

Analysis, Random Walks and Groups
Spring 2019

Week 5 tutorial

1. Let $p \geq 7$. For $0 < \alpha < 1$ define

$$\mu = \alpha\delta_1 + (1 - \alpha)\delta_{-1}.$$

Compute the convolutions $\mu * \mu$ and $\mu * \mu * \mu$.

2. Prove the following identities for the convolution: for all $f, g, h : \mathbb{Z}_p \rightarrow \mathbb{C}$ we have

(a) **Commutativity:** $f * g = g * f$

(b) **Associativity:** $f * (g * h) = (f * g) * h$

(c) **Linearity:** if $\alpha, \beta \in \mathbb{C}$, then $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$

3. (a) Prove that for all $A, B \subset \mathbb{Z}_p$ the cardinalities

$$\max\{|A|, |B|\} \leq |A \oplus B| \leq |A||B|.$$

(b) Give examples of sets $A, B \subset \mathbb{Z}_p$ such that

$$|A \oplus B| = \max\{|A|, |B|\}.$$

(c) Give examples of sets $A, B \subset \mathbb{Z}_p$ which are not \mathbb{Z}_p such that

$$|A \oplus B| = |A||B|.$$

4. Prove that if $\mu, \nu : \mathbb{Z}_p \rightarrow [0, 1]$ are probability distributions, then the entropy

$$H(\mu * \nu) \leq H(\mu) + H(\nu).$$

Hint: Use the convexity of $\varphi(x) = -x \log x$.