Analysis, Random Walks and Groups
Spring 2020

Week 6 tutorial

Given a probability distribution μ on $\mathbb{Z}_p$ and a mapping $f : \mathbb{Z}_p \to \mathbb{C}$, define the **multiplier operator**

$$M_\mu f(t) = f * \mu(t), \quad t \in \mathbb{Z}_p.$$

Thus $M_\mu f$ is again a function $\mathbb{Z}_p \to \mathbb{C}$ with added convolution from μ. Multiplier operators are an important class of operators in harmonic analysis which appear commonly in the study of PDEs, fractals and signals.

A complex number $\lambda \in \mathbb{C}$ is an **eigenvalue** of M_μ if there exists a non-zero $\psi : \mathbb{Z}_p \to \mathbb{C}$ (called **eigenfunction** of M_μ) such that

$$M_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

The **spectrum** $\sigma(M_\mu)$ of M_μ is then the collection of all eigenvalues

$$\sigma(M_\mu) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } M_\mu\}$$

1. Given a probability distribution μ on $\mathbb{Z}_p$, prove that for each $k \in \mathbb{Z}_p$, the Fourier transform $\widehat{\mu}(k)$ is an eigenvalue of the multiplier operator M_μ.

Hint: Use convolution theorem, Fourier series, and attempt to prove the function $\psi_k(t) = e^{2\pi i k/tp}$, $t \in \mathbb{Z}_p$, is an eigenfunction of M_μ with eigenvalue $\widehat{\mu}(k)$.

Solution. For a fixed $k \in \mathbb{Z}_p$ denote

$$\psi_k(t) = e^{2\pi i k/tp}, \quad t \in \mathbb{Z}_p.$$

Notice that $\psi_k \neq 0$ for all $k \in \mathbb{Z}_p$. We claim that ψ_k is an eigenfunction of M_μ with eigenvalue $\widehat{\mu}(k)$.

By the convolution theorem we have for all $\ell \in \mathbb{Z}_p$ that

$$\widehat{\psi_k}(\ell)\widehat{\mu}(\ell) = \widehat{\psi_k * \mu}(\ell) = \widehat{M_\mu \psi_k}(\ell).$$

Fix $t \in \mathbb{Z}_p$. Hence by the Fourier series for $M_\mu \psi(t)$ at t we have that

$$M_\mu \psi(t) = \frac{1}{p} \sum_{\ell \in \mathbb{Z}_p} \widehat{M_\mu \psi_k}(\ell) e^{2\pi i \ell t/p} = \frac{1}{p} \sum_{\ell \in \mathbb{Z}_p} \widehat{\psi_k}(\ell)\widehat{\mu}(k) e^{2\pi i \ell t/p}. \tag{0.1}$$

Here we have that

$$\widehat{\psi_k}(k) = \sum_{t \in \mathbb{Z}_p} \psi_k(t) e^{-2\pi i k t/p} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i k/tp} e^{-2\pi i k/tp} = \sum_{t \in \mathbb{Z}_p} 1 = p$$

and if $\ell \neq k$ we have

$$\widehat{\psi}(\ell) = \sum_{t \in \mathbb{Z}_p} \psi_k(t) e^{-2\pi i \ell t/p} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i k/tp} e^{-2\pi i \ell/tp} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i (k-\ell)t/p} = \frac{1 - e^{2\pi i (k-\ell)}}{1 - e^{2\pi i (k-\ell)/p}} = 0$$

by the exponential sum formula (which we can use as $k \neq \ell$). Hence by (0.1) we have

$$M_\mu \psi(t) = \frac{1}{p} \widehat{\mu}(k) p e^{2\pi i k t/p} + 0 = \widehat{\mu}(k) \psi_k(t)$$

so $\widehat{\mu}(k)$ is an eigenvalue of M_μ (i.e. $\widehat{\mu}(k) \in \sigma(M_\mu)$).

2. Conversely, establish that if $\lambda \in \sigma(M_\mu)$, then $\lambda = \widehat{\mu}(k)$ for some $k \in \mathbb{Z}_p$.

In particular, together with Question 1, this proves that the spectrum agrees with the Fourier coefficients of μ:

$$\sigma(M_\mu) = \{\widehat{\mu}(k) : k \in \mathbb{Z}_p\}.$$

Hint: Use convolution theorem, Fourier series, and homogeneity of Fourier transform: $\widehat{\lambda f} = \lambda \widehat{f}$ for all $\lambda \in \mathbb{C}$.

Solution. Let $\lambda \in \sigma(M_\mu)$. Then there exists $\psi : \mathbb{Z}_p \to \mathbb{C}$ which is non-zero such that

$$M_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p. \tag{0.2}$$

Fix $k \in \mathbb{Z}_p$ such that $\widehat{\psi}(k) \neq 0$. Such k exists since $\psi \neq 0$: indeed, if $\widehat{\psi}(k) = 0$ for all $k \in \mathbb{Z}_p$ we would have by the Fourier series for all $t \in \mathbb{Z}_p$ that

$$\psi(t) = \frac{1}{p} \sum_{k \in \mathbb{Z}_p} \widehat{\psi}(k) e^{2\pi i k t/p} = 0$$

so $\psi(t) = 0$ for all $t \in \mathbb{Z}_p$. Thus we know that $\widehat{\psi}(k) \neq 0$ for this $k \in \mathbb{Z}_p$.

Taking now Fourier transform from both sides of (0.2) gives us

$$\widehat{\psi * \mu}(k) = \widehat{\lambda\psi}(k).$$

By the convolution theorem and the homogeneity of Fourier transform we have

$$\widehat{\psi}(k)\widehat{\mu}(k) = \lambda\widehat{\psi}(k).$$

Dividing $\widehat{\psi}(k) \neq 0$ gives us

$$\widehat{\mu}(k) = \lambda$$

as claimed.

3. Define the iteration $M_\mu^n f = M_\mu(M_\mu^{n-1} f)$ with $M_\mu^0 f = f$ for $n \geq 1$. Prove that the L^1 norm

$$\|M_\mu^n f\|_1 \leq \sqrt{p} \sqrt{\sum_{k \in \mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}.$$

Hint: Use Cauchy-Schwartz for the map $t \mapsto |f * \mu^{*n}(t)|$ and the constant function 1, and then apply Plancherel's theorem and the convolution theorem

Solution. We have that

$$\|M_\mu^n f\|_1 = \sum_{t \in \mathbb{Z}_p} |f * \mu^{*n}(t)|$$

By Cauchy-Schwartz applied with $t \mapsto |f * \mu^{*n}(t)|$ and the constant function 1, we have

$$\sum_{t \in \mathbb{Z}_p} |f * \mu^{*n}(t)| = \langle |f * \mu^{*n}|, 1 \rangle \leq \|f * \mu^{*n}\|_2 \|1\|_2$$

Here

$$\|1\|_2 = \sqrt{\sum_{t \in \mathbb{Z}_p} 1^2} = \sqrt{p}$$

and by the Plancherel's theorem and convolution theorem we obtain

$$\|f * \mu^{*n}\|_2 = \|\widehat{f * \mu^{*n}}\|_2 = \|\widehat{f}\widehat{\mu^{*n}}\|_2 = \|\widehat{f}\widehat{\mu}^n\|_2 = \sqrt{\sum_{k \in \mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}$$

so the claim follows.