0C2 Exercise Sheet 3
Further differentiation

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1. Calculate the differential dy when $y = \sin(x)\ln(\ln(x))$

   Solution. By the chain rule and product rule we have

$$dy = \cos(x)\ln(\ln(x)) + \sin(x) \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}\, dx = \cos(x)\ln(\ln(x)) + \frac{\sin(x)}{x\ln(x)}\, dx$$

2. Use implicit differentiation to find derivative $\frac{dy}{dx}$ as a function of x when $y = \cos^{-1}(x)$.

   Solution. Taking cos from both sides of $y = \cos^{-1}(x)$ gives

$$\cos(y) = x.$$

   Taking d/dx from both sides gives us by the chain rule

$$-\sin(y)\frac{dy}{dx} = 1.$$

   Solving $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}.$$

   Use $y = \cos^{-1}(x)$ to obtain

$$\frac{dy}{dx} = -\frac{1}{\sin(\cos^{-1}(x))}.$$

3. Use implicit differentiation to find derivative $\frac{dy}{dx}$ as a function of x when $x^2 + y^2 = 3$ with $x, y \geq 0$.

   Solution. Taking d/dx from both sides of $x^2 + y^2 = 3$ gives us by the chain rule

$$2x + 2y\frac{dy}{dx} = 0.$$

   Solving $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = -\frac{x}{y}.$$

   Solving y from $x^2 + y^2 = 3$ gives us $y = \pm\sqrt{3 - x^2}$. Since $y \geq 0$ we can just choose $y = \sqrt{3 - x^2}$. Hence the final answer is

$$\frac{dy}{dx} = -\frac{x}{\sqrt{3 - x^2}}.$$

4. Use logarithmic differentiation to find derivative $\frac{dy}{dx}$ as a function of x when $y = \frac{\ln(x)}{\sin(x)}$.

Solution. Take ln from both sides of $y = \frac{\ln(x)}{\sin(x)}$ which gives

$$\ln y = \ln \frac{\ln(x)}{\sin(x)} = \ln(\ln(x)) - \ln(\sin(x)).$$

Taking d/dx from both sides gives us by the chain rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} - \frac{1}{\sin(x)} \cdot (-\cos(x)) = \frac{1}{x \ln x} + \frac{\cos(x)}{\sin(x)} = \frac{1}{x \ln x} + \cot(x).$$

Solving $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = y\Big(\frac{1}{x \ln x} + \cot(x)\Big).$$

Plug in $y = \frac{\ln(x)}{\sin(x)}$ gives us

$$\frac{dy}{dx} = \frac{\ln(x)}{\sin(x)}\Big(\frac{1}{x \ln x} + \cot(x)\Big).$$

5. Use logarithmic differentiation to find derivative $\frac{dy}{dx}$ as a function of x when $y = 2^{2^x}$.

Solution. Take ln from both sides of $y = 2^{2^x}$ to obtain

$$\ln y = \ln 2^{2^x} = 2^x \ln 2.$$

Take ln again to obtain

$$\ln \ln y = \ln(2^x \ln 2) = \ln 2^x + \ln \ln 2 = x \ln 2 + \ln \ln 2.$$

Taking d/dx from both sides gives us by the chain rule two times

$$\frac{1}{\ln y}\frac{1}{y}\frac{dy}{dx} = \ln 2.$$

Solving $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = (y \ln y) \ln 2.$$

Plug in $y = 2^{2^x}$ gives us

$$\frac{dy}{dx} = (2^{2^x} \ln 2^{2^x}) \ln 2 = 2^{2^x} 2^x (\ln 2)^2.$$

6. Use parametric differentiation to find derivative $\frac{dy}{dx}$ as a function of t when $x = t^2$ and $y = \cos(t)$.

Solution. We have

$$\frac{dx}{dt} = 2t$$

and

$$\frac{dy}{dt} = -\sin(t).$$

Hence when $\sin(t) \neq 0$ we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{2t}{\sin(t)}.$$

7. Use parametric differentiation to find derivative $\dfrac{dy}{dx}$ as a function of t when $x = \cos(t)$ and $y = \sin(t)$.

   Solution. We have

$$\frac{dx}{dt} = \sin(t)$$

   and

$$\frac{dy}{dt} = -\cos(t).$$

   Hence when $\cos(t) \neq 0$ we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\sin(t)}{-\cos(t)} = -\tan(t).$$

8. Find the Taylor expansion of $\tan(x)$ near $x = 0$ up to degree 3.

   Solution. We have by derivative rules for cosine, chain rule and the product rule that

   - $f(x) = \tan(x)$;
   - $f^{(1)}(x) = \sec^2(x)$;
   - $f^{(2)}(x) = 2\sec(x)\sec(x)\tan(x) = 2\sec^2(x)\tan(x)$;
   - $f^{(3)}(x) = 2(2\sec^2(x)\tan(x))\tan(x) + 2\sec^2(x)\sec^2(x) = 4\sec^2(x)\tan^2(x) + 2\sec^4(x)$.

   Thus

   - $f(0) = \tan(0) = 0$;
   - $f^{(1)}(0) = \sec^2(0) = \frac{1}{\cos^2(0)} = 1$;
   - $f^{(2)}(0) = 2\sec^2(0)\tan(0) = 0$;
   - $f^{(3)}(0) = 4\sec^2(0)\tan^2(0) + 2\sec^4(0) = 0 + 2 = 2$.

   Therefore, by the Taylor series as $x \to 0$ we obtain

$$\tan(x) \approx f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = 0 + x + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3.$$

9. Using the Taylor expansion for $\frac{1}{1+x^2}$, find the Taylor expansion for $\tan^{-1}(x)$ around 0.

in the lecture notes we have proved that the Taylor series of $1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \ldots$$

Substitute x^2 here gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \ldots$$

From integral tables (or using differential tables) we know that

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$

for a constant C. Therefore

$$\begin{aligned}\tan^{-1}(x) &= \int (1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \ldots)\, dx - C \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots - C\end{aligned}$$

But as $\tan^{-1}(0) = 0$ (since the angle $\theta \in (-\pi/2, \pi/2)$ that gives $\tan(\theta) = 0$ is $\theta = 0$). Hence

$$0 = \tan^{-1}(0) = 0 - \frac{1}{3}0^3 + \frac{1}{5}0^5 - \frac{1}{7}0^7 + \cdots + \frac{(-1)^n}{2n+1}0^{2n+1} + \cdots - C = -C$$

so the constant C must be $C = 0$. Therefore the Taylor series around 0 is

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \ldots.$$

10. Calculate the following limits by using Taylor expansions:

(i) $\lim_{x\to 0} \frac{\sqrt{1+2x^2} - 1}{x^2}$

(ii) $\lim_{x\to 0} \frac{\sqrt[3]{1+x} - 1}{\sqrt[5]{1-x} - 1}$

Solution. In both problems, let us first use the following property: Fix $\alpha > 0$. The Taylor series of $(1+x)^\alpha$ as $x \to 0$ is

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \ldots$$

which follows by differentiating the function $(1+x)^\alpha$ in total n times.

(i) Here set $\alpha = 1/2$ so we have

$$(1+2x^2)^{1/2} = 1 + \frac{1}{2}(2x^2) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2x^2)^2 + \cdots + \frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n!}(2x^2)^n + \ldots$$

Thus after simplifying we obtain

$$\begin{aligned}\frac{\sqrt{1+2x^2}-1}{x^2} &= \frac{1+\frac{1}{2}(2x^2)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2x^2)^2+\dots+\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}(2x^2)^n+\dots-1}{x^2}\\ &= \frac{\frac{1}{2}(2x^2)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2x^2)^2+\dots+\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n)}{n!}(2x^2)^n+\dots-1}{x^2}\\ &= 1-x^2+x^4-\dots.\end{aligned}$$

Hence letting $x \to 0$ gives us

$$\lim_{x\to 0}\frac{\sqrt{1+2x^2}-1}{x^2} = 1.$$

(ii) From the above formula with $\alpha = 1/3$, we have as $x \to 0$ that

$$(1+x)^{1/3} = 1+\frac{1}{3}x+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x^2+\dots+\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}x^n+\dots$$

and similarly with $\alpha = 1/5$ as $x \to 0$ that

$$(1-x)^{1/5} = 1+\frac{1}{5}(-x)+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2}(-x)^2+\dots+\frac{\frac{1}{5}(\frac{1}{5}-1)\dots(\frac{1}{5}-n+1)}{n!}(-x)^n+\dots$$

Thus after simplifying we obtain

$$\begin{aligned}\frac{\sqrt[3]{1+x}-1}{\sqrt[5]{1-x}-1} &= \frac{1+\frac{1}{3}x+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x^2+\dots+\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}x^n+\dots-1}{1+\frac{1}{5}(-x)+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2}(-x)^2+\dots+\frac{\frac{1}{5}(\frac{1}{5}-1)\dots(\frac{1}{5}-n+1)}{n!}(-x)^n+\dots-1}\\ &= \frac{\frac{1}{3}x+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x^2+\dots+\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}x^n+\dots}{\frac{1}{5}(-x)+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2}(-x)^2+\dots+\frac{\frac{1}{5}(\frac{1}{5}-1)\dots(\frac{1}{5}-n+1)}{n!}(-x)^n+\dots}\\ &= \frac{x(\frac{1}{3}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x+\dots+\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}x^{n-1}+\dots)}{x(-\frac{1}{5}+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2}(-x)+\dots+\frac{\frac{1}{5}(\frac{1}{5}-1)\dots(\frac{1}{5}-n+1)}{n!}(-x)^{n-1}+\dots)}\\ &= \frac{\frac{1}{3}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}x+\dots+\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}x^{n-1}+\dots}{-\frac{1}{5}+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2}(-x)+\dots+\frac{\frac{1}{5}(\frac{1}{5}-1)\dots(\frac{1}{5}-n+1)}{n!}(-x)^{n-1}+\dots}\end{aligned}$$

Hence letting $x \to 0$ gives us

$$\lim_{x\to 0}\frac{\sqrt{1+2x^2}-1}{x^2} = \frac{\frac{1}{3}}{-\frac{1}{5}} = -\frac{5}{3}.$$