Analysis, Random Walks and Groups
Spring 2019

Week 6 tutorial

1. Let $X_1, X_2, \ldots,$ be the random walk driven by $\mu_\alpha = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Compute the probabilities
(1) $\mathbb{P}(X_1 = 1, X_2 = 2)$.
(2) $\mathbb{P}(X_1 = 0, X_2 = 2)$.
Compare these to the probabilities
(a) $\mathbb{P}(1 \oplus t_2 = 2)$.
(b) $\mathbb{P}(0 \oplus t_2 = 2)$.

Solution. First of all, we have seen already in lectures that the convolution

$$\mu * \mu(t) = \frac{1}{4}\delta_{-2} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2$$

Thus
(1)

$$\mathbb{P}(X_1 = 1, X_2 = 2) = \mu(1)\mu * \mu(2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8};$$

(2)

$$\mathbb{P}(X_1 = 0, X_2 = 2) = \mu(0)\mu * \mu(2) = 0.$$

On the other hand,
(a)

$$\mathbb{P}(1 \oplus t_2 = 2) = \mathbb{P}(t_2 = 1) = \mu(1) = \frac{1}{2};$$

(b)

$$\mathbb{P}(0 \oplus t_2 = 2) = \mathbb{P}(t_2 = 2) = \mu(2) = 0.$$

2. Let $\mu_\alpha = \alpha\delta_0 + (1-\alpha)\delta_1$ on $\mathbb{Z}_5$ for some $0 < \alpha \le 1$. For which α is μ_α ergodic? Explain your answer. Compute $d(\mu_\alpha * \mu_\alpha, \lambda)$ as a function of α.

Solution. If $\alpha = 1$, the support of μ_α is $\{0\}$, which is a trivial subgroup of $\mathbb{Z}_p$, in particular by the subgroup characterisation of ergodicity $\mu_\alpha = \delta_0$ cannot be ergodic.

If $\alpha < 1$, the support of μ_α is $\{0, 1\}$. Since 5 is a prime number, the only subgroups of $\mathbb{Z}_5$ are $\{0\}$ and $\mathbb{Z}_5$ and so $\{0, 1\}$ cannot be a coset of any of these subgroups. Hence by the subgroup characterisation μ_α is ergodic.

The convolution

$$\mu_\alpha * \mu_\alpha(t) = \sum_{s \in \mathbb{Z}_p} \mu_\alpha(t \ominus s)\mu_\alpha(s) = \alpha\mu_\alpha(t) + (1-\alpha)\mu_\alpha(t \ominus 1).$$

We have $t \ominus 1 = 0$ when $t = 1$ and $t \ominus 1 = 1$ when $t = 2$. Thus we have

$$\mu_\alpha(t \ominus 1) = \alpha\delta_0(t \ominus 1) + (1-\alpha)\delta_1(t \ominus 1) = \alpha\delta_1(t) + (1-\alpha)\delta_2(t).$$

Hence the convolution

$$\mu_\alpha * \mu_\alpha(t) = \alpha[\alpha\delta_0(t) + (1-\alpha)\delta_1(t)] + (1-\alpha)[\alpha\delta_1(t) + (1-\alpha)\delta_2(t)],$$

which equals to

$$\alpha^2\delta_0(t) + 2\alpha(1-\alpha)\delta_1(t) + (1-\alpha)^2\delta_2(t).$$

By the L^1 identity for the total variation distance we have

$$d(\mu_\alpha * \mu_\alpha, \lambda) = \frac{1}{2}\sum_{t \in \mathbb{Z}_p} |\mu_\alpha * \mu_\alpha(t) - \lambda(t)|,$$

which, since $\mu_\alpha * \mu_\alpha(t) = 0$ when $t \neq 0$ and when $t = 1$ we have

$$\frac{|\alpha^2 - 1/5| + |2\alpha(1-\alpha) - 1/5| + |(1-\alpha)^2 - 1/5| + 2/5}{2},$$

which is our function of α.

3. Let μ be a probability distribution on $\mathbb{Z}_4$, which is not a Dirac mass, and assume that the support

$$\mathrm{spt}(\mu) = \{t \in \mathbb{Z}_4 : \mu(t) > 0\}$$

is a coset of a proper non-trivial subgroup of $\mathbb{Z}_4$. Is there a limit

$$\mu_\infty = \lim_{n\to\infty} \mu^{*n}?$$

What is it? No proofs necessary, just have a think how to maybe prove this.

Solution. The only proper subgroup of $\mathbb{Z}_4$ is $\Gamma = \{0, 2\}$. The only coset of this is $\Gamma \oplus 1 = \{1, 3\}$. If we consider first the case μ is concentrated on Γ, as μ is not a Dirac mass, then μ^{*n} can only have a limit

$$\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$$

that is, ν is the uniform measure on the subgroup Γ

Formal proof: If μ is concentrated on Γ, then we could identify Γ with $\mathbb{Z}_2$ as follows. First of all, group theoretically Γ is **isomorphic** to $\mathbb{Z}_2$, that is there exists a bijection $\varphi : \mathbb{Z}_2 \to \Gamma$ such that $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$, $a, b, \in \mathbb{Z}_2$. Then

$$\nu = \varphi_*\lambda,$$

where λ is the uniform measure on $\mathbb{Z}_2$ and $\varphi_*\lambda$ is the **push forward distribution** (recall earlier exercises on Wasserstein distance), which is defined by

$$\varphi_*\lambda(A) = \lambda(\varphi^{-1}A), \quad A \subset \Gamma.$$

Since the push forward under inverse φ^{-1} of μ, that is, $\varphi_*^{-1}\mu$ is ergodic (it is not concentrated on any proper subgroup of $\mathbb{Z}_2$ as it is not a Dirac mass), the iterated convolutions

$$(\varphi_*^{-1}\mu)^{*n} \to \lambda$$

as $n \to \infty$ in $\mathbb{Z}_2$ by the subgroup characterisation of ergodicity. Since $\lambda = \varphi_*^{-1}\nu$, this gives that $\mu^{*n} \to \nu$ in Γ so the limit $\mu_\infty = \nu$.

As for the coset $\Gamma \oplus 1$ we have an issue: if μ is supported on $\Gamma \oplus 1 = \{1, 3\}$ the support

$$\mathrm{spt}(\mu * \mu) = \mathrm{spt}(\mu) \oplus \mathrm{spt}(\mu) = \{1, 3\} \oplus \{1, 3\} = \{0, 2\}$$

On on the other hand

$$\mathrm{spt}(\mu * \mu * \mu) = \mathrm{spt}(\mu) \oplus \mathrm{spt}(\mu) \oplus \mathrm{spt}(\mu) = \{1, 3\} \oplus \{1, 3\} \oplus \{1, 3\} = \{0, 2\} \oplus \{1, 3\} = \{1, 3\}$$

and again

$$\mathrm{spt}(\mu * \mu * \mu * \mu) = \{1, 3\} \oplus \{1, 3\} = \{0, 2\}$$

so the support of μ^{*n} alternates between $\{0, 2\}$ and $\{1, 3\}$, which are disjoint. In particular for even n and odd n we get measures μ^{*n} that have never the same support. Thus it is impossible for μ^{*n} to have a limit as $n \to \infty$.