Analysis, Random Walks and Groups
Spring 2019

Week 3 tutorial

Let $0 < \alpha < 1$, integer $p \geq 2$ and define the following probability distribution on $\mathbb{Z}_p$:

$$\mu_\alpha = \alpha\delta_1 + (1-\alpha)\delta_{-1}.$$

1. Find the probabilities of the events:

(a) *"a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is even"*

(b) *"a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is prime"*

Solution 1. For the first event, define the set

$$A = \{t \in \mathbb{Z}_p : t \text{ is even}\}.$$

Then we are asking the measure $\mu_\alpha(A)$. The answer will depend on whether p is even or odd.

Note that $-1 = p - 1$. Thus:

- If p is odd, then $p-1$ is even, that is, $\delta_{p-1}(A) = 1$, so

$$\mu_\alpha(A) = \alpha \cdot 0 + (1-\alpha) \cdot 1 = 1 - \alpha.$$

Hence if p is odd, then the answer is a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is a even with probability $1 - \alpha$.

- If p is even, then $p-1$ is odd, that is, $\delta_{p-1}(A) = 0$, so

$$\mu_\alpha(A) = \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0.$$

Hence if p is even, then the answer is a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is a even with probability 0.

Now for the second event, define the set

$$B = \{t \in \mathbb{Z}_p : t \text{ is prime}\}.$$

We need to find the measure $\mu_\alpha(B)$.

Note that primes are always strictly greater than 1, so $1 \notin B$.

- If $p-1$ is a prime, then $\delta_{p-1}(B) = 1$ so

$$\mu_\alpha(B) = \alpha \cdot 0 + (1-\alpha) \cdot 1 = 1 - \alpha.$$

Hence if $p-1$ is a prime, then the answer is a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is a prime with probability $1 - \alpha$.

- If $p-1$ is not a prime, then $\delta_{p-1}(B) = 1$ so

$$\mu_\alpha(B) = \alpha \cdot 0 + (1-\alpha) \cdot 0 = 0.$$

Hence if $p-1$ is not a prime, then the answer is a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is a prime with probability 0.

2. Define a function $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$f(t) = \begin{cases} +1; & t \text{ is even;} \\ -1; & t \text{ is odd.} \end{cases}$$

Find the integral (i.e. expectation) $\mu_\alpha(f)$ of f.

Solution 2. Firstly we know that $-1 = p-1$. Thus the value of $\mu_\alpha(p-1)$ depends whether p is even or odd.

Case 1: p is odd. Then $p-1$ is even so

$$\mu_\alpha(f) = \sum_{t \in \mathbb{Z}_p} f(t)\mu_\alpha(t) = -1 \cdot \alpha + 1 \cdot (1-\alpha) = 2 - \alpha.$$

Case 2: p is even. Then $p-1$ is odd so

$$\mu_\alpha(f) = \sum_{t \in \mathbb{Z}_p} f(t)\mu_\alpha(t) = -1 \cdot \alpha + (-1) \cdot (1-\alpha) = -1.$$

The number $\mu_\alpha(f)$ tells us the average sign of a random number $t \in \mathbb{Z}_p$ (where sign is $+1$ if even and -1 if odd).

3. Define a function $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$f(t) = \begin{cases} +1; & t \text{ is even;} \\ -1; & t \text{ is odd.} \end{cases}$$

Find the integral (i.e. expectation) $\lambda(f)$ of f with respect to the uniform measure.

Solution 3. If p is even, we did this already in the lectures, so there are $p/2$ even and $p/2$ odd numbers in $\{0, 1, \dots, p-1\}$. Thus

$$\lambda(f) = \sum_{t \in \mathbb{Z}_p} f(t)\lambda(t) = \frac{p}{2} \cdot \frac{1}{p} - \frac{p}{2} \cdot \frac{1}{p} = 0.$$

If p is odd, then $p-1$ is even (and we define 0 to be even), so there are in total $(p-1)/2$ odd numbers and $(p-1)/2+1$ even numbers. Hence

$$\lambda(f) = \sum_{t \in \mathbb{Z}_p} f(t)\lambda(t) = \Big(\frac{p-1}{2} + 1\Big) \cdot \frac{1}{p} - \frac{p-1}{2} \cdot \frac{1}{p} = \frac{1}{p}.$$