Analysis, Random Walks and Groups
Spring 2019

Week 9 tutorial - SOLUTIONS

1. Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ in $\mathbb{Z}_4$. Find an upper bound for the mixing time $n_{\text{mix}}(1/100)$ for μ, that is, after how many convolutions μ^{*n} is the total variation distance

$$d(\mu^{*n}, \lambda) < \frac{1}{100}?$$

Solution. By the Upper Bound Lemma, we have

$$d(\mu^{*n}, \lambda) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \{0\}} |\widehat{\mu}(k)|^{2n}}.$$

In the Week 7 exercise 2 (the measure ν there), we computed that

$$|\widehat{\mu}(1)| = \frac{\sqrt{2}}{2}, \quad |\widehat{\mu}(2)| = 0, \quad |\widehat{\mu}(3)| = \frac{\sqrt{2}}{2}.$$

Hence

$$\frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \{0\}} |\widehat{\mu}(k)|^{2n}} \leq \frac{1}{2}\sqrt{\Big(\frac{\sqrt{2}}{2}\Big)^{2n} + \Big(\frac{\sqrt{2}}{2}\Big)^{2n}} = \Big(\frac{\sqrt{2}}{2}\Big)^{n+1}.$$

Now, to have

$$\Big(\frac{\sqrt{2}}{2}\Big)^{n+1} < \frac{1}{100}$$

after taking logarithms, we have

$$(n+1)\log\Big(\frac{\sqrt{2}}{2}\Big) < \log\Big(\frac{1}{100}\Big)$$

and as $\frac{\sqrt{2}}{2} < 1$, the logarithm is negative, so

$$n > \frac{\log 100}{\log 2 - \log\sqrt{2}} - 1 \approx 12.2877.$$

Hence $n_{\text{mix}}(1/100) \leq 13$.

2. Define the subgroup $\Gamma := \{0, 2\} \subset \mathbb{Z}_4$. Let μ be any probability distribution on $\mathbb{Z}_4$ with support $\text{spt}(\mu) = \Gamma$. Define the uniform measure on Γ by

$$\nu_\Gamma(t) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2.$$

Prove the following version of the Upper Bound Lemma:

$$d(\mu^{*n}, \nu_\Gamma) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \Gamma} |\widehat{\mu}(k)|^{2n}}$$

Solution.
By the L^1 identity

$$4d(\mu^{*n}, \nu_\Gamma)^2 = \Big(\sum_{t=0}^{p-1} |\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2.$$

Since $\lambda(t) = 1/p$ for all $t \in \mathbb{Z}_p$, we have

$$\Big(\sum_{t=0}^{p-1} |\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2 = p^2\Big(\sum_{t=0}^{p-1} \lambda(t)|\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2.$$

Using the definition of the inner product for the functions

$$f(t) := \lambda(t), \quad \text{and} \quad g(t) := |\mu^{*n}(t) - \nu_\Gamma(t)|, \quad t \in \mathbb{Z}_p,$$

and Cauchy-Schwartz Inequality we obtain

$$\Big(\sum_{t=0}^{p-1}\lambda(t)|\mu^{*n}(t)-\nu_\Gamma(t)|\Big)^2 = |\langle f,g\rangle|^2 \leq \|f\|_2^2\|g\|_2^2.$$

The L^2 norms here are

$$\|f\|_2^2 = \sum_{t\in\mathbb{Z}_p}\lambda(t)^2 = \sum_{t\in\mathbb{Z}_p}p^{-2} = p^{-1}$$

and by definition of g:

$$\|g\|_2^2 = \sum_{t\in\mathbb{Z}_p}|\mu^{*n}(t)-\nu_\Gamma(t)|^2.$$

Hence we have proved

$$4d(\mu^{*n},\lambda)^2 \leq p\sum_{t\in\mathbb{Z}_p}|\mu^{*n}(t)-\nu_\Gamma(t)|^2 = p\|\mu^{*n}-\nu_\Gamma\|_2^2$$

By Plancherel's Theorem, we have that

$$p\|\mu^{*n}-\nu_\Gamma\|_2^2 = \|\widehat{\mu^{*n}-\nu_\Gamma}\|_2^2 = \|\widehat{\mu^{*n}}-\widehat{\nu_\Gamma}\|_2^2 = \sum_{k=0}^{p-1}|\widehat{\mu^{*n}}(k)-\widehat{\nu_\Gamma}(k)|^2.$$

Computing Fourier transform of ν_Γ we see that

$$\widehat{\nu_\Gamma}(k) = \frac{1}{2}(1+e^{-\pi ik}) = \begin{cases}1, & k\in\Gamma;\\ 0, & k\notin\Gamma.\end{cases}$$

On the other hand, by the Convolution Theorem we have

$$\widehat{\mu^{*n}}(k) = \widehat{\mu}(k)^n.$$

As $\operatorname{spt}\mu = \{0,2\}$ we know that there exists $0<\alpha<1$ such that $\mu = \alpha\delta_0 + (1-\alpha)\delta_2$. Thus

$$\widehat{\mu}(k) = \alpha + (1-\alpha)e^{-\pi ik}.$$

Thus

$$\widehat{\mu}(0) = 1 \quad\text{and}\quad \widehat{\mu}(2) = 1.$$

Hence the difference

$$\widehat{\mu^{*n}}(k) - \widehat{\nu_\Gamma}(k) = \begin{cases}0, & k\in\Gamma;\\ \widehat{\mu^{*n}}(k), & k\notin\Gamma.\end{cases}$$

This gives

$$\sum_{k=0}^{p-1}|\widehat{\mu^{*n}}(k)-\widehat{\nu_\Gamma}(k)|^2 = \sum_{k\in\mathbb{Z}_p\setminus\Gamma}|\widehat{\mu}(k)|^{2n}.$$

Dividing by 4 and taking square roots from both sides gives the claim.

3. In the previous Exercise 2., after how many convolutions μ^{*n} is the total variation distance

$$d(\mu^{*n},\nu_\Gamma) < \frac{1}{100}?$$

Solution. We have that $\mu = \alpha\delta_0 + (1-\alpha)\delta_1$ for some $0<\alpha<1$ since $\operatorname{spt}\mu = \Gamma = \{0,2\}$. Hence

$$\widehat{\mu}(k) = \alpha + (1-\alpha)e^{-\pi ik}$$

Thus

$$\widehat{\mu}(1) = \widehat{\mu}(3) = 2\alpha - 1$$

so by the previous exercise

$$d(\mu^{*n},\nu_\Gamma) \leq \frac{1}{2}\sqrt{\sum_{k\in\mathbb{Z}_4\setminus\Gamma}|\widehat{\mu}(k)|^{2n}} \leq \frac{1}{2}\sqrt{2|2\alpha-1|^{2n}} = \frac{\sqrt{2}}{2}|2\alpha-1|^n.$$

Thus after taking logarithms we know that (when $\alpha\neq 1/2$) that

$$\frac{\sqrt{2}}{2}|2\alpha-1|^n < \frac{1}{100}$$

if and only if

$$n > \frac{\log(1/(50\sqrt{2}))}{\log|2\alpha - 1|}$$

so if $\alpha \neq 1/2$ and

$$n \geq \left\lceil \frac{\log(1/(50\sqrt{2}))}{\log|2\alpha - 1|} \right\rceil$$

then

$$d(\mu^{*n}, \nu_\Gamma) < \frac{1}{100}.$$

When $\alpha = 1/2$, then $\mu = \nu_\Gamma$, so as Γ is a subgroup we have for all $n \in \mathbb{N}$ that

$$\mu^{*n} = \nu_\Gamma^{*n} = \nu_\Gamma,$$

which implies

$$d(\mu^{*n}, \nu_\Gamma) = 0.$$

Thus $n \geq 1$ is enough (recall that $\mu^{*1} = \mu$).