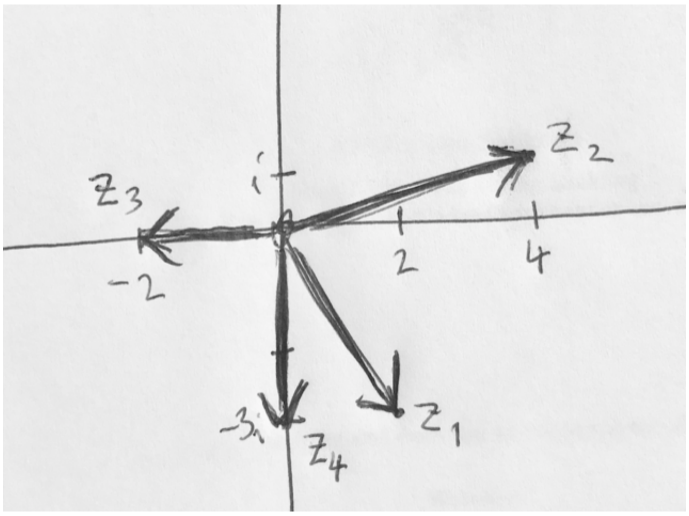
SOLUTIONS to 0C2 Exercise Sheet 1
Complex numbers

In the following exercises 1,2 and 3, define complex numbers:

$$z_1 = 2 - 3i, \quad z_2 = 4 + i, \quad z_3 = -2, \quad z_4 = -3i.$$

1. Draw z_1, z_2, z_3 and z_4 on the complex plane. What are the real and imaginary parts of z_1, z_2, z_3 and z_4?

   Solution:



The real parts are

$$\mathrm{Re}\, z_1 = 2, \quad \mathrm{Re}\, z_2 = 4, \quad \mathrm{Re}\, z_3 = -2, \quad \mathrm{Re}\, z_4 = 0.$$

The imaginary parts are

$$\mathrm{Im}\, z_1 = -3, \quad \mathrm{Im}\, z_2 = 1, \quad \mathrm{Im}\, z_3 = 0, \quad \mathrm{Im}\, z_4 = -3.$$

2. For the z_1, z_2, z_3 and z_4 defined above, compute

   (a) $z_1 + z_2$

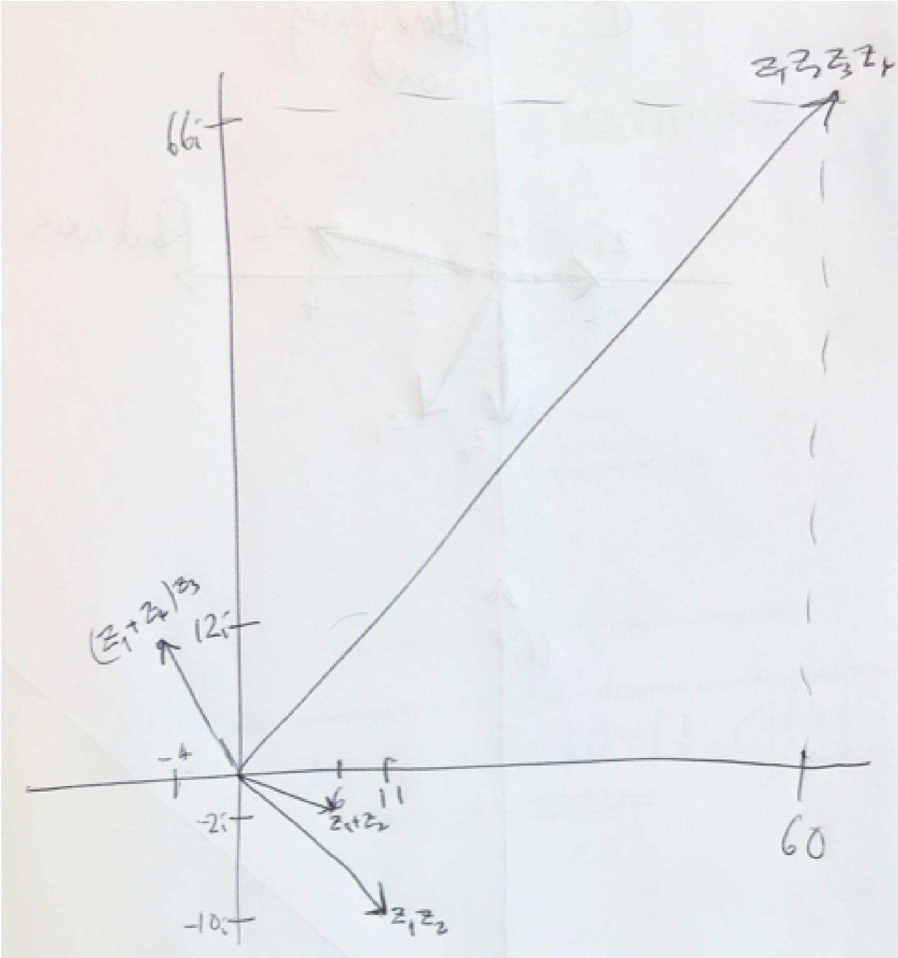
   (b) $z_1 z_2$

   (c) $(z_1 + z_4) z_3$

   (d) $z_1 z_2 z_3 z_4$

   and draw them on the complex plane.

   Solution:



(a) $z_1 + z_2 = (2 - 3i) + (4 + i) = 6 - 2i.$

(b) Using $i^2 = -1$ we get

$$\begin{aligned} z_1 z_2 &= (2-3i)(4+i) \\ &= 2 \cdot 4 + 2 \cdot i + (-3i) \cdot 4 + (-3i) \cdot i \\ &= 8 + 2i - 12i - 3i^2 \\ &= 8 + 2i - 12i + 3 \\ &= 11 - 10i. \end{aligned}$$

(c)

$$\begin{aligned} (z_1 + z_4)z_3 &= ((2-3i) + (-3i))(-2) \\ &= (2 - 6i)(-2) \\ &= -4 + 12i. \end{aligned}$$

(d) From (b) we see that

$$z_1 z_2 = 11 - 10i.$$

Hence using $i^2 = -1$ we obtain

$$z_1 z_2 z_3 z_4 = (11 - 10i)(-2)(-3i) = (11 - 10i)(6i) = 66i - 60i^2 = 60 + 66i.$$

3. Compute the modulus and complex conjugates of z_1, z_2, z_3 and z_4.

Solution:

Using the formula $|z| = \sqrt{a^2 + b^2}$ for $z = a + bi$ we obtain

$$|z_1| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$$|z_2| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$|z_3| = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$$

$$|z_4| = \sqrt{0^2 + (-3)^2} = \sqrt{9} = 3$$

Using the formula $\overline{z} = a - bi$ for $z = a + bi$ we obtain

$$\overline{z_1} = 2 + 3i$$

$$\overline{z_2} = 4 - i$$

$$\overline{z_3} = -2$$

$$\overline{z_4} = 3i$$

4. Find all the solutions $x \in \mathbb{C}$ to the quadratic equation

$$x^2 + 2x + 2 = 0.$$

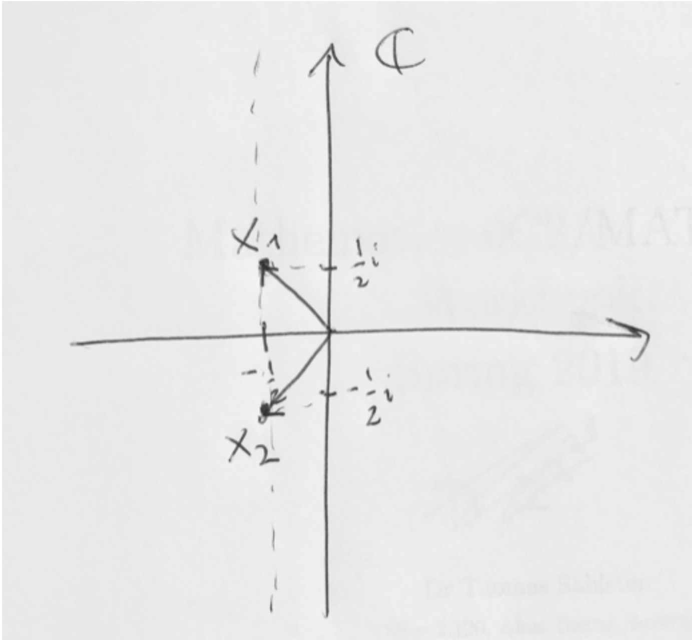
Draw them on the complex plane and compute their modulus and complex conjugates.

Solution: Using the quadratic formula the two solutions in $\mathbb{C}$ are:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 2} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm \sqrt{4}\sqrt{-1}}{4} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{1}{2}i.$$

Thus there are two solutions

$$x_1 = -\frac{1}{2} + \frac{1}{2}i \quad \text{and} \quad x_2 = -\frac{1}{2} - \frac{1}{2}i$$



Moduli are:

$$|x_1| = \sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{\sqrt{8}}$$

and

$$|x_2| = \sqrt{(-\frac{1}{2})^2 + (-\frac{1}{2})^2} = \frac{1}{\sqrt{8}}$$

Notice that they are equal! Moreover, we see that the complex conjugates

$$\overline{x_1} = -\frac{1}{2} - \frac{1}{2}i = x_2$$

and

$$\overline{x_2} = -\frac{1}{2} + \frac{1}{2}i = x_1$$

so we can obtain x_1 from x_2 by taking the complex conjugate.

5. Find the argument $\operatorname{Arg} z$ of the complex number

$$z = 1 + i.$$

Solution: We need to solve the trigonometric equation

$$\tan\theta = \frac{1}{1} = 1.$$

The solution θ between 0 and 2π to this equation is $\theta = \pi/4$. Hence

$$\operatorname{Arg} z = \pi/4.$$

6. Find the polar coordinate form of the complex number

$$z = 1 + i.$$

What is the exponential form of z? Compute the power z^9 using the exponential form.

Solution: To find the polar coordinate form of z, we need to find the modulus $|z|$ and argument $\operatorname{Arg} z$. In the previous exercise we already saw the argument

$$\operatorname{Arg} z = \pi/4$$

so we just need to find the modulus $|z|$. We have

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus the polar coordinate form of z is given by

$$z = \sqrt{2} \cdot [\cos(\pi/4) + i \sin(\pi/4)].$$

For the exponential form of z, recall that the exponential function

$$e^{i\theta} = \cos\theta + i \sin\theta$$

so the exponential form of z is given by

$$z = \sqrt{2} \cdot e^{i\pi/4}.$$

Using the exponential form of z, we see that

$$z^9 = (\sqrt{2})^9 \cdot (e^{i\pi/4})^9 = (\sqrt{2})^9 \cdot e^{i9\pi/4}.$$

Note that this is not the final answer! Since $9/4 = 2.25$ we see that the angle $9\pi/4$ satisfies $2\pi < 9\pi/4 < 4\pi$ so $\operatorname{Arg}(z^9)$ is not equal to $9\pi/4$. Hence we need to substract 2π to get the argument $\operatorname{Arg}(z^9)$. Thus we have

$$\operatorname{Arg}(z^9) = 9\pi/4 - 2\pi = 5\pi/4.$$

Hence the exponential form of z^9 is

$$z^9 = (\sqrt{2})^9 \cdot e^{i5\pi/4}.$$

7. Find the polar coordinate form of the complex number

$$z = 2 - 7i.$$

What is the exponential form of z?

Solution: To find the polar coordinate form of z, we need to find the modulus $|z|$ and argument $\operatorname{Arg} z$. For the modulus, we have

$$|z| = \sqrt{2^2 + (-7)^2} = \sqrt{53}.$$

For the argument $\operatorname{Arg} z$, we need to solve the trigonometric equation

$$\tan\theta = \frac{-7}{2} = -3.5.$$

We cannot find a simple principal solution to this equation. Using a calculator, we see that the angle

$$\theta = \tan^{-1}(-3.5) \approx -1.29 < 0$$

(in radians) solves the above trigonometric equation. However, $\theta \approx -1.29$ is not the argument $\operatorname{Arg} z$ since we defined the argument to be always between 0 and 2π, in particular it cannot be negative -1.29. How to now find the $\operatorname{Arg} z$?

The angle

$$\theta = \tan^{-1}(-3.5) \approx -1.29 < 0$$

is formally the clockwise taken angle, so to get the right angle between 0 and 2π, we need to *add* 2π to get an angle between 0 and 2π:

$$\operatorname{Arg}(z) = \tan^{-1}(-3.5) + 2\pi.$$

Hence the exponential form is given by

$$z = |z|e^{i\operatorname{Arg}(z)} = \sqrt{53}e^{i(\tan^{-1}(-3.5)+2\pi)}.$$