation of pore radii, \( CV(r) \) is constant, \( i.e. \) the standard deviation of pore radii is proportional to the mean.

### 3.3.6 Mean pore dimensions – Another way

The derivation that we have just considered was first presented in 1966 and has formed the basis of all subsequent attempts to model pore size in random and near-random planar fibrous networks. Work carried out in Manchester has recently shown that the assumption that the two exponential distributions used in the derivation are independent and identical is incorrect. In fact, the lengths of adjacent polygon sides are correlated and arise from different gamma distributions. We will not consider this further at this point; the derivation given above serves as a good approximation and has served scientists well in guiding their thinking. The realisation that our existing models are not fully valid has however given rise to a new line of thought in modelling the void structure of stochastic fibrous networks. We shall close the course by considering two recent results, derived in Manchester, that give the expected dimensions of pores in the plane and in the direction perpendicular to it.

**In-plane pore dimension** Consider a two dimensional random fibre network with mean coverage \( \bar{c} \) formed from fibres of width \( \omega \). The expected number of crossings per unit area is given by Equation (32),

\[ \bar{n}_{\text{cross}} \approx \frac{\bar{c}^2}{\pi \omega^2} \] (32)
Now, since every crossing is associated with four pores, and the expected number of sides per polygon is four also, we can assume that the number of pores per unit area is approximately equal to the number of crossings, \( \bar{n}_{\text{pores}} \approx \bar{n}_{\text{cross}} \) (64)

The mean pore area is given by the fractional open area \( \epsilon \) divided by the number of crossings per unit area:

\[
\bar{a} = \frac{\epsilon}{\bar{n}_{\text{pores}}} = \frac{\epsilon}{\bar{n}_{\text{cross}}} = \frac{\pi \epsilon \omega^2}{\epsilon^2} \tag{65}
\]

We bear in mind that the result of increasing fibre width at a constant total fibre length per unit area is that the smallest pores become closed and other pores become smaller. A consequence of this is that the distribution of pore areas, and hence the mean pore area, are unaffected by fibre width.

By definition, the fractional open area, \( \epsilon \) is the fraction of the network with coverage zero \( \epsilon^{-\bar{c}} \). So

\[
\bar{c} = \log(1/\epsilon) \tag{66}
\]

and we have

\[
\bar{a} = \frac{\pi \epsilon \omega^2}{(\log(1/\epsilon))^2} \tag{67}
\]

If we consider the mean pore dimension to be the diameter of a circle with area \( a \), then we have

\[
\bar{d} = \sqrt{\frac{4 a}{\pi}} = \frac{2 \sqrt{\epsilon} \omega}{\log(1/\epsilon)} \tag{68}
\]

Equation (68) can be applied to an elemental section in the plane of the network and it tells us that in a network of given porosity \( \epsilon \), the expected size of pores is directly proportional to the width of fibres. This is perhaps a little counter intuitive, but arises because if we fix porosity and increase width, then we must reduce the total fibre length per unit area.

**Pore height** Consider now a thicker fibre network of mean coverage \( \bar{c} \gg 1 \). To a good approximation, the expected number of pores above a point in the plane of the network is \( \bar{c} - 1 \). If we denote the total thickness of the network \( z \), then the expected thickness of the network is

\[
\bar{z} = \bar{c} \bar{t} + (\bar{c} - 1) \bar{h} \tag{69}
\]
where $\bar{t}$ is the expected thickness of the fibres perpendicular to the plane and $\bar{h}$ is the expected height of a pore.

By definition, the porosity of the network is

$$\epsilon = \frac{(\bar{c} - 1) \bar{h}}{\bar{z}}$$  \hspace{1cm} (70)

Substitution of Equation (69) into Equation (70) yields, on manipulation,

$$\bar{h} = \frac{\bar{c}}{\bar{c} - 1} \frac{\epsilon}{1 - \epsilon} \bar{t}$$  \hspace{1cm} (71)

So for a network of given mean coverage, the expected pore height is a function of porosity and the mean thickness of fibres only. This is a rather similar observation as that made for pore dimensions in the plane of the network. We close the course with a final exercise from which you should draw your own conclusions.

**Exercise**

Generate a plot of $\bar{h}$ as given by Equation (71) against $\bar{d}$ as given by Equation (68). You will need to make some assumptions, make sure that you can justify them.