Analysis, Random Walks and Groups
Spring 2020

Week 5 tutorial

We say that a probability distribution μ on $\mathbb{Z}_p$ has a **spectral gap** if

$$|\widehat{\mu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}.$$

1. Find the Fourier transform of the probability distribution $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ in $\mathbb{Z}_4$ (all the values of $\widehat{\mu}(k)$) and use this to prove that μ does not have a spectral gap.

Notice also that μ is supported on a subgroup $\{0, 2\}$ so it cannot be ergodic. Later we will see that in general spectral gap implies ergodicity!

Solution. The Fourier transform of μ at $k \in \mathbb{Z}_4$ is

$$\widehat{\mu}(k) = \sum_{t \in \mathbb{Z}_4} \mu(t) e^{-2\pi ikt/4} = \frac{1}{2}e^0 + \frac{1}{2}e^{4\pi ik/4} = \frac{1}{2}(1 + e^{k\pi i}).$$

We see that

$$e^{0\pi i} = 1, e^{1\pi i} = -1, e^{2\pi i} = 1, e^{3\pi i} = -1.$$

Thus

$$\widehat{\mu}(0) = 1, \widehat{\mu}(1) = 0, \widehat{\mu}(2) = 1, \widehat{\mu}(3) = 0.$$

Hence

$$|\widehat{\mu}(k)| = 1$$

for some $k \in \mathbb{Z}_4 \setminus \{0\}$ (value $k = 2$). In particular, μ does not have a spectral gap.

2. Same as question 1 but consider $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Prove that ν has a spectral gap.

Notice also that the support $\operatorname{spt} \nu = \{0, 1\}$ which is not a coset of a proper subgroup of $\mathbb{Z}_4$ so ν is ergodic.

Solution. The Fourier transform of ν at $k \in \mathbb{Z}_4$ is

$$\widehat{\nu}(k) = \sum_{t \in \mathbb{Z}_4} \mu(t) e^{-2\pi ikt/4} = \frac{1}{2}e^0 + \frac{1}{2}e^{2\pi ik/4} = \frac{1}{2}(1 + e^{k\pi i/2}).$$

We see that

$$e^{0\pi i/2} = 1, e^{1\pi i/2} = i, e^{2\pi i/2} = -1, e^{3\pi i/2} = -i.$$

Thus

$$\widehat{\nu}(0) = 1, \widehat{\nu}(1) = \frac{1}{2}(1 + i), \widehat{\nu}(2) = 0, \widehat{\nu}(3) = \frac{1}{2}(1 - i).$$

Hence

$$|\widehat{\nu}(1)| = \frac{\sqrt{2}}{2} < 1, |\widehat{\nu}(1)| = 0 < 1, |\widehat{\nu}(3)| = \frac{\sqrt{2}}{2} < 1$$

Thus

$$|\widehat{\nu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}$$

so ν has a spectral gap.

3. Define the Laplace operator Δ for functions $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$\Delta f(t) = \frac{f(t \oplus 1) + f(t \ominus 1)}{2} - f(t), \quad t \in \mathbb{Z}_p.$$

We say that $\psi : \mathbb{Z}_p \to \mathbb{C}$ is an **eigenfunction** of the Laplacian with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta\psi(t) = \lambda\psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

Prove that the function

$$\psi_k(t) := e^{2\pi ikt/p}, \quad t \in \mathbb{Z}_p$$

is an eigenfunction of the Laplacian with eigenvalue $\lambda_k = \cos(2\pi k/p) - 1$.

Solution. By definition of the Laplacian, we have

$$\Delta\psi_k(t) = \frac{1}{2}(\psi_k(t \oplus 1) + \psi_k(t \ominus 1)) - \psi_k(t) = \frac{1}{2}(e^{2\pi ik(t\oplus 1)/p} + e^{2\pi ik(t\ominus 1)/p}) - \psi_k(t)$$

We notice that
$$e^{2\pi ik(t\oplus 1)/p} = e^{2\pi i(kt\oplus k)/p} = \psi_k(t)e^{2\pi ik/p}$$
and
$$e^{2\pi ik(t\ominus 1)/p} = e^{2\pi i(kt\ominus k)/p} = \psi_k(t)e^{-2\pi ik/p}.$$
Hence
$$\frac{1}{2}(e^{2\pi ik(t\oplus 1)/p} + e^{2\pi ik(t\ominus 1)/p}) = \frac{e^{2\pi ik/p} + e^{-2\pi ik/p}}{2} \cdot \psi_k(t) = \cos(2\pi k/p)\psi_k(t)$$
so with $\lambda_k = \cos(2\pi k/p) - 1$ we have
$$\Delta\psi_k(t) = \lambda_k\psi_k(t)$$
for all $t \in \mathbb{Z}_p$. Thus ψ_k is an eigenfunction of the Laplacian with eigenvalue λ_k.