**Analysis, Random Walks and Groups**
Spring 2020

Week 6 tutorial

Given a probability distribution $\mu$ on $\mathbb{Z}_p$ and a mapping $f : \mathbb{Z}_p \to \mathbb{C}$, define the **multiplier operator**

$$M_\mu f(t) = f * \mu(t), \quad t \in \mathbb{Z}_p.$$

Thus $M_\mu f$ is again a function $\mathbb{Z}_p \to \mathbb{C}$ with added convolution from $\mu$. Multiplier operators are an important class of operators in harmonic analysis which appear commonly in the study of PDEs, fractals and signals.

A complex number $\lambda \in \mathbb{C}$ is an **eigenvalue** of $M_\mu$ if there exists a non-zero $\psi : \mathbb{Z}_p \to \mathbb{C}$ (called **eigenfunction** of $M_\mu$) such that

$$M_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

The **spectrum** $\sigma(M_\mu)$ of $M_\mu$ is then the collection of all eigenvalues

$$\sigma(M_\mu) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } M_\mu\}$$

**1.** Given a probability distribution $\mu$ on $\mathbb{Z}_p$, prove that for each $k \in \mathbb{Z}_p$, the Fourier transform $\widehat{\mu}(k)$ is an eigenvalue of the multiplier operator $M_\mu$.

*Hint: Use convolution theorem, Fourier series, and attempt to prove the function $\psi_k(t) = e^{2\pi i k/tp}, \quad t \in \mathbb{Z}_p$, is an eigenfunction of $M_\mu$ with eigenvalue $\widehat{\mu}(k)$.*

**2.** Conversely, establish that if $\lambda \in \sigma(M_\mu)$, then $\lambda = \widehat{\mu}(k)$ for some $k \in \mathbb{Z}_p$.

In particular, together with Question 1, this proves that the spectrum agrees with the Fourier coefficients of $\mu$:

$$\sigma(M_\mu) = \{\widehat{\mu}(k) : k \in \mathbb{Z}_p\}.$$

*Hint: Use convolution theorem, Fourier series, and homogeneity of Fourier transform: $\widehat{\lambda f} = \lambda \widehat{f}$ for all $\lambda \in \mathbb{C}$.*

**3.** Define the iteration $M_\mu^n f = M_\mu(M_\mu^{n-1} f)$ with $M_\mu^0 f = f$ for $n \geq 1$. Prove that the $L^1$ norm

$$\|M_\mu^n f\|_1 \leq \sqrt{p} \sqrt{\sum_{k \in \mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}.$$

*Hint: Use Cauchy-Schwartz for the map $t \mapsto |f * \mu^{*n}(t)|$ and the constant function 1, and then apply Plancherel's theorem and the convolution theorem*