Analysis, Random Walks and Groups
Spring 2019

Week 10 tutorial - SOLUTIONS

1. Let $X_1, X_2, \ldots$ be the random walk on S_{52} driven by the probability distribution μ describing the weak Borel shuffle (recall the first tutorial of the course). Write down the formula for this measure μ. Then, let $e \in S_{52}$ be the identity permutation. Apply the right convolution $\mu *_R \mu$ in the group S_{52} to compute the probability

$$\mathbb{P}(X_2 = e).$$

Solution. Define a permutation

$$\sigma_k(j) = \begin{cases} k & \text{if } j = 0; \\ j-1 & \text{if } 1 \leq j \leq k; \\ j & \text{if } k < j \leq 51; \end{cases}$$

Then define the random permutation $\sigma \in S_{52}$ by choosing $k \in \{0, 1, \ldots, 52\}$ uniformly with probability $1/52$ and setting $\sigma = \sigma_k$. Hence the probability distribution on S_{52} can be defined formally for all $\sigma \in S_{52}$ as

$$\mu(\sigma) = \begin{cases} \frac{1}{52}, & \text{if } \sigma = \sigma_k \text{ for some } k = 0, 1, 2, \ldots, 51; \\ 0, & \text{otherwise.} \end{cases}$$

We have that the probability

$$\mathbb{P}(X_2 = e) = \mu * \mu(e)$$

By the definition of the right convolution we have

$$\mu * \mu(e) = \sum_{\sigma \in S_{52}} \mu(e\sigma^{-1})\mu(\sigma)$$

We note that $e\sigma^{-1} = \sigma^{-1}$ since e is the neutral element of the group S_{52}. Since $\mu(\sigma) = 0$ for all $\sigma \neq \sigma_k$ for some $k = 0, 1, 2, \ldots, 51$, we have

$$\sum_{\sigma \in S_{52}} \mu(e\sigma^{-1})\mu(\sigma) = \sum_{k=0}^{52} \mu(\sigma_k^{-1})\mu(\sigma_k).$$

When $k = 0$ we see that $\sigma_0^{-1} = \sigma_0$ so

$$\mu(\sigma_0^{-1}) = \mu(\sigma_0) = \frac{1}{52}.$$

When $k = 1$ we see that $\sigma_1^{-1} = \sigma_1$ so

$$\mu(\sigma_1^{-1}) = \mu(\sigma_1) = \frac{1}{52}.$$

However, if $k \geq 2$, then σ_k^{-1} can never be any of the permutations σ_ℓ, $\ell = 0, 1, 2, \ldots, 51$ because $\sigma_k^{-1}(k) = 0$ and we have $\sigma_\ell(1) = 0$ when $\ell \geq 1$, which would force $k = 1$. Thus

$$\mu(\sigma_k^{-1}) = 0, \quad \text{for all } k = 2, 3, \ldots, 51.$$

This implies that

$$\sum_{k=0}^{52} \mu(\sigma_k^{-1})\mu(\sigma_k) = \mu(\sigma_0^{-1})\mu(\sigma_0) + \mu(\sigma_1^{-1})\mu(\sigma_1) = \frac{1}{52^2} + \frac{1}{52^2} = \frac{2}{52^2}.$$

Thus we have

$$\mathbb{P}(X_2 = e) = \mu * \mu(e) = \frac{2}{52^2}.$$

2. Prove the Upper Bound Lemma in $\mathbb{Z}_2^d$.

Solution. Let $\mu : \mathbb{Z}_2^d \to [0,1]$ be a probability distribution and $\lambda(t) = 1/2^d$, $t \in \mathbb{Z}_2^d$, the uniform distribution on $\mathbb{Z}_2^d$. Fix $n \in \mathbb{N}$. We claim that

$$d(\mu^{*n}, \lambda) \le \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_2^d \setminus \{0\}} |\widehat{\mu}(k)|^{2n}}.$$

By the L^1 identity, we have

$$4d(\mu^{*n}, \lambda)^2 = \Big(\sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|\Big)^2.$$

Since $\lambda(t) = 1/2^d$ for all $t \in \mathbb{Z}_2^d$, we have

$$\Big(\sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|\Big)^2 = 2^{2d}\Big(\sum_{t \in \mathbb{Z}_2^d} \lambda(t)|\mu^{*n}(t) - \lambda(t)|\Big)^2.$$

Using the definition of the inner product for the functions

$$f(t) := \lambda(t), \quad \text{and} \quad g(t) := |\mu^{*n}(t) - \lambda(t)|, \quad t \in \mathbb{Z}_2^d,$$

and Cauchy-Schwartz Inequality we obtain

$$\Big(\sum_{t \in \mathbb{Z}_2^d} \lambda(t)|\mu^{*n}(t) - \lambda(t)|\Big)^2 = |\langle f, g\rangle|^2 \le \|f\|_2^2 \|g\|_2^2.$$

The L^2 norms here are

$$\|f\|_2^2 = \sum_{t \in \mathbb{Z}_2^d} \lambda(t)^2 = \sum_{t \in \mathbb{Z}_2^d} 2^{-2d} = 2^{-d}$$

and by definition of g:

$$\|g\|_2^2 = \sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|^2.$$

Hence we have proved

$$4d(\mu^{*n}, \lambda)^2 \le 2^d \sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|^2 = 2^d \|\mu^{*n} - \lambda\|_2^2$$

By Plancherel's Theorem, we have that

$$2^d \|\mu^{*n} - \lambda\|_2^2 = \|\widehat{\mu^{*n} - \lambda}\|_2^2 = \|\widehat{\mu^{*n}} - \widehat{\lambda}\|_2^2 = \sum_{k \in \mathbb{Z}_2^d} |\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k)|^2.$$

In $\mathbb{Z}_2^d$ we have that

$$\widehat{\lambda}(k) = \begin{cases} 1, & k = 0; \\ 0, & k \ne 0. \end{cases}$$

On the other hand, as μ^{*n} is a probability distribution, the Fourier transform

$$\widehat{\mu^{*n}}(0) = \sum_{t \in \mathbb{Z}_2^d} \mu^{*n}(t) = 1.$$

Hence the difference

$$\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k) = \begin{cases} 0, & k = 0; \\ \widehat{\mu^{*n}}(k), & k \ne 0. \end{cases}$$

Moreover, by the Convolution Theorem we have

$$\widehat{\mu^{*n}}(k) = \widehat{\mu}(k)^n.$$

Thus

$$\sum_{k \in \mathbb{Z}_2^d} |\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k)|^2 = \sum_{k \in \mathbb{Z}_2^d \setminus \{0\}} |\widehat{\mu}(k)|^{2n}.$$

Dividing by 4 and taking square roots from both sides gives the claim.