Solutions for MATH19832 Mathematics 0C2, May 2019

Question 1a

Learning Outcome	**Solution**

ILO1 – Define complex numbers and sketch them using the Argand Diagram
ILO2 – Perform arithmetic operations on complex numbers and compute their moduli, arguments and conjugates

Unseen but similar to tutorial exercises. ILO1 is tested at a low level, ILO2 is tested at a low level.

Using $i^2 = -1$ we obtain

$$\begin{aligned} z_1 z_2 z_3 &= (1+i)(-2-i)i \\ &= (-2-i-2i-i^2)i \\ &= (-2-3i+1)i \\ &= (-1-3i)i \\ &= -i-3i^2 \\ &= 3-i. \end{aligned}$$

Hence the real part $\operatorname{Re}(z_1 z_2 z_3) = 3$.

[correct answer 5 marks]

Question 1b

Learning Outcome	**Solution**

ILO2 – Perform arithmetic operations on complex numbers and compute their moduli, arguments and conjugates
ILO5 – Define binomial coefficients, write binomial formula and apply it in integration exercises

Unseen. ILO2 and ILO5 are tested at a medium level.

By the binomial formula

$$(1+i)^5 = \sum_{r=0}^{5} \binom{5}{r} 1^{5-r} i^r = \sum_{r=0}^{5} \binom{5}{r} i^r.$$

From Pascal's triangle we have that

$$\sum_{r=0}^{5} \binom{5}{r} i^r = i^0 + 5i^1 + 10i^2 + 10i^3 + 5i^4 + i^5$$

Since $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, we obtain

$$i^0 + 5i^1 + 10i^2 + 10i^3 + 5i^4 + i^5 = 1 + 5i - 10 - 10i + 5 + i = -4 - 4i.$$

Thus

$$(1+i)^5 = -4 - 4i,$$

so the imaginary part $\operatorname{Im}[(1+i)^5] = -4$.

[correct answer 5 marks]

Question 2a

Learning Outcome	**Solution**

ILO2 – Perform arithmetic operations on complex numbers and compute their moduli, arguments and conjugates

Similar problem in examples sheets. ILO2 is tested at a low level, ILO3 is tested at a medium level.

ILO3 – Express complex numbers in their polar and exponential forms and perform computations using these expressions

The modulus of z is

$$|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

As z is in the first quadrant, the argument of z can be found from the solution in $[0, \pi/2]$ to

$$\cos\theta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

A solution to this in $[0, \pi/2]$ is $\theta = \pi/4$. Hence the argument of z is $\pi/4$. This gives the exponential form of z as

$$z = 2\sqrt{2}e^{i\pi/4}.$$

[correct answer 5 marks]

Question 2b

Learning Outcome

ILO3 – Express complex numbers in their polar and exponential forms and perform computations using these expressions

Solution

Similar problem in examples sheets. ILO3 is tested at a medium level.

Taking to the power 9, we obtain

$$z^9 = 2^9\sqrt{2}^9 e^{i9\pi/4}.$$

The angle $2\pi < 9\pi/4 < 4\pi$ so to get the argument of z^9 in $[0, 2\pi)$ we need to substract 2π to obtain that the argument of z^9 is

$$9\pi/4 - 2\pi = 9\pi/4 - 4\pi/2 = 5\pi/4.$$

[correct answer 5 marks]

Question 3a

Learning Outcome

ILO4 – Define arithmetic, geometric and binomial sequences, evaluate their sums and compute convergent series

Solution

Seen in lectures. ILO4 is tested at a low level.

The first term of the arithmetic progression is $a = 1$. The difference of the arithmetic progression is $d = 2$. By the arithmetic progression sum formula

$$S_n = \frac{1}{2}n\Big(2 \cdot 1 + (n-1) \cdot 2\Big) = n^2.$$

[correct answer 5 marks]

Question 3b

Learning Outcome

ILO4 – Define arithmetic, geometric and binomial sequences, evaluate their sums and compute convergent series

Solution

Similar problem in examples sheets (there done with even numbers). ILO4 is tested at a low level.

The sum $S_n \geq 1000$ as long as $n^2 \geq 1000$ so we have to have $n \geq \sqrt{1000} \approx 31.6$. Hence after $n = 32$ terms the sum is at least 1000.

[correct answer 5 marks]

Question 4a

Learning Outcome	**Solution**
ILO6 – Write Taylor and Maclaurin Series and apply them to compute limits	*This example was done during lectures. ILO6 is tested at a low level.*

We have

$$f^{(1)}(x) = \cos(x), \quad f^{(2)}(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x) = f(x).$$

Evaluating at $x = 0$ gives $f(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$. Therefore the Taylor expansion up to degree 3 of $f(x)$ as $x \to 0$ is

$$\sin(x) \approx x - \frac{1}{3!}\, x^3 = x - \frac{x^3}{6}$$

[correct answer 5 marks]

Question 4b

Learning Outcome	**Solution**
ILO6 – Write Taylor and Maclaurin Series and apply them to compute limits	*Unseen but similar analysis done for different function during lectures. ILO6 is tested at a medium level.*

Since as $x \to 0$ we have

$$\sin(x) \approx x - \frac{1}{3!}\, x^3$$

we obtain

$$\frac{\sin(x)}{x} \approx \frac{x - x^3/6}{x} = 1 - \frac{x^2}{6}.$$

As $x \to 0$, we see that $1 - \frac{x^2}{6} \to 0$ so this implies

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

[correct answer 5 marks]

Question 5a

Learning Outcome	**Solution**
ILO7 – Apply implicit, logarithmic and parametric differentiation in differentiation exercises	*Unseen but similar to example sheets (different functions). ILO7 is tested at a low level.*

By the product and chain rules the derivative

$$\frac{dy}{dx} = e^x \sin(\sin(x)) + e^x \frac{d}{dx}[\sin(\sin(x))] = e^x \sin(\sin(x)) + e^x \cos(\sin(x)) \cos(x).$$

Thus the differential is

$$dy = [e^x \sin(\sin(x)) + e^x \cos(\sin(x)) \cos(x)]\, dx.$$

[correct answer 5 marks]

Question 5b

Learning Outcome

ILO7 – Apply implicit, logarithmic and parametric differentiation in differentiation exercises

Solution

Unseen but similar to example sheets (different functions). ILO7 is tested at a medium level.

Differentiate both sides of

$$x^2y^3 = 1$$

to obtain using the product rule

$$2xy^3 + x^2 \cdot 2y^2 \frac{dy}{dx} = 0$$

Moving terms gives us as $x > 0$ that

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Substitution of $y = x^{-2/3}$ gives us

$$\frac{dy}{dx} = -\frac{x^{-2/3}}{x} = -x^{1/2}.$$

[correct answer 5 marks]

Question 6a

Learning Outcome

ILO7 – Apply implicit, logarithmic and parametric differentiation in differentiation exercises

Solution

Done in lectures. ILO7 is tested at a medium level.

Take logarithms from both sides:

$$\ln y = \ln x^x.$$

Logarithm rule $\log(a^b) = b \log a$ applied gives

$$\ln y = x \ln x.$$

Differentiate both sides and use the chain rule and product rule

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Multiply both sides by y:

$$\frac{dy}{dx} = y \ln x + y.$$

Input $y = x^x$ to get:

$$\frac{dy}{dx} = x^x \ln x + x^x$$

as claimed.

[correct answer 5 marks]

Question 6b

Learning Outcome

ILO7 – Apply implicit, logarithmic and parametric differentiation in differentiation exercises

Solution

Done in lectures. ILO7 is tested at a low level.

Using the parametric differentiation, we first differentiate both parametrisations with respect to t:

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = \cos t.$$

Therefore

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = \frac{1}{-\tan t}$$

for those t when $\tan t \neq 0$.

[correct answer 5 marks]

Question 7a

Learning Outcome

ILO8 – Write integration by parts and integration by substitution formulae and apply them in integration exercises

Solution

Similar problem in lectures. ILO8 is tested at a low level.

We take $u = \ln(x)$ and $dv = dx$ so that $du = \frac{1}{x}dx$ and $v = x$. Applying the integration by parts formula gives

$$\int \ln(x)\ dx = x\ln(x) - \int x\,\frac{dx}{x} = x\ln(x) - \int\ dx = x\ln(x) - x\ + C\,.$$

[correct answer 5 marks]

Question 7b

Learning Outcome

ILO9 – Compute examples of improper integrals

Solution

Unseen. ILO9 is tested at a medium level.

Fix a real number $b > 0$. The antiderivative of x^{-3} if $-\frac{1}{2}x^{-2}$ so

$$\int_1^b x^{-3}\,dx = \Big[-\frac{1}{2}x^{-2}\Big]_{x=1}^{x=b} = -\frac{1}{2}b^{-2} - (-\frac{1}{2}1^{-1}) = \frac{1}{2} - \frac{1}{2b^2}.$$

Now if we increase $b \to \infty$, we see that

$$\frac{1}{2} - \frac{1}{2b^2} \to \frac{1}{2}.$$

Therefore, the improper integral exists and

$$\int_1^\infty x^{-3}\,dx = \frac{1}{2}.$$

[correct answer 5 marks]

Question 8a

Learning Outcome

ILO10 – Express improper rational functions as proper rational functions

ILO11 – Find partial fraction coefficients for proper rational functions

Solution

Seen in lectures. ILO10 is tested at a medium level, ILO11 is tested at a medium level.

Step 1: Using polynomial long division we find that

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2} = (x-2)\ + \frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2}\,.$$

Step 2: To factorise the cubic denominator $D(x) = x^3 + 4x^2 + 5x + 2$, we begin by trying to spot a linear factor by checking the possible factors of the constant term. Since $D(-1) = 0$, we note that by the remainder theorem $x+1$ is a factor. Hence

$$\begin{aligned} x^3+4x^2+5x+2 &= (x+1)(x^2+ax+b) \\ &= (x+1)(x^2+3x+2) \\ &= (x+1)(x+1)(x+2) \\ &= (x+1)^2(x+2)\,. \end{aligned}$$

Step 3: Using the factorisation of the denominator we find

$$\frac{x^2+7x+8}{x^3+4x^2+5x+2} = \frac{x^2+7x+8}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}\,.$$

Step 4: Multiplying both sides of this last equation by x^3+4x^2+5x+2 gives

$$\begin{aligned} x^2+7x+8 &= \frac{A(x+1)^2(x+2)}{(x+1)} + \frac{B(x+1)^2(x+2)}{(x+1)^2} + \frac{C(x+1)^2(x+2)}{(x+2)} \\ &= A(x+1)(x+2) + B(x+2) + C(x+1)^2\,. \end{aligned}$$

Step 5: Setting $x=-1$ gives $2 = B$. Setting $x=-2$ gives $-2 = C$. Comparing the coefficients of x^2 gives $1 = A + C$, and hence $A = 3$. Thus

$$\frac{x^2+7x+8}{x^3+4x^2+5x+2} = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

Conclusion:

$$\frac{x^4+2x^3-2x^2-x+4}{x^3+4x^2+5x+2} = x-2 \;+ \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

[correct answer 8 marks]

Question 8b

Learning Outcome

ILO12 – Apply the algorithms of simplifying improper rational functions to compute their integrals

Solution

Seen in lectures. ILO13 is tested at a medium level.

From 7a we see that

$$\frac{x^4+2x^3-2x^2-x+4}{x^3+4x^2+5x+2} = x-2 \;+ \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

Altogether:

$$\begin{aligned} \int \frac{x^4+2x^3-2x^2-x+4}{x^3+4x^2+5x+2}\;dx &= \int \left(x-2+\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{x+2}\right)\,dx \\ &= \int (x-2)\;dx + 3\int \frac{dx}{(x+1)} + 2\int (x+1)^{-2}\;dx \\ &\quad -2\int \frac{dx}{(x+2)} \\ &= \frac{1}{2}x^2 - 2x + 3\;\ln|x+1| - \frac{2}{(x+1)} \\ &\quad -2\;\ln|x+2|\;+C\,. \end{aligned}$$

[correct answer 2 marks]

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