0C2 Exercise Sheet 2
Sequences and series: SOLUTIONS

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1. Complete the following sentences using the phrases:
sum, difference, first term, last term, $\mathbf{a_k}$, $\mathbf{a_k = a + (k-1)d}$, $\mathbf{d = a_{k+1} - a_k}$, $\mathbf{\frac{n}{2}}$

The kth term of the sequence $a_1, a_2, a_3, \ldots$ is denoted by ____________. We say that the sequence is an arithmetic progression if each pair of consecutive terms have a common ____________, d. That is $d =$ ____________, for all k. If the arithmetic progression has ____________ a, then the kth term can be expressed as ____________. The ____________ of the first n terms of an arithmetic progression can be found by adding together the ____________ and the ____________, and multiplying the result by ____________.

Solution. a_k, difference, $d = a_{k-1} - a_k$, first term, $a_k = a + (k-1)d$, sum, first term, last term.

2. Find the 100^{th} term of the following arithmetic progression:
$\frac{7}{2}$, $\frac{19}{4}$, 6, $\frac{29}{4}$, ...
What is the sum of the first 100 terms?

Solution. $a = 7/2$, $d = 5/4$ so $a_{100} = a + 99d = 7/2 + 99 \cdot 5/4 = 509/4$. Moreover, the sum

$$S_{100} = \frac{100}{2}(a + a_{100}) = \frac{100}{2}(a + a_{100}) = \frac{13075}{2}.$$

3. If the first term of an arithmetic progression is 1, and the sum of the first 5 terms is 37, what is the common difference?

Solution. $a = 1$, $S_5 = 37$. We need to find difference d. By formula

$$S_5 = \frac{5}{2}(2a + 4d) = \frac{5}{2}(2 + 4d) = 5 + 10d.$$

Thus $37 = 5 + 10d$, from which we can solve $d = 3$.

4. Complete the following sentences using the phrases:
ratio, $\mathbf{a_k = ar^{k-1}}$, $\mathbf{r = \frac{a_{k+1}}{a_k}}$, **sum,** $\mathbf{\sum_{k=1}^{n} a_k}$, $\mathbf{a\frac{1-r^k}{1-r}}$, **first term,** $\mathbf{\frac{a}{1-r}}$

We say that the sequence $a_1, a_2, a_3, \ldots$ is a geometric progression if each pair of consecutive terms have a common ______________ r. That is ______________. If the geometric progression has first term a and common ______________ r, then the kth term can be expressed as ______________. The ______________ of the first n terms of a geometric progression is given by ______________. The sum of the first n terms of a geometric progression with ______________ a and common ______________ $r \neq 1$ is given by ______________. If $-1 < r < 1$, then the (infinite) sum of all terms in the geometric progression is ______________.

Solution. ratio, $r = \frac{a_{k+1}}{a_k}$, ratio, $a_k = ar^{k-1}$, sum, $\sum_{k=1}^{n} a_k$, first term, ratio, $a\frac{1-r^k}{1-r}$, $\frac{a}{1-r}$.

5. Find the sum of the first 10 terms of the following geometric progression:
$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$
What is the sum of the infinite progression?

Solution. $a = 1$, $r = 2/3$, so

$$S_{10} = a\frac{1-r^{10}}{1-r} = \frac{1-(2/3)^{10}}{1-2/3} = 3 - \frac{2^{10}}{3^9}.$$

Series

$$\sum_{k=1}^{\infty} = \frac{a}{1-r} = \frac{1}{1-2/3} = 3.$$

6. Find the sum of the first 8 terms of the following geometric series:
$1 + 3 + 9 + 27 + 81 + \ldots$
After how many terms is the sum greater than a million?

Solution. $a = 1$, $r = 3$ so

$$S_8 = a\frac{1-r^8}{1-r} = \frac{1-3^8}{1-3} = 3280$$

and

$$S_n = a\frac{1-r^n}{1-r} = \frac{1-3^n}{1-3} = \frac{3^n-1}{2}$$

For $S_n > 1000000$ we need

$$\frac{3^n-1}{2} > 1000000$$

which is equivalent to

$$3^n > 2000001$$

Taking base 3 logarithms gives

$$n > \log_3(2000001) \approx 13.2$$

so if $n \geq 14$ we have $S_n > 1000000$.

7. Use the Binomial Theorem to expand the following:
(i) $(a+b)^7$, (ii) $(\sqrt{3}-\sqrt{2})^4$.
(Note: for part (ii), give the exact answer using a radical; don't give approximate values.)

Solution. (i) The 7th row of Pascal's triangle is $1, 7, 21, 35, 35, 21, 7, 1$ so by Binomial thorem

$$(a+b)^7 = \sum_{k=0}^{7} \binom{7}{k} a^k b^{7-k},$$

is equal to

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^7 + b^7.$$

(ii) the fourth row of Pascal's triangle is $1, 4, 6, 4, 1$ so

$$\begin{aligned}(\sqrt{3}-\sqrt{2})^4 &= \sqrt{3}^4 - 4\sqrt{3}^3\sqrt{2} + 6\sqrt{3}^2\sqrt{2}^2 - 4\sqrt{3}\sqrt{2}^3 + \sqrt{2}^4 \\ &= 49 - 20\sqrt{6}.\end{aligned}$$

8. Expand $(1+x)^6$, and use this to approximate $(1.01)^6$ to four decimal places.

Solution. Sixth row of Pascal's triangle: $1, 6, 15, 20, 15, 6, 1$. Hence

$$(1+x)^6 = \sum_{k=0}^{6} \binom{6}{k} 1^k x^{6-k} = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

Setting $x = 0.01$ we see that

$$(1.01)^6 = (1+x)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

which is equal to

$$(0.01)^6 + 6(0.01)^5 + 15(0.01)^4 + 20(0.01)^3 + 15(0.01)^2 + 6(0.01) + 1$$

We only care about the first 4 digits, so we can drop the term

$$(0.01)^6 + 6(0.01)^5 + 15(0.01)^4 + 20(0.01)^3$$

as these all begin with at least 4 decimal digits (open up each power 0.01 using a calculator). Hence the answer is

$$\approx 15(0.01)^2 + 6(0.01) + 1 = 1.0615$$

in four decimal places.