Analysis, Random Walks and Groups
Spring 2020

Week 5 tutorial

We say that a probability distribution μ on $\mathbb{Z}_p$ has a **spectral gap** if

$$|\widehat{\mu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}.$$

1. Find the Fourier transform of the probability distribution $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ in $\mathbb{Z}_4$ (all the values of $\widehat{\mu}(k)$) and use this to prove that μ does not have a spectral gap.

Notice also that μ is supported on a subgroup $\{0,2\}$ so it cannot be ergodic. Later we will see that in general spectral gap implies ergodicity!

2. Same as question 1 but consider $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Prove that ν has a spectral gap.

Notice also that the support $\operatorname{spt} \nu = \{0,1\}$ which is not a coset of a proper subgroup of $\mathbb{Z}_4$ so ν is ergodic.

3. Define the Laplace operator Δ for functions $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$\Delta f(t) = \frac{f(t \oplus 1) + f(t \ominus 1)}{2} - f(t), \quad t \in \mathbb{Z}_p.$$

We say that $\psi : \mathbb{Z}_p \to \mathbb{C}$ is an **eigenfunction** of the Laplacian with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta\psi(t) = \lambda\psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

Prove that the function

$$\psi_k(t) := e^{2\pi i k t/p}, \quad t \in \mathbb{Z}_p$$

is an eigenfunction of the Laplacian with eigenvalue $\lambda_k = \cos(2\pi k/p) - 1$.