Mathematics 0C2/MATH19832
Workbook
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Including examples and exercises to be completed during class

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Contents

Welcome

Welcome to the *Mathematics 0c2* course in the University of Manchester! This course continues directly from the Mathematics 0c1 course in the previous year. We will continue where that course ended, in particular, we will give further techniques in differentiation and integration, which can be useful in applications. Moreover, we will introduce notions like sequences and series, which appear commonly in mathematics and nature. We will also define the notion of a complex number, which extend our notion of real numbers and have some beautiful geometric properties and appears commonly in problems in physics.
This workbook contains examples and exercises, which we will intend to fill in the lectures.

There are 13 Intended Learning Outcomes (ILOs) for this course, which are listed as follows. On completion of this unit successful students will be able to:

(ILO1) Define complex numbers and sketch them using the Argand Diagram

(ILO2) Perform arithmetic operations on complex numbers and compute their moduli, arguments and conjugates

(ILO3) Express complex numbers in their polar and exponential forms and perform computations using these expressions

(ILO4) Define arithmetic, geometric and binomial sequences, evaluate their sums and compute convergent series

(ILO5) Define binomial coefficients, write binomial formula and apply it in integration exercises

(ILO6) Write Taylor and Maclaurin Series and apply them to compute limits

(ILO7) Apply implicit, logarithmic and parametric differentiation in differentiation exercises

(ILO8) Write integration by parts and integration by substitution formulae and apply them in integration exercises

(ILO9) Compute examples of improper integrals

(ILO10) Express improper rational functions as proper rational functions

(ILO12) Find partial fraction coefficients for proper rational functions

(ILO13) Apply the algorithms of simplifying improper rational functions to compute their integrals

It is helpful to follow them as the assessment (class tests and final exam) will be based on them and you can come back to this list to check what is expected and if there is something to be improved. **Good luck with the course!** *–Tuomas, Manchester, January 2019*

Chapter 1

Complex numbers

1.1 Problem of solving $ax^2 + bx + c = 0$

Example

Consider the following quadratic equation:

$$x^2 + 2x + 3 = 0. \tag{1.1}$$

Which numbers x solve this equation?

We can attempt to solve equation (1.1) by adding -2 to both sides, which gives

$$x^2 + 2x + 1 = -2.$$

Completing the square (that is, using binomial formula), gives us

$$x^2 + 2x + 1 = (x + 1)^2$$

so we are trying to find those numbers x such that

$$(x + 1)^2 = -2. \tag{1.2}$$

At this stage, normally, we would not take "square roots from both sides", but this is not allowed. We would not be able to proceed since the square

$$(x + 1)^2 \geq 0$$

for real numbers x so it is not possible for $(x + 1)^2$ to equal a negative number -2.

Formally, however, we can use a new class of numbers, called **complex numbers**, to solve this issue. The basic idea is to "extend" real numbers $\mathbb{R}$ to a large collection numbers $\mathbb{C}$ where we could, for example, solve our problematic equation (1.2). Thus if we can formally define "$\sqrt{-2}$", then taking square roots from both sides in (1.2) gives:

$$x + 1 = \pm\sqrt{-2}$$

Hence the solution to (1.2) (and so (1.1)) is

$$x = -1 \pm \sqrt{-2},$$

what ever this "$\sqrt{-2}$" means.

Let us now look at this mysterious $\sqrt{-2}$. Formally we will define $\sqrt{-2}$ to be the number $\sqrt{2}i$, where $\sqrt{2}$ is the usual square root of 2 (it satisfies $\sqrt{2}^2 = 2$) and i is the so called *imaginary unit* that satisfies $i^2 = -1$.

We will more generally define complex numbers as follows:

Definition (Complex numbers)

A **complex number** z is an expression

$$z = a + bi,$$

where $a, b \in \mathbb{R}$ are real numbers and the quantity i satisfies $i^2 = -1$. The quantity i is called **imaginary unit**. We write $z \in \mathbb{C}$ if z is a complex number.

Here we also say $a \in \mathbb{R}$ is the **real part** of z, written

$$a = \operatorname{Re} z;$$

and $b \in \mathbb{R}$ is the **imaginary part** of z, written

$$b = \operatorname{Im} z.$$

Example

Looking at the above example (1.1), we saw that the solutions to (1.1) were

$$x = -1 \pm \sqrt{-2},$$

and we see that we must have

$$\sqrt{-2} = \sqrt{2}i$$

since

$$-2 = \sqrt{-2}^2 = (\sqrt{2}i)^2 = 2i^2 = -2.$$

Thus the solutions to (1.2) are complex numbers

$$x = -1 \pm \sqrt{2}i.$$

Thus, for example, if $x = -1 + \sqrt{2}i$, then $\operatorname{Re} x = -1$ and $\operatorname{Im} z = \sqrt{2}$.

Given the complex numbers, we can now solve much more quadratic equations than before as we can express the solutions using the imaginary unit i. As in the case of real numbers, we can use the quadratic formula to solve any quadratic equation

Solving quadratic equations

If we want to solve a quadratic equation

$$ax^2 + bx + c = 0$$

we can use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\pm$ mean we have two solutions: either with sign $+$ or sign -1.

Example

In the above quadratic equation

$$x^2 + 2x + 3 = 0$$

we can find the solution by applying directly the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = -1 \pm \sqrt{2}i$$

by choosing $a = 1$, $b = 2$ and $c = 3$ instead of doing the completing square trick above.

The quadratic formula will be available in the exam in the formula book, so you don't need to memorise the values but more to learn how to apply it and that the solution could be allowed to be a complex number.

Example

Find all the solutions to the quadratic equation:

$$2x^2 + 3x + 5 = 0.$$

We will use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 2, b = 3, c = 5$ and obtain

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 5}}{2 \cdot 2} = \frac{-3 \pm \sqrt{-31}}{4} = -\frac{3}{4} \pm \frac{\sqrt{31}}{4}i.$$

1.2 Arithmetic operations on complex numbers

Like with the real numbers, we can perform arithmetic operations on the complex numbers. However, things become a bit more complicated here due to the identity $i^2 = -1$ and one can obtain some quite strange and even counterintuitive outcomes when, say, multiplying complex numbers.

The sum of complex numbers is defined as a sum of the real and imaginary parts of the complex numbers:

Definition (Sum of complex numbers)

Let z_1, z_2 be complex numbers, that is, $z_1, z_2 \in \mathbb{C}$. Then for some real numbers $a_1, b_1 \in \mathbb{R}$ we have

$$z_1 = a_1 + b_1 i$$

and for some real numbers $a_2, b_2 \in \mathbb{R}$ we have

$$z_2 = a_2 + b_2 i.$$

The **sum** of the complex numbers z_1 and z_2 is defined by

$$z_1 + z_2 := (a_1 + a_2) + (b_1 + b_2)i.$$

Thus the sum $z_1 + z_2$ is a complex number with the real part

$$\operatorname{Re}(z_1 + z_2) = a_1 + a_2$$

and imaginary part

$$\operatorname{Im}(z_1 + z_2) = b_1 + b_2.$$

Example

Let

$$z_1 = 2 + 3i \quad \text{and} \quad z_2 = -2 + 7i.$$

Then the sum

$$z_1 + z_2 = (2 - 2) + (3 + 7)i = 0 + 10i = 10i.$$

The sum of complex numbers is more complicated. We see that a complex number is always of the form $z = a + bi$ for some real numbers $a, b \in \mathbb{R}$, so we can formally define their product by using the binomial formula:

Definition (Product of complex numbers)

Let z_1, z_2 be complex numbers, that is, $z_1, z_2 \in \mathbb{C}$. Then for some real numbers $a_1, b_1 \in \mathbb{R}$ we have

$$z_1 = a_1 + b_1 i$$

and for some real numbers $a_2, b_2 \in \mathbb{R}$ we have

$$z_2 = a_2 + b_2 i.$$

The **product** of the complex numbers z_1 and z_2 is defined by:

$$z_1 z_2 := (a_1 + b_1 i)(a_2 + b_2 i).$$

Opening this up and using $i^2 = -1$ we get a formula for the product

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 b_1 + a_1 b_2 i + a_2 b_1 i + a_2 b_2 i^2 \\ &= (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1) i. \end{aligned}$$

Thus the sum $z_1 z_2$ is a complex number with the real part

$$\operatorname{Re}(z_1 z_2) = a_1 b_1 - a_2 b_2$$

and imaginary part

$$\operatorname{Im}(z_1 z_2) = a_1 b_2 + a_2 b_1.$$

Unless asked/specified directly, you can either use the the definition

$$z_1 z_2 := (a_1 + b_1 i)(a_2 + b_2 i).$$

or apply the formula directly:

$$z_1 z_2 = (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1) i.$$

Example

Let

$$z_1 = 2 + 3i \quad \text{and} \quad z_2 = -2 + 7i.$$

Then using the definition the product is and using $i^2 = -1$ we have

$$\begin{aligned} z_1 z_2 &= (2 + 3i)(-2 + 7i) \\ &= 2 \cdot (-2) + 2 \cdot 7i + 3 \cdot (-2)i + 3 \cdot 7 \cdot i^2 \\ &= -4 + 14i - 6i - 21 \\ &= -25 + 8i. \end{aligned}$$

The product of two complex numbers may actually become a real number, as the following example shows:

Example

Let

$$z_1 = 2 + 3i \quad \text{and} \quad z_2 = 2 - 3i.$$

Then using the definition the product is and using $i^2 = -1$ we have

$$\begin{aligned} z_1 z_2 &= (2 + 3i)(2 - 3i) \\ &= 2 \cdot 2 + 2 \cdot (-3)i + 3 \cdot 2i + 3 \cdot (-3) \cdot i^2 \\ &= 4 - 6i + 6i + 9 \\ &= 13. \end{aligned}$$

This is because z_2 is a "*complex conjugate*" of z_1, but we will define this later in the course more precisely.

Let us finally consider a more complicated example involving both sums and products:

Example (Sums and products)

Let

$$z_1 = 2 + 3i \quad \text{and} \quad z_2 = 2 - 7i \quad \text{and} \quad z_3 = 1 + 2i.$$

What are the real and imaginary parts of the complex number

$$(z_1 + z_2)z_3?$$

To find the real and imaginary parts, this means that we need to find $a, b \in \mathbb{R}$ such that

$$(z_1 + z_2)z_3 = a + bi.$$

To do this, we have two steps:

(1) firstly open up the definitions of sum $z_1 + z_2$:

$$z_1 + z_2 = (2 + 2) + (3 - 7)i = 4 - 4i.$$

(2) secondly apply the product formula for $(4 - 4i)z_3$:

$$\begin{aligned} (4 - 4i)z_3 &= (4 - 4i)(1 + 2i) \\ &= 4 \cdot 1 + 4 \cdot 2i + -4 \cdot 1i + -4 \cdot 2 \cdot i^2 \\ &= 4 + 8i - 4i + 8 \\ &= 12 + 4i. \end{aligned}$$

Thus the real part is $\mathrm{Re}\,((z_1 + z_2)z_3) = 12$ and imaginary part is $\mathrm{Im}\,((z_1 + z_2)z_3) = 4$.

1.3 Geometric way to represent complex numbers

Recall from basic trigonometry the double angle formula for cosine

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

The proof using properties of trigonometric identities looked quite notorious and hard to remember. Complex numbers and their arithmetics (products and sums) turn out to provide a very efficient and simple way to prove such an identity (and many other trigonometric identities) with just few lines and give a very geometric way to understand why it is true.

To do this however, we need to first define geometrically what complex numbers are using the so called **complex plane** $\mathbb{C}$ or the **Argand's Diagram**. Given a complex number $z = a + bi$, we can think z as a point (a, b) in two dimensional plane with first coordinate given by the real part a and the second coordinate given by the imaginary part b, see Figure 1.1. Here we can visualise each point $z = a + bi \in \mathbb{C}$ as an "arrow" (or vector) starting from 0 all the way to the point $z = a + bi$.

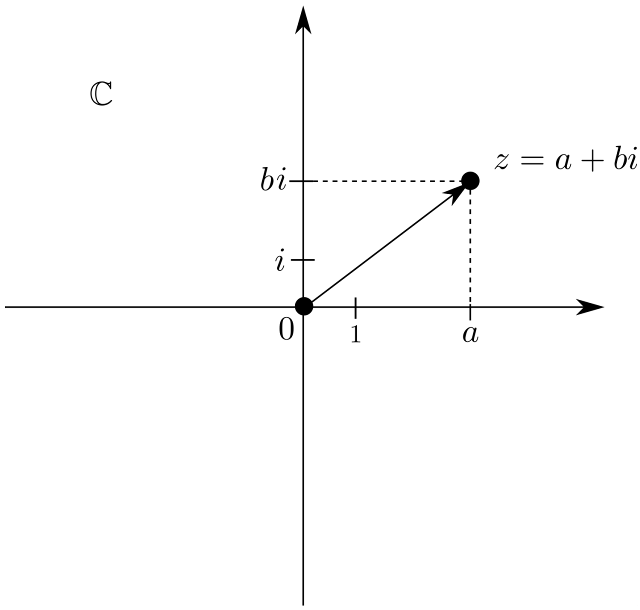


Figure 1.1: The complex plane or the Argand's Diagram: each complex number $z = a + bi$ is represented as the pair (a, b) in a two dimensional plane.

The first coordinates $a = \mathrm{Re}(z)$ of complex numbers $z = a + bi \in \mathbb{C}$ form the **real axis** of $\mathbb{C}$ and the second coordinates $b = \mathrm{Im}(z)$ of the complex numbers $z = a + bi \in \mathbb{C}$ form the **imaginary axis** of $\mathbb{C}$, in Figure 1.1 they are the two axis of the plane.

The benefit of this geometric picture of the complex numbers is that now we can visually imagine sums and products of complex numbers. In Figure 2 we see that summing complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ together produces a complex number $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ where we add the x-coordinates together and then add the y-coordinates together, see Figure 1.2.

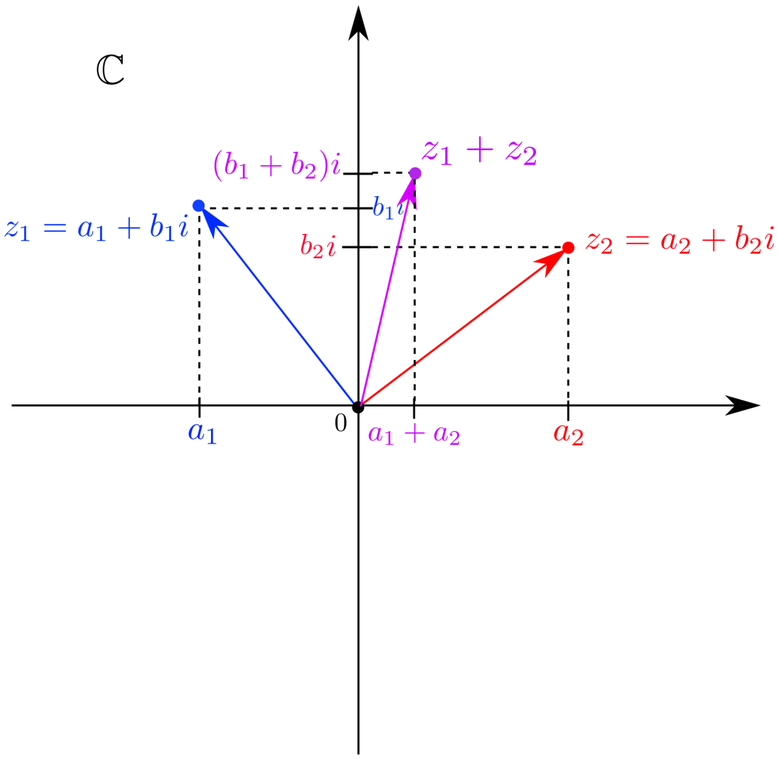


Figure 1.2: The sum $z_1 + z_2$ (purple) of two complex numbers z_1 (blue) and z_2 (red). We obtain the sum by formally adding the real parts $a_1 + a_2$ and the imaginary parts $b_1 + b_2$.

Example

Draw the numbers

$$z_1 = -2 + 3i, \quad z_2 = -3i \quad \text{and} \quad z_3 = 4$$

in the complex plane, compute their sum

$$z_1 + z_2 + z_3$$

and draw the sum in the complex plane below:

Taking products of complex numbers is visually a bit harder to understand. We see in a basic example in Figure 1.3 that multiplying complex numbers z_1 (pointing at 45 degree angle to real axis) and z_2 (pointing upwards) produces a number z_1z_2 pointing 135 degrees angle to real axis.

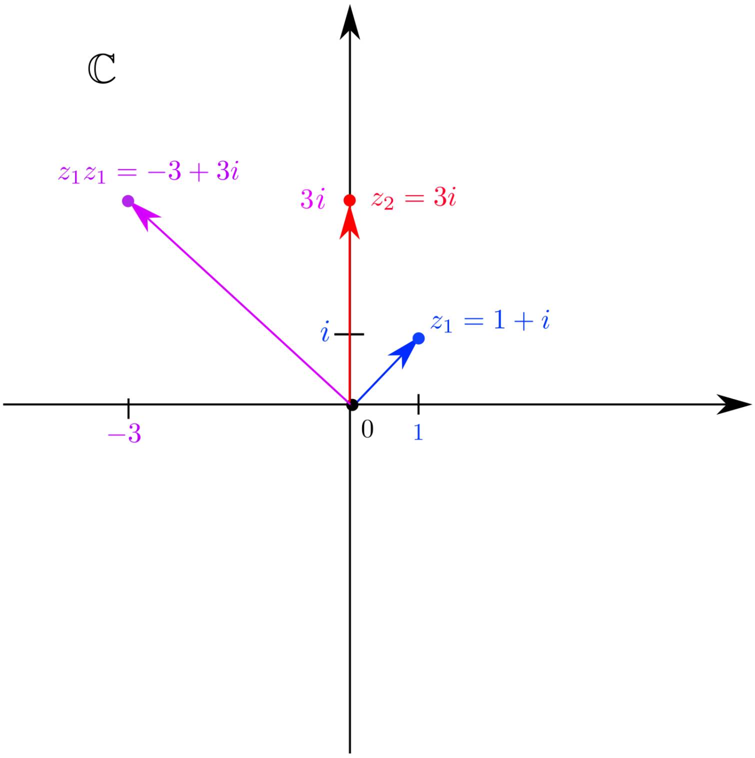


Figure 1.3: Example of the product z_1z_2 (purple) with $z_1 = 1 + i$ (blue) and $z_2 = 3i$ (red). Then $z_1z_2 = (1 + i)3i = 3i + 3i^2 = -3 + 3i$.

Later in the course we will see that taking a product of two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ corresponds to a kind of "rotating" with some angle θ and multiplying the complex number with a real number $r \in \mathbb{R}$. However, to make this more precise, we will need to talk about **"modulus"** and **"argument"** of complex numbers.

Example

Draw the complex numbers

$$z_1 = -2 + 3i, \quad z_2 = -3i \quad \text{and} \quad z_3 = 4$$

in the complex plane, compute and draw their product $z_1 z_2 z_3$ below:

1.4 Modulus and complex conjugate

We can associate to every real number $a \in \mathbb{R}$ the **absolute value**

$$|a|,$$

which is defined by

$$|a| = \begin{cases} a, & a \geq 0; \\ -a, & a < 0. \end{cases}$$

Note that $|a| \geq 0$ for every real number $a \in \mathbb{R}$. Absolute value $|a|$ tells us formally "how far a is from the origin 0".

A similar idea can be done in the complex plane for complex numbers $z = a + bi$. Here the notion of "modulus" is the analogue of absolute value:

Definition (Modulus $|z|$)

Let $z = a + bi \in \mathbb{C}$ be a complex number. Then we define the **modulus** of the complex number z by

$$|z| := \sqrt{a^2 + b^2}.$$

Notice that $|z| \geq 0$ as $a^2 + b^2 \geq 0$ for any real numbers $a, b \in \mathbb{R}$.

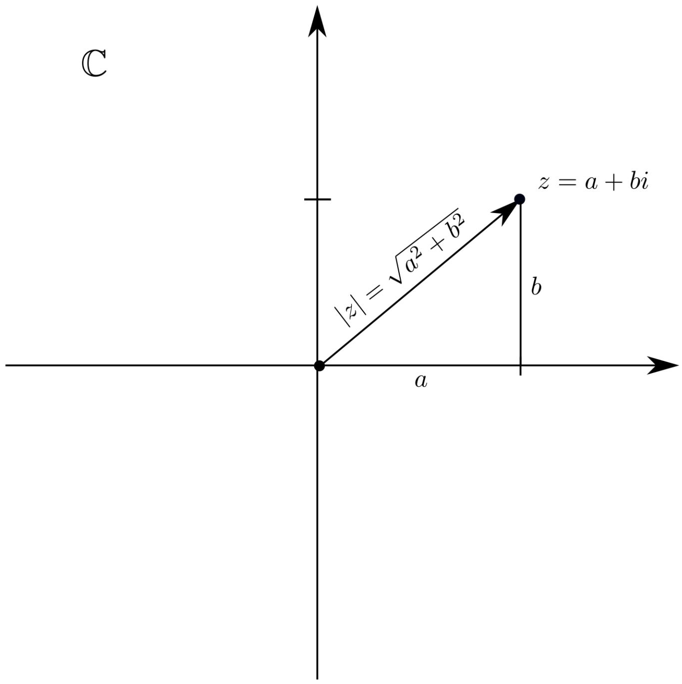


Figure 1.4: Visualisation of the modulus $|z|$ of $z = a + bi \in \mathbb{C}$. It tells us the length of the vector z due to Pythagoras theorem.

Notice that if $z = a + bi$ is a real number, that is, the imaginary part $b = 0$, then

$$|z| = \sqrt{a^2 + b^2} = \sqrt{a^2} = |a|,$$

the absolute value of the real number $a \in \mathbb{R}$. Thus $|z|$ naturally extends the notion of absolute value to all complex numbers. Visually we can imagine the modulus to be **length** of the vector $z = a + bi$ imagined in the complex plane. The reason for this is due to the theorem of Pythagoras: if $a, b \geq 0$, then the point $z = a + bi$ gives an edge of a right triangle with height b and length a. Then if c is the length of the hypotenuse of this right triangle, then Pythagoras theorem says that

$$c^2 = a^2 + b^2.$$

Thus

$$c = \sqrt{a^2 + b^2} = |z|.$$

See Figure 1.4 for a visualisation of this.

Example

Let $z = -1 + 4i$, then the modulus

$$|z| = \sqrt{(-1)^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}.$$

There is also an important operation on complex numbers $z = a + bi \in \mathbb{C}$, which is called *complex conjugation*, which is useful for computing modulus.

Definition (Complex conjugate $\overline{z}$)

Let $z = a + bi \in \mathbb{C}$ be a complex number. Then we define the **complex conjugate** of the complex number z by

$$\overline{z} := a - bi.$$

If $z = a + bi$ is a complex number, then geometrically the complex conjugate $\overline{z}$ gives the point on the plane with y-coordinate swapped signs, see Figure 1.5.

Example

Let $z = 2 + 3i$, then the complex conjugate

$$\overline{z} = 2 - 3i.$$

Theorem

Let $z = a + bi \in \mathbb{C}$ be a complex number. Then the modulus satisfies

$$|z|^2 = z\overline{z}.$$

Proof
Note that $|z|^2 = a^2 + b^2$. Thus by the definition of the product of complex numbers and $i^2 = -1$ we obtain:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + abi - bai - b^2 i^2 = a^2 + b^2 = |z|^2.$$

□

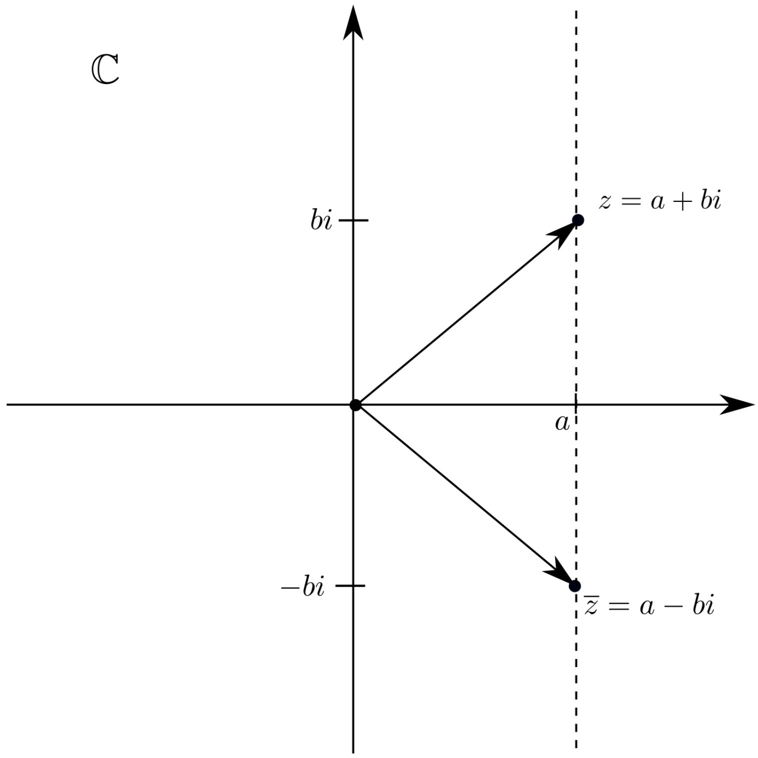


Figure 1.5: Visualisation of the complex conjugate $\overline{z} = a - bi$ of $z = a + bi \in \mathbb{C}$. We simply just swap the signs of the imaginary part, or more geometrically, reflect along the Real axis.

Example

Compute the modulus of $z = 2 + 3i$. There are two ways to solve this.

(1) We can either use directly the formula

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

(2) or compute the complex conjugate $\overline{z} = 2 - 3i$ and then compute

$$|z| = \sqrt{z\overline{z}} = \sqrt{(2 + 3i)(2 - 3i)} = \sqrt{2^2 + 2 \cdot 3i - 3 \cdot 2i - 3^2 i^2} = \sqrt{13}.$$

1.5 Trigonometry and complex numbers

When we have a complex number

$$z = a + bi$$

for some real $a, b \in \mathbb{R}$, then we now see that geometrically a is the x coordinate of z in the complex plane and b is the y coordinate of z in the complex plane. Moreover, we see that the modulus $|z|$ is the distance from the origin 0, of the length of the "vector" joining 0 to z, see Figure 1.6. The line from z to 0 has some angle θ with respect to the real axis. We call this θ the **argument** $\operatorname{Arg} z$ of z.

Definition (Argument $\operatorname{Arg} z$)

Let $z = a + bi \in \mathbb{C}$ be a complex number that is not 0. Then the angle θ formed by the line joining z to 0 is the **argument** $\operatorname{Arg} z$ of z.

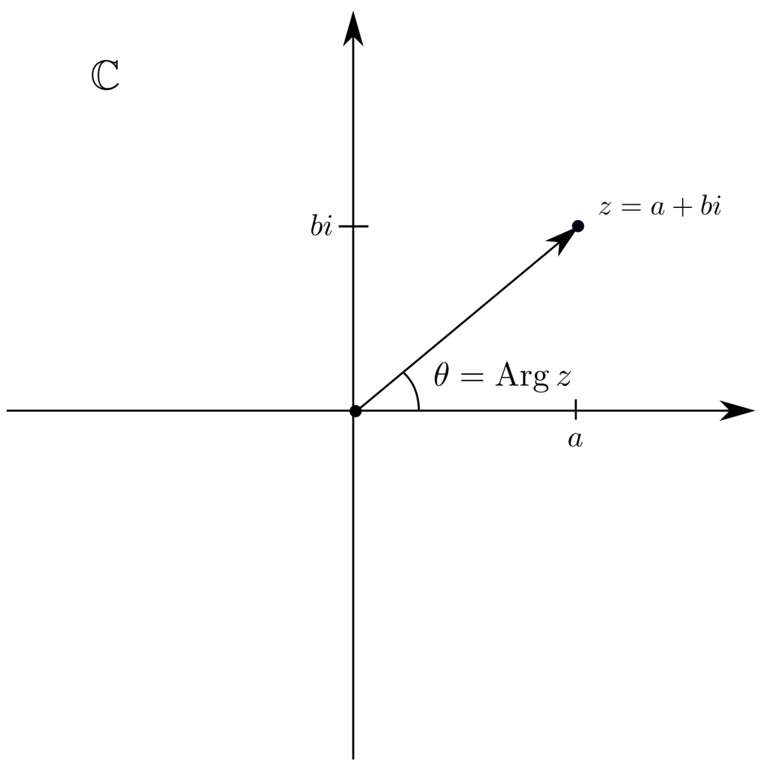


Figure 1.6: Visualisation of the argument $\theta = \operatorname{Arg} z$ of $z = a + bi \in \mathbb{C}$. It tells us the angle to the real axis.

Equivalently we can define the argument $\theta = \operatorname{Arg} z$ of z to be the principal solution θ to the trigonometric equation

$$\tan \theta = \frac{b}{a}.$$

This is because θ is an angle in the right triangle with opposite side length b and adjacent side length a.

Indeed, this follows from the definition of cosine and sine, the points $(0,0)$, $(a,0)$ and $(0,b)$ for form a triangle with angle θ and hypothenuse of length $r = |z|$, see Figure 1.7. Then b is the length of the side opposite to the angle to the hypotenuse and a is the length of the

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{r}$$

and

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{r}$$

so we obtain

$$a = r\cos\theta$$

and

$$b = r\sin\theta.$$

This implies that

$$z = a + bi = r\cos\theta + [r\sin\theta]i = r(\cos\theta + i\sin\theta).$$

This is the so called "polar coordinate" form of z.

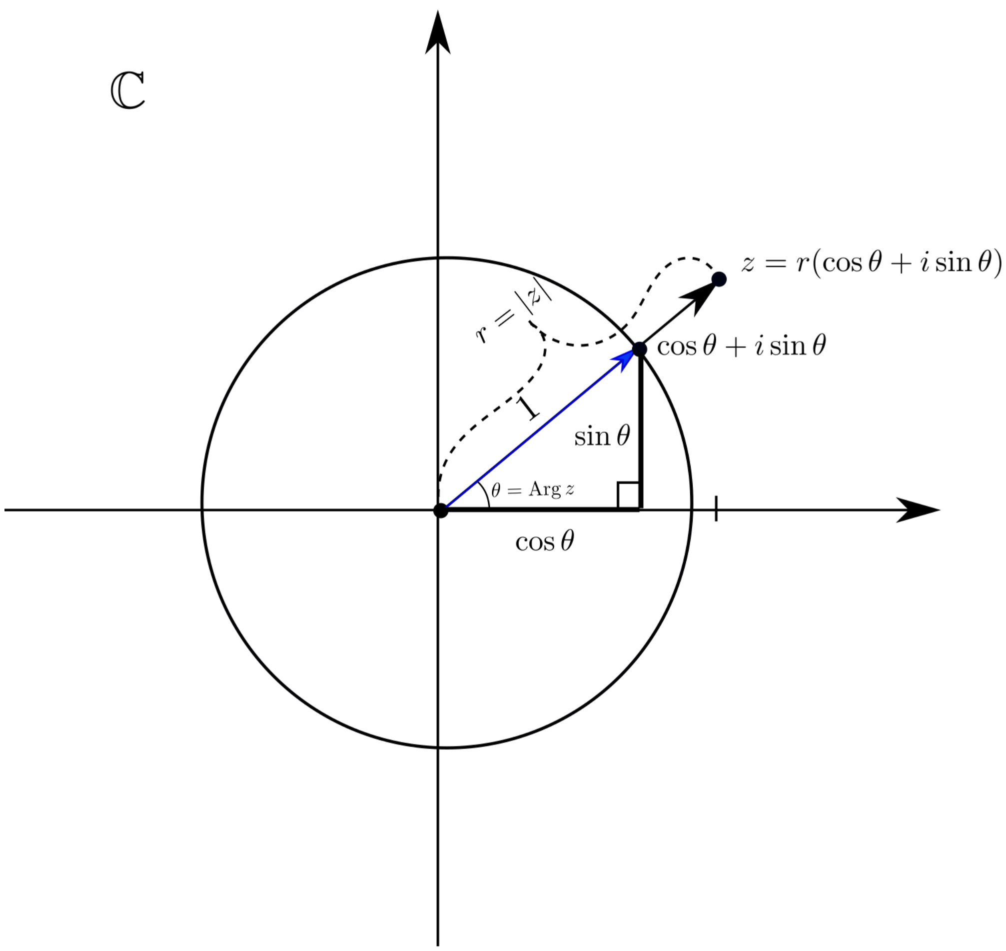


Figure 1.7: Polar coordinate form of a complex number z. If we know the modulus $|z|$ and the argument $\theta = \operatorname{Arg} z$, we can write $z = r(\cos\theta + i\sin\theta)$.

Definition (Polar coodinate form of complex numbers)

If z is a complex number with modulus $r = |z|$ and argument $\theta = \operatorname{Arg} z$, then

$$z = r(\cos\theta + i\sin\theta).$$

This is called the **polar coordinate form of z.**

Example

Consider the complex number

$$z = i.$$

Then i forms an angle of $\pi/2$ (or 90 degrees) with the real axis in the complex plane:

so the argument

$$\operatorname{Arg} z = \pi/2$$

and modulus

$$|z| = \sqrt{0^2 + 1^2} = 1$$

so the polar coordinate form is

$$z = 1 \cdot (\cos(\pi/2) + i\sin(\pi/2)).$$

Example

Consider the complex number

$$z = -2.$$

Then -2 forms an angle of π (or 180 degrees) with the real axis in the complex plane:

so the argument

$$\operatorname{Arg} z = \pi$$

and modulus

$$|z| = \sqrt{(-2)^2 + 0^2} = 2$$

so the polar coordinate form is

$$z = 1 \cdot (\cos(\pi) + i \sin(\pi)).$$

Example

Consider the complex number

$$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Then z forms an angle of $\pi/4$ (or 45 degrees) with the real axis in the complex plane:

Thus the argument

$$\operatorname{Arg} z = \pi/4$$

and modulus

$$|z| = \sqrt{\frac{1}{\sqrt{2}^2} + \frac{1}{\sqrt{2}^2}} = \sqrt{1/2 + 1/2} = \sqrt{1} = 1$$

so the polar coordinate form is

$$z = 1 \cdot (\cos(\pi/4) + i \sin(\pi/4)).$$

In practise often the polar coordinate form can be tedious to write down. Thus one defines the complex exponential function notation to ease this. However, it turns out that in this form it satisfies many of the usual properties of exponentiation.

Definition (Exponential function)

Given an angle $0 \leq \theta \leq 2\pi$, we define the **complex exponential function**:

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

With this in mind, we can use the polar coordinate form

$$z = r(\cos\theta + i\sin\theta)$$

of complex numbers, where $r = |z|$ and $\theta = \operatorname{Arg} z$ to write the exponential form:

Definition (Exponential form of complex numbers)

If $z \in \mathbb{C}$ is a complex number with modulus $|z|$ and argument $\theta = \operatorname{Arg} z$, its **exponential form** is defined by

$$z = re^{i\theta}.$$

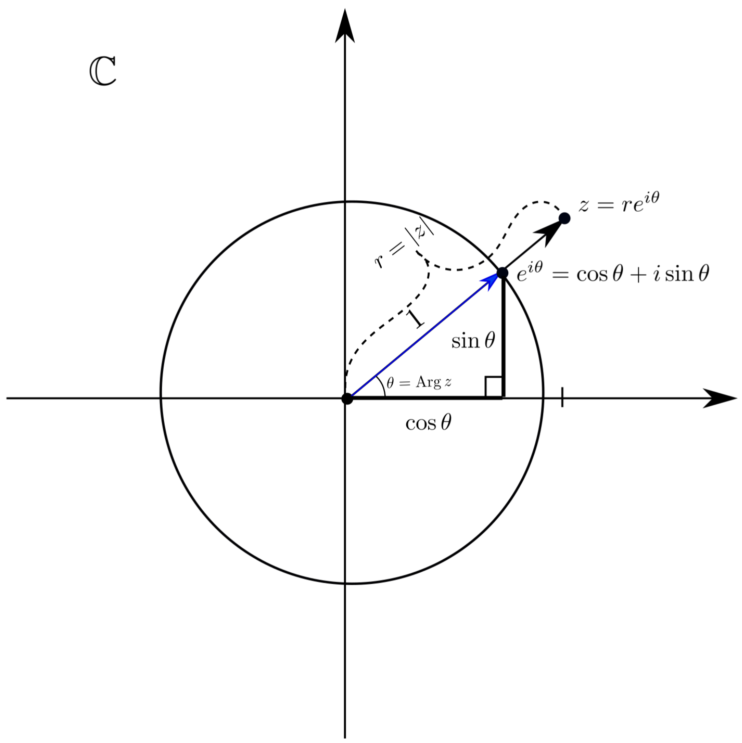


Figure 1.8: Exponential form of a complex number z. If we know the modulus $|z|$ and the argument $\theta = \operatorname{Arg} z$, we can write $z = re^{i\theta}$.

A famous application of the exponential form of complex numbers is the following formula involving 1, 0, e, i and π in a very simple and elegant way:

Example ("The Most Beautiful Formula in Mathematics")

The following formula holds:

$$e^{i\pi} + 1 = 0.$$

This means that the exponential form for -1 should be $e^{i\pi}$:

$$-1 = e^{i\pi}.$$

This is true as $\sin\pi = 0$ and $\cos\pi = -1$ so

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

Example (Important, gives steps on finding polar/exponential form!)

Determine the exponential form of $z = \frac{\sqrt{3}}{4} + \frac{1}{4}i$.

Step 1. Compute the modulus $r = |z|$. We have

$$|z| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{\frac{3}{16} + \frac{1}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Step 2. Compute the argument $\theta = \operatorname{Arg} z$. First of all, by definition, the argument θ is a solution to the equation

$$\tan\theta = \frac{b}{a},$$

where

$$b = \operatorname{Im}(z) = \frac{1}{4} \quad \text{and} \quad a = \operatorname{Re}(z) = \frac{\sqrt{3}}{4}.$$

Since $\frac{b}{a} = \frac{1}{\sqrt{3}}$ we need to solve the equation

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

We can see from formula book that a radian angle θ solving this trigonometric equation is

$$\theta = \frac{\pi}{6},$$

that is, the angle of 30 degrees. Hence $\operatorname{Arg} z = \frac{\pi}{6}$ in radians.

Step 3. Write the answer. By definition, the exponential form of z is

$$z = re^{i\theta} = \frac{1}{3}e^{i\frac{\pi}{6}}.$$

How about now considering products $z_1 z_2$ of complex numbers z_1 and z_2? We found that they have a very complicated form: if

$$z_1 = a_1 + b_1 i \quad \text{and} \quad z_2 = a_2 + b_2 i.$$

then recall that the product of the complex numbers z_1 and z_2 is defined by:

$$z_1 z_2 := (a_1 + b_1 i)(a_2 + b_2 i).$$

Opening this up and using $i^2 = -1$ we get a formula for the product

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 b_1 + a_1 b_2 i + a_2 b_1 i + a_2 b_2 i^2 \\ &= (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1) i. \end{aligned}$$

Thus the sum $z_1 z_2$ is a complex number with the real part

$$\operatorname{Re}(z_1 z_2) = a_1 b_1 - a_2 b_2$$

and imaginary part

$$\operatorname{Im}(z_1 z_2) = a_1 b_2 + a_2 b_1.$$

Using the exponential forms of z_1 and z_2, however, it is easier to see what happens when taking a product. If $r_1 = |z_1|$ is the modulus of z_1 and $r_2 = |z_2|$ of z_2 respectively, and $\theta_1 = \operatorname{Arg} z_1$ is the argument of z_1 and $\theta_2 = \operatorname{Arg} z_2$ is the argument of z_2, then the exponential forms of z_1 and z_2 are

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}.$$

Thus the product

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

This means that the product $z_1 z_2$ has modulus

$$|z_1 z_1| = r_1 r_2 = |z_1||z_2|.$$

How about the argument? Note that if

$$\theta_1 + \theta_2 > 2\pi$$

so the angle is bigger than the possible angle we could have between $z_1 z_2$ and the real axis (maximum is 2π). Hence we have if $0 \leq \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq 2\pi$, we have

$$\operatorname{Arg}(z_1 z_2) = \theta_1 + \theta_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

and if $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq 2\pi > 2\pi$ we have

$$\operatorname{Arg}(z_1 z_2) = \theta_1 + \theta_2 - 2\pi = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi.$$

Thus we have the following theorem:

Theorem (Modulus and argument of products)

If z_1 and z_2 are complex numbers, then the modulus

$$|z_1 z_1| = |z_1||z_2|$$

and the argument

$$\operatorname{Arg}(z_1 z_2) = \begin{cases} \operatorname{Arg} z_1 + \operatorname{Arg} z_2, & \text{if } 0 \leq \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq 2\pi; \\ \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi, & \text{if } \operatorname{Arg} z_1 + \operatorname{Arg} z_2 > 2\pi. \end{cases}$$

Thus taking products of complex numbers z_1 and z_2 means we multiply their lengths together and sum the angles to the real line together. Let us demonstrate Theorem 1.5 in an example, which may give an idea why we need to substract the 2π.

Example

Find the exponential form of

$$z = 2 + 2i$$

and use this to compute z^{10}. What is the modulus and argument of z^{10}?

Solution: We need to find $r \geq 0$ and $0 \leq \theta \leq 2\pi$ such that

$$z = re^{i\theta}.$$

Here $r = |z|$ is the modulus and $\theta = \operatorname{Arg} z$ is the argument.
• For the modulus, we use definition

$$r = |z| = \sqrt{2^2 + 2^2} = \sqrt{8}.$$

• For the argument, we need to solve in $0 \leq \theta \leq 2\pi$ the trigonometric equation

$$\tan\theta = \frac{\operatorname{Im} z}{\operatorname{Re} z} = \frac{2}{2} = 1.$$

From a formula book we see that a solution $\theta = \pi/4$.
• Thus the exponential form is $z = \sqrt{8}e^{i\pi/4}$.
• Thus the power is

$$z^{10} = \sqrt{8}^{10} e^{10i\pi/4} = 8^{10/2} e^{i(5\pi/2)} = 8^5 e^{i(5\pi/2)} = 32768 e^{i(5\pi/2)}.$$

We see that the modulus of z^{10} is $|z^{10}| = 32768$ but the argument is not $5\pi/2$ since $5\pi/2 > 2\pi$. We see that $2\pi < 5\pi/2 \leq 4\pi$ so the argument to be between 0 and 2π, we need to substract 2π. Hence

$$\operatorname{Arg}(z^{10}) = 5\pi/2 - 2\pi = (5/2 - 2)\pi = \pi/4.$$

To explain a bit more about the reason we substract 2π, recall from trigonometry the formulae:

$$\sin(\theta \pm 2\pi) = \sin\theta \quad \text{and} \quad cos(\theta \pm 2\pi) = \cos\theta$$

we also have

$$e^{i(\theta \pm 2\pi)} = e^{i\theta}.$$

Hence in the previous example we had the exponential form

$$z^{10} = 32768e^{i(5\pi/2)} = 32768e^{i(5\pi/2-2\pi)} = 32768e^{i\pi/4}.$$

When finding an exponential form of a complex number, we want to express the exponential form in terms of the angle being between 0 and 2π.

Let us now turn on the original question we had about the double angle formula for cosine, which we can prove now using the exponential form notation for complex numbers:

Example (Double angle formula)

For all $0 \le \theta \le \pi$ we have

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

How to prove this?

Step 1. First of all, the following expression for cosine holds:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Indeed, since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ we have

$$e^{i\theta} + e^{-i\theta} = [\cos\theta + i\sin\theta] + [\cos(-\theta) + i\sin(-\theta)] = 2\cos(\theta).$$

Step 2. Apply Step 1 to prove the double angle formula:

$$\begin{aligned}
\cos(2\theta) &= \frac{e^{2\theta i} + e^{-2\theta i}}{2} \\
&= \frac{(e^{\theta i})^2 + (e^{-2\theta i})^2}{2} \\
&= \frac{(e^{\theta i})^2 + 2 + (e^{-2\theta i})^2}{2} - 1 \\
&= \frac{(e^{\theta i})^2 + 2e^{i\theta}e^{-i\theta} + (e^{-2\theta i})^2}{2} - 1 \\
&= \frac{(e^{i\theta} + e^{-i\theta})^2}{2} - 1 \\
&= 2\Big(\frac{e^{i\theta} + e^{-i\theta}}{2}\Big)^2 - 1 \\
&= 2\cos^2(\theta) - 1.
\end{aligned}$$

Chapter 2

Sequences and Series

Revision from last semester..

Powers:

For a, b real numbers and r, s rational numbers such that a^r, a^s etc. are defined:

$a^0 = 1 \qquad a^1 = a$

$a^r a^s = a^{r+s} \qquad a^r/a^s = a^{r-s} \qquad a^{-s} = 1/a^s$

$(a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r \qquad (a/b)^r = a^r/b^r$

Logarithms:
For $a > 0, a \neq 1$, $a^x = y \Leftrightarrow x = \log_a(y)$.

$\log_a(1) = 0 \qquad \log_a(a) = 1$

$\log_a(a^x) = x \qquad a^{\log_a(b)} = b$

$\log_a(bc) = \log_a(b) + \log_a(c) \qquad \log_a(b/c) = \log_a(b) - \log_a(c)$

$\log_a(b^y) = y \log_a(b) \qquad \log_b(c) = \dfrac{\log_a(c)}{\log_a(b)}$

2.1 Notations

Definition (Sequences)

A **sequence** is an ordered list of numbers $a_1, a_2, a_3, \ldots$, each of which is called a **term**. The kth term of the sequence is a_k.
A sequence may have finitely many terms or infinitely many terms.

Example

$$2, 5, 10, 17, 26, ..., k^2 + 1, ..., 10001$$

is a finite sequence of 100 numbers.
If k is a whole number between 1 and 100, then the kth term of this sequence is $a_k = k^2 + 1$.

Definition (Sums and series)

(1) A **sum** of the n numbers $a_1, a_2, a_3, \ldots, a_n$ is often written by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

(2) A **series** of a sequence $a_1, a_2, a_3, \ldots$ is formally the infinite sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \ldots.$$

Notice that series is formally a 'limit' of the finite sums S_n when n increases to infinity.

Exercise

Let a_k be the sequence $1, 2, 3, \ldots$. Then $a_k = k$ for all k. Hence the sum up to term n is

$$S_n = \sum_{k=1}^{n} a_k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

(check this!). What do you think is the series $\sum_{k=1}^{\infty} a_k$? We will come back to this case a bit later in this section but the answer can be quite surprising, check for example the video `https://www.youtube.com/watch?v=w-I6XTVZXww`

2.2 Arithmetic Progressions

Definition (Differences)

Given a sequence

$$a_1, a_2, a_3, a_4, ..., a_k, a_{k+1}, ...,$$

(finite or infinite), we can associate to it the corresponding sequence of **differences**:

$$d_1 = a_2 - a_1, \quad d_2 = a_3 - a_2, \quad d_3 = a_4 - a_3, \ \ldots, \quad d_k = a_{k+1} - a_k, \ \ldots .$$

The sequence $d_1, d_2, d_3, \ldots$ describes the "growth" or "rate of change" of the original sequence $a_1, a_2, a_3, \ldots$. Notice that this is a similar idea to taking the derivative[1] of a function. You may find it useful to think of a sequence as a function of k where k only takes discrete values ($k = 1, 2, 3, \ldots$).

Example

If $d_k = 0$ for all k, this means that $a_{k+1} - a_k = 0$, or in other words, all the terms a_k are the same, $a_k = a$. This is the analog of a constant function.
(Recall that if a function $f(x)$ has derivative 0 everywhere, then it must be a constant.)

In this section we shall be interested in sequences whose differences d_k are not necessarily all zero, but are the *same for all* k. This means that the sequence has "uniform growth". Such sequences are analogous to linear functions and there is a special name for sequences of this type.

Definition (Arithmetic progression)

An **arithmetic progression** is a sequence $a_1, a_2, a_3, \ldots$ in which the corresponding differences $d_k = a_{k+1} - a_k$ are all the same.
We call this value the **common difference** d.

An arithmetic progression with common difference d and first term a has the form

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \ldots, \quad a + (k-1)d, \ldots$$

Each term is obtained by adding d to the previous term.
The kth term of this sequence is $a_k = a + (k-1)d$.

[1] Recall from last semester that the derivative is defined via the differences $f(x+h) - f(x)$; we take the limit of $\big(f(x+h) - f(x)\big)/h$ when $h \to 0$. Unlike this situation, for sequences, the "step size" is fixed (it is always 1) so there is no way of taking such a limit, and we have to deal with the differences $a_{k+1} - a_k$ directly.

Example

Each pair of consecutive terms in the sequence

$$5, 7, 9, 11, 13, 15, ..., 77$$

differs by 2.
This is an arithmetic progression with first term $a = 5$ and common difference $d = 2$.

We can calculate the number of terms in the sequence as follows:
Since $a = 5$ and $d = 2$, we know that the last term of this sequence can be written as $77 = 5 + (k - 1) \times 2$, where k is the number of terms.
Rearranging this equation gives $k - 1 = 72/2$, and hence $k = 37$.

Theorem 1

The sum of the first n terms of an arithmetic progression with first term a and common difference d is

$$S_n = \sum_{k=1}^{n} a_k = \frac{1}{2} n\Big(2a + (n-1)d\Big) .$$

Proof
Write out the sum twice: once forwards, once backwards, and then add them together:

$$\begin{array}{lllllllll} S_n & = & a & + & (a+d) & + \cdots + & (a+(n-2)d) & + & (a+(n-1)d) \\ S_n & = & (a+(n-1)d) & + & (a+(n-2)d) & + \cdots + & (a+d) & + & a \\ \hline 2S_n & = & (2a+(n-1)d) & + & (2a+(n-1)d) & + \cdots + & (2a+(n-1)d) & + & (2a+(n-1)d) \end{array}$$

Note that each term on the right-hand side of the last line is equal to $2a + (n - 1)d$. Since there are n of these we see that $2S_n = n(2a + (n - 1)d)$, and dividing this by two gives the result. □

Remark

We can rewrite the formula for the sum of the first n terms of an arithmetic progression in the following easy to remember form:

$$S_n = \sum_{k=1}^{n} a_k = \frac{1}{2}\, n(a_1 + a_n)\,.$$

We add together the first term and the last term, multiply the result by the number of terms and then divide by 2.
(The first term is a, whilst the last (nth) term is $a_n = a + (n-1)d$. So their sum is equal to the bracketed expression in the previous result.)

Example

Find the sum:

$$5 + 7 + 9 + 11 + 13 + 15 + \cdots + 77\,.$$

Solution: We have seen that this is an arithmetic progression with $n = 37$ terms whose first term is 5 whose last term is 77, so $S_{37} = \dfrac{1}{2} 37\,(5 + 77) = 1517$.

Example

Find the sum of the first n natural numbers:

$$1 + 2 + 3 + 4 + \cdots + (n-1) + n$$

Solution: The terms in this sum form an arithmetic progression with common difference 1. The first term is 1 and the last term is n. Therefore the sum of the first n terms will be $S_n = \frac{1}{2}n(n+1)$.

Example

The 8^{th} term of an arithmetic progression is $-\frac{1}{2}$, and the 13^{th} term is -8.
Find the first term, the common difference, and the sum of the first 15 terms.
Solution: We can solve for a and d:

$$\begin{array}{lclllr}
13^{\text{th}} \text{ term} & = & a_{13} & = a + (13-1)d & = a + 12d & = -8 \\
8^{\text{th}} \text{ term} & = & a_8 & = a + (8-1)d & = a + 7d & = -\frac{1}{2} \\
\hline
 & & & & 5d & = -\frac{15}{2}.
\end{array}$$

Dividing the last equation through by 5 gives $d = -\frac{3}{2}$.
Substituting this value back into the first equation and rearranging gives $a = 10$.
The sum of the first 15 terms is $S_{15} = \dfrac{15}{2}(2a + 14d) = \dfrac{15}{2}\left(20 + 14(-\dfrac{3}{2})\right) = -\dfrac{15}{2}$.

2.3 Geometric progressions

Definition (Ratios)

Consider a sequence

$$a_1, a_2, a_3, a_4, ..., a_k, a_{k+1}, ...,$$

(finite or infinite), such that all the terms are non-zero ($a_k \neq 0$ for all k). Then, instead of considering the differences $d_k = a_{k+1} - a_k$, as we did in the previous section, let's consider the *ratios*:

$$r_1 = a_2/a_1, \quad r_2 = a_3/a_2, \quad r_3 = a_4/a_3, \ \ldots, \quad r_k = a_{k+1}/a_k, \ \ldots$$

(Note that we need all of the terms in our original sequence to be non-zero so that we can divide by these numbers.)

If all the ratios are 1, then $a_1 = a_2 = a_3 = \ldots$, so we again arrive at a constant sequence. The next most interesting case is when the ratios r_k are all the same. There is a special name for sequences of this type.

Definition (Geometric progression)

A **geometric progression** is a sequence of non-zero terms $a_1, a_2, a_3, \ldots$ in which the corresponding ratios $r_k = a_{k+1}/a_k$ are all the same.
We call this value the **common ratio** r.

A geometric progression with common ratio r and first term a has the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots \quad ar^{k-1}, \quad \ldots$$

Each term is obtained by multiplying the previous term by r.
The kth term of this sequence is $a_k = ar^{k-1}$.

(Notice that this definition is similar to that of an arithmetic progression considered before. Here we measure the "growth" of a sequence in a multiplicative way, rather than additively.)

Example

$$3, 6, 12, 24, 48, 96, ...$$

is a geometric progression with first term $a = 3$ and common ratio $r = 2$.

Example

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

is a geometric progression with first term $a = 1$ and common ratio $r = \frac{1}{2}$.

Theorem 2

The sum of the first n terms of a geometric progression is

$$S_n = \sum_{k=1}^{n} a_k = \begin{cases} \dfrac{a\,(1 - r^n)}{1 - r} & \text{if } r \neq 1, \\ na & \text{if } r - 1. \end{cases}$$

Proof

$$\begin{array}{llllllllll} S_n & = & a & + & ar & + & ar^2 & + \cdots + ar^{n-2} & + & ar^{n-1}. \\ rS_n & = & ar & + & ar^2 & + & ar^3 & + \cdots + ar^{n-1} & + & ar^n. \\ \hline (1-r)S_n & = & \multicolumn{8}{l}{a - ar^n = a(1 - r^n).} \end{array}$$

If $r \neq 1$ then we can divide both sides through by $1 - r$ to obtain the result. If $r = 1$ then all terms of this geometric progression are equal, and the sum of the first n terms is n times the first term a. □

Example

In the following geometric progression

$$3, 6, 12, 24, 48, 96, ...$$

after how many terms will the **sum** be greater than 9999 ?
Solution: We have $a = 3$ and $r = 2$. The formula for the sum of the first n terms is

$$S_n = \frac{3 \times (1 - 2^n)}{1 - 2} = 3\left(2^n - 1\right) .$$

We want to find the smallest number n for which $3\left(2^n - 1\right) > 9999$.
Dividing both sides through by 3 and adding 1 to both sides, we see that this is equivalent to finding the smallest value of n for which $2^n > 3334$.
We can use logarithms to solve this:

$$2^n > 3334 \Leftrightarrow \ln(2^n) > \ln(3334) \Leftrightarrow n\ \ln(2) > \ln(3334) \Leftrightarrow n > \frac{\ln(3334)}{\ln(2)} .$$

Using a calculator we find that

$$n > \frac{\ln(3334)}{\ln(2)} \approx \frac{8.112}{0.693} \approx 11.703 .$$

Taking $n = 12$ gives $2^{12} = 4096 > 3334$. Note that $2^{11} = 2048 < 3334$, so $n = 12$ is indeed the smallest value of n for which $2^n > 3334$.
Thus after exactly 12 terms, the sum will be greater than 9999.

Example

Calculate the sum of the first six terms of a geometric progression with $a = 1$ and $r = 10$.
Solution: Using the formula:

$$S_6 = \frac{1 \times (1 - 10^6)}{1 - 10} = \frac{-999999}{-9} = 111111 .$$

Alternatively, we may notice that this sum is just

$$1 + 10 + 100 + 1000 + 10000 + 100000 = 111111 .$$

Example

Calculate the sum of the first 8 terms of a geometric progression with $a = 1$ and $r = \frac{1}{2}$.
Solution: Using the formula:

$$S_8 = \frac{1 \times \left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}} = \frac{\frac{2^8-1}{2^8}}{\frac{1}{2}} = \frac{2 \times (2^8 - 1)}{2^8} = \frac{510}{256} \approx 1.992\,(3d.p.).$$

In fact, if we continue adding terms of the above progression, we get values closer and closer to 2. This gives an example of a *convergent series*, as we shall explain below.

2.4 Convergence of series

Definition (Convergent and divergent series)

A **series** is an expression of the form $a_1 + a_2 + \cdots + a_k + \cdots$, representing the **sum** of terms in a sequence.

The notation

$$\sum_{k=1}^{\infty} a_k \quad = \quad a_1 + a_2 + \cdots + a_k + \ldots$$

is also used.

We say that the series $\sum_{k=1}^{\infty} a_k$ is **convergent** if the terms of the sequence of partial sums

$$a_1, \quad (a_1 + a_2), \quad (a_1 + a_2 + a_3), \quad \ldots \quad , (a_1 + a_2 + \cdots + a_n), \ldots$$

approach a fixed value as $n \to \infty$.

Otherwise we say that the sequence is **divergent**.

Remark

The notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + +a_n$$

is also used for finite sums.

We give some examples of the compact notation for series and finite sums, using the examples we have seen thus far.

Example

The finite sequence $2, 5, 10, 17, 26, ..., k^2 + 1, ..., 10001$ has 100 terms. Since the kth term of this sequence is $a_k = k^2 + 1$, we can use the notation

$$\sum_{k=1}^{100} (k^2 + 1)$$

to represent the sum of the terms in this sequence.

Example

The finite arithmetic progression $5, 7, 9, 11, 13, 15, \ldots, 77$ has 37 terms. Since the kth term of this sequence is $a_k = 5 + 2(k-1)$, we can use the notation

$$\sum_{k=1}^{37} (5 + 2(k-1))$$

to represent the sum of the terms in this sequence.

A general arithmetic progression has the form $a, a+d, a+2d, \ldots$, with kth term $a_k = a + (k-1)d$. We use the notation

$$\sum_{k=1}^{n} (a + (k-1)d)$$

to represent the sum of the first n terms. The corresponding infinite series

$$\sum_{k=1}^{\infty} (a + (k-1)d)$$

is **divergent** unless $a = d = 0$.

Example

Consider the infinite arithmetic progression

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

with first term 1 and common difference 1. The corresponding sequence of partial sums is:

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots\ ,\ \frac{1}{2}n(n-1),\ \ldots$$

These values do not approach a fixed value as $n \to \infty$. Thus the series $\sum_{k=1}^{\infty} k$ is divergent.

Example

The infinite geometric progression with first term 1 and common difference 10 has kth term $a_k = 10^{(k-1)}$. We use the notation

$$\sum_{k=1}^{n} 10^{k-1} = \frac{1-10^n}{1-10}$$

to represent the sum of the first n terms in this sequence. The sequence of partial sums is

$$1,\ 11,\ 111,\ 1111,\ 11111,\ 111111,\ \ldots,\ \frac{1-10^n}{1-10},\ \ldots$$

Since the terms do not approach a fixed value for $n \to \infty$, the infinite series

$$\sum_{k=1}^{\infty} 10^{k-1}$$

is **divergent**

Recall that a general geometric progression has the form $a, ar, ar^2, \ldots$, with kth term $a_k = ar^{k-1}$. We use the notation $\sum_{k=1}^{n} ar^{k-1}$ to represent the sum of the first n terms. The corresponding infinite series is $\sum_{k=1}^{\infty} ar^{k-1}$.

Theorem 3

The sum of an infinite geometric progression with first term a and common ratio r satisfying $|r| < 1$ is

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}\,.$$

Proof

We have $S_n = \dfrac{a\,(1-r^n)}{1-r}$, and we note that as n gets larger, the value of r^n becomes smaller, since $|r| < 1$. We can say that as $n \to \infty$, $r^n \to 0$. Thus, letting $n \to \infty$ in the formula for S_n gives

$$\sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} S_n = \frac{a\,(1-0)}{1-r} = \frac{a}{1-r}.$$

□

Example

Using the previous result we find that the sum of the infinite geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$ is

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \frac{a}{1-r} = \frac{1}{0.5} = 2.$$

2.5 The Binomial Theorem

The binomial theorem is used when we want to multiply out expressions of the form $(a+b)^n$. We shall limit ourselves to the case where n is a positive integer, but there are analogues for negative, and non-integer values of n.
We can obtain the following directly (by multiplying out the brackets):

$$\begin{array}{lcl}
(a+b)^1 & = & a+b \\
(a+b)^2 & = & a^2+2ab+b^2 \\
(a+b)^3 & = & a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 & = & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 & = & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
\multicolumn{3}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

Each term of the above expansions is of the form $a^{n-r}b^r$, where r takes the values $0, 1, ..., n$ successively. Notice that the sum of the two exponents is $(n-r)+r =$ n. The coefficients of the terms $a^{n-r}b^r$ occurring in the expansion of $(a+b)^n$ are called **binomial coefficients**. These coefficients may be obtained using Pascal's Triangle:

Pascal's Triangle

$$\begin{array}{lcccccccccccccccccc}
\text{Row } 0 & & & & & & & & & 1 & & & & & & & & \\
\text{Row } 1 & & & & & & & & 1 & & 1 & & & & & & & \\
\text{Row } 2 & & & & & & & 1 & & 2 & & 1 & & & & & & \\
\text{Row } 3 & & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
\text{Row } 4 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
\text{Row } 5 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
\text{Row } 6 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
\text{Row } 7 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
\text{Row } 8 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & & & & &
\end{array}$$

The rule for forming the numbers in Pascal's triangle is:
Each number is the sum of the two nearest neighbors in the previous row.

Notice that the nth row of Pascal's triangle contains $n+1$ numbers.
We label the entries of the nth row as follows:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n-1}, \binom{n}{n}.$$

For example the sixth row of Pascal's triangle contains the numbers:
$\binom{6}{0} = 1$, $\binom{6}{1} = 6$, $\binom{6}{2} = 15$, $\binom{6}{3} = 20$ $\binom{6}{4} = 15$, $\binom{6}{5} = 6$, $\binom{6}{6} = 1$.

Notice that:

$$\binom{n}{0} = \binom{n}{n} = 1\,, \quad \text{and} \quad \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \text{for all } 0 < r < n\,.$$

The mathematical notation $\binom{n}{r}$ is pronounced "n choose r" because these numbers count the different possible ways of choosing r objects from a choice of n. You may come across other notations for these numbers (e.g. $\binom{n}{r}$ or nCr or ${}^nC_r$) on your calculator.

Theorem 4 (The Binomial Theorem)

If n is a positive integer, and a and b are real numbers then

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \,,$$

where the coefficients $\binom{n}{r}$ are the numbers in Pascal's triangle defined above.

Proof
We have seen that this is true for some small values of n.
Since $(a+b)^{n+1} = (a+b)^n(a+b)$, we shall first assume that the formula holds for some value of n and by multiplying through by $a+b$ deduce that the corresponding formula must also hold for $n+1$.
If

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + \binom{n}{n} b^n,$$

then multiplying both sides by $(a+b)$ gives

$$(a+b)^{n+1} = (a+b)\left[\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + \binom{n}{n} b^n\right].$$

The right hand side can be rewritten as

$$\begin{aligned}&\left(\binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \cdots + \binom{n}{r} a^{n+1-r}b^r + \cdots + \binom{n}{n} ab^n\right)\\ &+\left(\binom{n}{0} a^n b + \binom{n}{1} a^{n-1}b^2 + \cdots + \binom{n}{r} a^{n-r}b^{r+1} + \cdots + \binom{n}{n} b^{n+1}\right)\end{aligned}$$

Gathering together terms involving the same powers of a and b gives

$$\begin{aligned}&\binom{n}{0} a^{n+1} + \left[\binom{n}{1} + \binom{n}{0}\right] a^n b + \left[\binom{n}{2} + \binom{n}{1}\right] a^{n-1}b^2 + \cdots\\ &+ \left[\binom{n}{r} + \binom{n}{r-1}\right] a^{n+1-r}b^r + \cdots + \binom{n}{n} b^{n+1}\\ =\;& a^{n+1} + \binom{n+1}{1} a^n b + \binom{n+1}{2} a^{n-1}b^2 + \cdots + \binom{n+1}{r} a^{n+1-r}b^r + \cdots + b^{n+1}\,,\end{aligned}$$

which is the desired formula for $n+1$. Therefore by starting from the known cases and increasing the power by one at each step, we can prove the formula for all $n = 1, 2, 3, 4, \ldots$ using the above method. □

The previous result confirms our observation that the binomial coefficients are the numbers occurring in Pascal's triangle. Using Pascal's triangle is the quickest way of finding the binomial coefficients for small values of n. However, it is useful to have a non-recursive formula for binomial coefficients more generally. We need one more definition.

Definition (Factorial)

The number denoted $k!$, pronounced k **factorial**, is the product of all integers from k down to 1, i.e.,

$$k! = k \times (k-1) \times (k-2) \times \cdots \times 3 \times 2 \times 1\,.$$

Convention: We also define $0! = 1$.

Example

$1! = 1$, $2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$, $4! = 4 \times 3 \times 2 \times 1 = 24$, $5! = 5 \times 4! = 120$.

Theorem 5

For every positive integer n and for all $r = 0, 1, ..., n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}\,.$$

Proof

We check that the RHS, i.e., the number $\frac{n!}{r!(n-r)!}$ satisfies the relations for the numbers in Pascal's Triangle. First note that

$$\frac{n!}{0!(n-0)!} = 1 \text{ and } \frac{n!}{n!(n-n)!} = 1.$$

Now for all positive n and $r = 1, 2, \ldots, n-1$, we have

$$\begin{aligned} \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} &= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{n!r}{r!(n-r+1)!} \\ &= \frac{(n+1)n!}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n+1-r)!}. \end{aligned}$$

Therefore we have shown that these numbers are calculated by exactly the same rules as the numbers in Pascal's triangle, i.e., $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. □

Corollary 6

It follows from the symmetry of the previous result that

$$\binom{n}{r} = \binom{n}{n-r}.$$

Example

In the case $n = 5$, we can compute the coefficients:

$$\begin{aligned}
\binom{5}{0} &= \frac{5!}{5!0!} &= \frac{120}{120 \times 1} &= 1 \\
\binom{5}{1} &= \frac{5!}{4!1!} &= \frac{120}{24 \times 1} &= 5 \\
\binom{5}{2} &= \frac{5!}{3!2!} &= \frac{120}{6 \times 2} &= 10 \\
\binom{5}{3} &= \frac{5!}{2!3!} &= \frac{120}{2 \times 6} &= 10 \\
\binom{5}{4} &= \frac{5!}{1!4!} &= \frac{120}{1 \times 24} &= 5 \\
\binom{5}{5} &= \frac{5!}{0!5!} &= \frac{120}{1 \times 120} &= 1
\end{aligned}$$

giving the fifth row of Pascal's triangle.

Example

$$\begin{aligned}
(a+b)^5 &= \binom{5}{0} a^5 b^0 + \binom{5}{1} a^4 b^1 + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^3 + \binom{5}{4} a^1 b^4 + \binom{5}{5} a^0 b^5 \\
&= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5.
\end{aligned}$$

When expressing the binomial coefficients using factorials, it can be convenient to get rid of common factors in the numerator and denominator of the fraction:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 3 \cdot 2 \cdot 1}{r!(n-r)\cdots 3 \cdot 2 \cdot 1} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Therefore the first few terms in the binomial expansion (for general n) are as follows:

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2}\,a^{n-2}b^2 + \frac{n(n-1)(n-2)}{6}\,a^{n-3}b^3 + \cdots.$$

We may substitute any expression for the values of a and b.

Example

To expand $(1-2x)^5$, we set $a=1$ and $b=-2x$, which gives:

$$\begin{aligned}(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\ (1-2x)^5 &= 1+5(-2x)+10(-2x)^2+10(-2x)^3+5(-2x)^4+(-2x)^5\\ &= 1-10x+40x^2-80x^3+80x^4-32x^5.\end{aligned}$$

Example

Approximate $(0.98)^5$ to 4 decimal places.
Solution: We can use the previous example, by substituting $x = 0.01$

$$(0.98)^5 = (1-2x)^5 = 1-0.1+0.004-0.00008+0.0000008-0.0000000032$$

Notice that the last two terms can be ignored as they are clearly insignificant to 4 decimal places. Thus

$$\begin{aligned}(0.98)^5 &\approx 1-0.1+0.004-0.00008\\ &\approx 0.9039 \text{ (to four decimal places)}.\end{aligned}$$

Example

Use the binomial theorem to expand out the brackets in the expression $(x+3)^8$
Solution:

$$\begin{aligned}(x+3)^8 &= \sum_{r=0}^{8}\binom{8}{r}x^{8-r}(3)^r\\ &= \binom{8}{0}x^8(3)^0+\binom{8}{1}x^7(3)^1+\binom{8}{2}x^6(3)^2+\binom{8}{3}x^5(3)^3+\binom{8}{4}x^4(3)^4\\ &\quad+\binom{8}{5}x^3(3)^5+\binom{8}{6}x^2(3)^6+\binom{8}{7}x^1(3)^7+\binom{8}{8}x^0(3)^8\\ &= (1\times 1)x^8+(8\times 3)x^7+(28\times 9)x^6+(56\times 27)x^5+(70\times 81)x^4\\ &\quad+(56\times 243)x^3+(28\times 729)x^2+(8\times 2187)x+(1\times 6561)\\ &= x^8+24x^7+252x^6+1512x^5+5670x^4+13608x^3+20412x^2\\ &\quad+17496x+6561\end{aligned}$$

Example

Find the coefficient of x^2 in the expansion of $(3x-\frac{1}{x})^8$.
Solution: Each term of the expansion will be of the form $\binom{n}{r}(3x)^r(\frac{-1}{x})^{8-r}$. Note that we can write this term as some coefficient times $x^{r-(8-r)}$. Since we want to find the coefficient of the x^2 term we must look for the value of r such that $r-(8-r)=2r-8=2$; it is easy to see that this is when $r=5$. The corresponding binomial coefficient is

$$\binom{8}{5}=\frac{8!}{3!5!}=\frac{8\cdot 7\cdot 6}{3!}=8\cdot 7=56\,,$$

and so we deduce that the x^2 term of the expansion will be

$$\binom{8}{5}(3x)^5\left(\frac{-1}{x}\right)^3=-56\times 3^5x^2=-13608x^2.$$

Thus the required coefficient is -13608.

Chapter 3

Further differentiation

3.1 Reminder of basic differentiation

Revision from last semester..

The **derivative** $f'(x)$ of a function $f(x)$ is given by

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \text{ where } h \text{ represents a small change in } x.$$

$f(x)$	$f'(x)$	Notes
m	0	m any constant
x^n	nx^{n-1}	n any (non-zero) real number
e^x	e^x	
$\ln(x)$	$\frac{1}{x}$	
$\sin(x)$	$\cos(x)$	for x in radians
$\cos(x)$	$-\sin(x)$	for x in radians
$\tan(x)$	$\sec^2(x)$	for x in radians
$\lambda f(x)$	$\lambda f'(x)$	(constant $\times$ function)
$f(x) + g(x)$	$f'(x) + g'(x)$	(sum of functions)
$f(u(x))$	$f'(u(x))u'(x)$	(function of a function; 'chain rule')
$u(x)\,v(x)$	$u'(x)v(x) + u(x)v'(x)$	(product of functions; 'product rule')
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$	(quotient of functions; 'quotient rule')

Exercise

For each of the following functions $f(x)$, find the derivative $f'(x)$:

→→→→→→→→→→→→→→→ **Differentiation** →→→→→→→→→→→→→→	
$f(x) = x^2 + \sin(2x)$	$f'(x) = 2x + 2\cos(2x)$ (Example)
$f(x) = x^3 + 2x + 5$	$f'(x) =$
$f(x) = e^{x+3}$	$f'(x) =$
$f(x) = \ln(x) + 7$	$f'(x) =$
$f(x) = 2\sin(3x) + 2x + 1$	$f'(x) =$
$f(x) = \cos(x^2)$	$f'(x) =$

Hint: You may need to use the **chain rule**!

We will also talk about *differentials* df instead of derivatives $f'(x)$, which can be computed by just first computing $f'(x)$ and then multiplying by dx:

Definition

The **differential** df of a function f is given by

$$df = f'(x)\,dx.$$

Differentials thus have the same table as derivatives:

Remark

f	differential df	Notes
m	$0\,dx$	m any constant
x^n	$nx^{n-1}\,dx$	n any (non-zero) real number
e^x	$e^x\,dx$	
$\ln(x)$	$\frac{1}{x}\,dx$	
$\sin(x)$	$\cos(x)\,dx$	for x in radians
$\cos(x)$	$-\sin(x)\,dx$	for x in radians
$\tan(x)$	$\sec^2(x)\,dx$	for x in radians
$\lambda f(x)$	$\lambda f'(x)\,dx$	(constant $\times$ function)
$f(x)+g(x)$	$[f'(x)+g'(x)]dx$	(sum of functions)
$f(u(x))$	$f'(u(x))u'(x)\,dx$	(function of a function; 'chain rule')
$u(x)\,v(x)$	$[u'(x)v(x)+u(x)v'(x)]\,dx$	(product of functions; 'product rule')
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x)-u(x)v'(x)}{v(x)^2}\,dx$	(quotient of functions; 'quotient rule')

Remark

Sometimes we also write the chain rule, product rule and quotient rule more simply using a shorter notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{df}\frac{dg}{dx} = f'(g(x))g'(x) \quad \text{(chain rule)}$$

$$d(uv) = du\,v + u\,dv \quad \text{(product rule)}$$

and

$$d\Big(\frac{u}{v}\Big) = \frac{du\,v - u\,dv}{v^2} \quad \text{(quotient rule)}$$

Check from the definition of differential why we can write in this way.

Using our table of standard derivatives together with the fundamental properties of differentiation (such as the product rule and the chain rule) it is possible to differentiate a wide range of functions.

Example

Find the differential of $y = x^2 \sin(x)$.
Solution: Let $u = x^2$ and $v = \sin(x)$, so that $du = 2xdx$ and $dv = \cos(x)dx$, giving

$$\begin{aligned} dy = d(uv) &= du\, v + u\, dv \\ &= (2x\ dx)\sin(x) + x^2\,(\cos(x)\ dx) \\ &= (2x\sin(x) + x^2\cos(x))\ dx\,. \end{aligned}$$

Example

Find the derivative of $y = x\ln(x)$.
Solution: Let $u = x$ and $v = \ln(x)$, so that $u' = 1$ and $v' = \frac{1}{x}$ giving

$$y' = (uv)' = u'\,v + u\,v' = 1\ \ln(x) + x\frac{1}{x} = \ln(x) + 1.$$

Example

Differentiate $y = \sin(x^2)$.
Solution: Let $u(x) = x^2$ so that $y(u) = \sin(u)$ and

$$\frac{dy}{du} = \cos(u)\,, \qquad du = 2x\,dx.$$

Using the chain rule gives

$$dy = \frac{dy}{du}\,du = (\cos(u))\,(2x\ dx) = 2x\cos(x^2)\ dx\,.$$

Example

Find the derivative of $y = e^{\sin(x)}$.
Solution: Let $u = \sin(x)$, so that $y = e^u$. Applying the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = e^u\cos(x) = e^{\sin(x)}\ \cos\,(x).$$

Example

Differentiate $y = \dfrac{e^{x^2}}{\sin(x)}$.

Solution: Let $u = e^{x^2}$ and $v = \sin(x)$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(\sin\ (x))(2xe^{x^2}) - (e^{x^2})(\cos(x))}{\sin^2(x)}$$
$$= (2x\sin(x) - \cos(x))\,\frac{e^{x^2}}{\sin^2\ (x)}$$

Example

Find the differential of $y = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$.

Solution:

$$\begin{aligned} dy &= \frac{d(\cos(x))\,\sin(x) - \cos(x)\,d(\sin(x))}{\sin^2(x)} && \text{(By the quotient rule)} \\ &= \frac{(-\sin(x)\ dx)\,\sin(x) - \cos(x)\ (\cos(x)\ dx)}{\sin^2(x)} && \text{(Using tables.)} \\ &= \frac{(-\sin^2(x) - \cos^2(x))\ dx}{\sin^2(x)} = \frac{-dx}{\sin^2(x)} && \text{(Using } \cos^2(x) + \sin^2(x) = 1) \\ &= -\text{cosec}^2(x)\,dx. \end{aligned}$$

Since the quotient rule is a consequence of the product rule and the chain rule, if we prefer, we can always apply the product rule and then chain rule separately:

Example

Let $y = \cot(x)$. We have:

$$\begin{aligned} dy &= d\left(\cos(x)\frac{1}{\sin(x)}\right) = d(\cos(x))\frac{1}{\sin(x)} + \cos(x)\,d\left(\frac{1}{\sin(x)}\right) && \text{(Product rule)} \\ &= (-\sin(x)\ dx)\,\frac{1}{\sin(x)} + \cos(x)\Big(-\frac{1}{\sin^2(x)}\,(\cos(x))\,dx\Big) && \text{(Tables + Chain rule)} \\ &= \left(-1 - \frac{\cos^2(x)}{\sin^2(x)}\right)\ dx \\ &= -(1 + \cot^2(x))\ dx\ = -\text{cosec}^2(x)dx && \text{(Trig. identities)} \end{aligned}$$

Example

Let $y = \frac{1}{w}$ for any function w of x. Now, applying the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dw}\,\frac{dw}{dx} = -w^{-2}\,\frac{dw}{dx} = -\frac{1}{w^2}\,\frac{dw}{dx}$$

Example

Differentiate $y = (1 + \cos(x^2))^6$.
Solution: Let $u = 1 + \cos(x^2)$. Then $y = u^6$ and by the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = 6u^5\frac{du}{dx}.$$

Notice that $u = 1 + \cos(x^2)$ is also a "function of a function". So in order to differentiate u with respect to x we may use the chain rule *again*.
Let $v = x^2$. Then $u = 1 + \cos(v)$ and by the chain rule:

$$\frac{du}{dx} = \frac{du}{dv}\,\frac{dv}{dx} = (-\sin(v))\;2x.$$

Putting this all together gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx}\\ &= 6u^5\;(-\sin(v))\;2x\\ &= 6(1+\cos(x^2))^5\;(-\sin(x^2))2x\\ &= -12x\,\sin\,(x^2)(1+\cos(x^2))^5\end{aligned}$$

3.2 Implicit differentiation

An expression of the form $y = f(x)$ describes y as an *explicit* function of x.

e.g., $y = x^2$, $y = \cos(x) + \sin(x)$, $y = \frac{1}{x+2}$ etc.

The dependence of y upon the variable x is given *explicitly* by an equation with y on one side, and an expression *in terms of the variable x only* on the other side.

We may encounter more complicated relationships between the variables x and y.

e.g. $x^2 + y^2 = 1$, $e^y = x^2 + 3$, $\ln(2y + 1) = \sin(x)$ etc.

Such an equation still describes a dependence of variable y upon variable x, and so defines y as a function of x *'implicitly'*. We obtain an explicit function if we can solve the equation for y. However it is possible to find the derivative $\frac{dy}{dx}$ **without** explicitly solving the equation.

Example

Find $\frac{dy}{dx}$ when $x^2 + y^2 = 1$.
Solution: By treating y as an implicit function of x, we can differentiate the equation of the circle directly to obtain:

$$2x + 2y\frac{dy}{dx} = 0\,.$$

Rearranging this gives:

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}\,.$$

Note that the right-hand side contains y; this may be eliminated from the formula by solving the equation for y, if required.
For example, when $y \geq 0$ we obtain

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1 - x^2}}\,.$$

Example

Find $\frac{dy}{dx}$ when $e^y = x^2 + 3$
Solution: By treating y as an implicit function of x, we obtain:

$$e^y\frac{dy}{dx} = 2x,$$

which rearranges to give

$$\frac{dy}{dx} = \frac{2x}{e^y} = \frac{2x}{x^2 + 3}$$

Example

Find $\frac{dy}{dx}$ when $\ln(2y-1) = \sin(x)$
Solution: By treating y as an implicit function of x, we obtain:

$$\frac{1}{2y-1} 2\frac{dy}{dx} = \cos(x),$$

which rearranges to give

$$\frac{dy}{dx} = \frac{1}{2}(2y-1)\cos(x) = \frac{1}{2}e^{\sin(x)}\cos(x)$$

Implicit differentiation is useful in finding the derivatives of inverse functions.

Example

Use implicit differentiation to find the derivative of $y = \sin^{-1}(x)$.
Solution: We have $\sin(y) = x$. Now treating y as an implicit function of x gives

$$\begin{aligned} \cos(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} \end{aligned}$$

This expression can be further simplified.
Using the identity $\cos^2(y) + \sin^2(y) = 1$, we note that $\cos(y) = \pm\sqrt{1-\sin^2(y)}$.
Since $-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$, it follows that $\cos(\sin^{-1}(x))$ is **positive**.
Thus $\cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$, giving

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Example

Use implicit differentiation to find the derivative of $y = \cos^{-1}(x)$.
Solution: We have $\cos(y) = x$. Now treating y as an implicit function of x gives

$$\begin{aligned} -\sin(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin(y)} = -\frac{1}{\sin(\cos^{-1}(x))} \end{aligned}$$

This expression can be further simplified.
Using the identity $\cos^2(y) + \sin^2(y) = 1$, we note that $\sin(y) = \pm\sqrt{1 - \cos^2(y)}$.
Since $0 \leq \cos^{-1}(x) \leq \pi$, it follows that $\sin(\cos^{-1}(x))$ is **positive**.
Thus $\sin(\cos^{-1}(x)) = \sqrt{1 - \cos^2(\cos^{-1}(x))} = \sqrt{1 - x^2}$, giving

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Example

Use implicit differentiation to find the derivative of $y = \tan^{-1}(x)$.
Solution: We have $\tan(y) = x$. Now treating y as an implicit function of x gives

$$\begin{aligned} \sec^2(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2(y)} = \cos^2(y) = \cos^2(\tan^{-1}(x)) \end{aligned}$$

This expression can be simplified using the identity $1 + \tan^2(y) = \sec^2(y)$ which rearranges to $\cos^2(y) = \dfrac{1}{1 + \tan^2(y)}$. Therefore

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \frac{1}{1 + x^2}.$$

Example

Find the derivative of $y = a^x$ where $a > 0$ and $a \neq 1$.
Solution: First note that we may write $\ln(y) = \ln(a^x) = x\ln(a)$. Now treating y as an implicit function of x gives

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \ln(a) \\ \frac{dy}{dx} &= \ln(a)y = \ln(a)a^x. \end{aligned}$$

Example

Find the differential of $y = x^\alpha$ where $x > 0$ and α is a real number.
Solution: First note that we may write $\ln(y) = \alpha \ln(x)$. Now treating y as an implicit function of x gives

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \alpha\frac{1}{x} \\ \frac{dy}{dx} &= \alpha\frac{y}{x} = \alpha\frac{x^\alpha}{x} = \alpha x^{\alpha-1}. \end{aligned}$$

Notice that the formula is the same as that for the powers x^n, where n is an integer.

Remark

We add the more fundamental examples to our table of standard derivatives:

Function	Derivative	Differential
$\cot(x)$	$-\mathrm{cosec}^2(x)$	$-\mathrm{cosec}^2(x)\,dx$
$\sin^{-1}(x)$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{dx}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{-\,dx}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\dfrac{1}{1+x^2}$	$\dfrac{dx}{1+x^2}$
x^α (where $x > 0$ and α is a real number)	$\alpha x^{\alpha-1}$	$\alpha\, x^{\alpha-1}\ dx$
a^x (where $a > 0$)	$a^x \ln(a)$	$a^x \ln(a)\ dx$

3.3 Logarithmic differentiation

Logarithmic differentiation is an application of implicit differentiation, which is applied to an expression $y = f(x)$, where the function $f(x)$, after taking, logarithms, will be easier to differentiate than without logarithms. The basic strategy is in the following.

Definition (Logarithmic differentiation)

1. Consider a function
$$y = f(x)$$
2. Take natural logarithms from both sides:
$$\ln|y| = \ln|f(x)|.$$
Solve this now using implicit differentiation.
3. In implicit differentiation, we first differentiate both sides and use the chain rule:
$$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)}.$$
4. Multiplying both sides by y, we have
$$\frac{dy}{dx} = y\frac{f'(x)}{f(x)}.$$
5. Inputting $y = f(x)$ gives us the derivative of y with respect to x:
$$\frac{dy}{dx} = f'(x).$$

It may look like this method does not do much, but in the following examples we see its use.

The first application is the product rule from differentiation.

Example (Product rule)

Let
$$y = f(x)g(x).$$
Prove using logarithmic differentiation that
$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x).$$

Solution: Take logarithms from both sides:
$$\ln|y| = \ln|f(x)g(x)|.$$
Logarithm transforms product into sums so we have:
$$\ln|y| = \ln|f(x)| + \ln|g(x)|.$$
Differentiate both sides and use chain rule:
$$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}.$$
Multiply both sides by y:
$$\frac{dy}{dx} = y\frac{f'(x)}{f(x)} + y\frac{g'(x)}{g(x)}.$$
Input $y = f(x)g(x)$ to get:
$$\frac{dy}{dx} = f(x)g(x)\frac{f'(x)}{f(x)} + f(x)g(x)\frac{g'(x)}{g(x)} = f'(x)g(x) + f(x)g'(x)$$
as claimed.

The second application is the quotient rule from differentiation.

Example (Quotient rule)

Let
$$y = \frac{f(x)}{g(x)}.$$
Prove using logarithmic differentiation that
$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Solution: Take logarithms from both sides:
$$\ln|y| = \ln\left|\frac{f(x)}{g(x)}\right|.$$
Logarithm transforms quotients into differences so we have:
$$\ln|y| = \ln|f(x)| - \ln|g(x)|.$$
Differentiate both sides and use chain rule:
$$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$
Multiply both sides by y:
$$\frac{dy}{dx} = y\frac{f'(x)}{f(x)} - y\frac{g'(x)}{g(x)}.$$
Input $y = f(x)g(x)$ to get:
$$\frac{dy}{dx} = \frac{f(x)}{g(x)}\frac{f'(x)}{f(x)} + \frac{f(x)}{g(x)}\frac{g'(x)}{g(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{g(x)^2}$$
as claimed.

Logarithmic differentiation can also be used to differentiate functions that would look very hard to differentiate otherwise:

Example (Differentiating x^x)

For $x > 0$ let
$$y = x^x.$$
Prove using logarithmic differentiation that
$$\frac{dy}{dx} = x^x \ln x + x^x.$$
Solution: Take logarithms from both sides:
$$\ln y = \ln x^x.$$
Logarithm rule $\log(a^b) = b \log a$ applied gives
$$\ln y = x \ln x.$$
Differentiate both sides and use the chain rule and product rule
$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$
Multiply both sides by y:
$$\frac{dy}{dx} = y \ln x + y.$$
Input $y = x^x$ to get:
$$\frac{dy}{dx} = x^x \ln x + x^x$$
as claimed.

Equipped with the previous examples, if you are courageous, now try to differentiate using logarithmic differentiation the following insane looking functions:
$$y = x^{x^{x^{x^x}}}$$
or
$$y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x+1}}}}.$$

3.4 Parametric differentiation

Parametric differentiation considers a situation where we have two functions

$$y = f(t)$$

and

$$x = g(t)$$

that depend on a parameter t, say, time. Then we would like to know, say, the derivative of y with respect to x

$$\frac{dy}{dx}$$

as a function of t. In real world situations these could appear, for example, in dynamical systems when parametrising the movement of pendulum in two dimensional plane, where $x = f(t)$ would tell the x-coordinate of the pendulum at time t and $y = g(t)$ the y-coordinate. Here the fraction dy/dx would tell the local proportions of the vertical and horizontal changes of the speed of the pendulum at the time t.

Definition (Parametric differentiation)

1. Consider functions
$$x = f(t) \quad \text{and} \quad y = g(t).$$
2. Differentiate both sides by t:
$$\frac{dx}{dt} = f'(t) \quad \text{and} \quad \frac{dy}{dt} = g'(t).$$
3. Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dy}{dt}} = \frac{f'(t)}{g'(t)}.$$
provided that $g'(t)$ is not 0.

Let us give in the following a couple of examples where we can use this method.

Example

We can parametrise the unit circle with the following points:

$$x = \cos t \quad \text{and} \quad y = \sin t,$$

when $0 \leq t \leq 2\pi$, recall for example the polar coordinate form of complex numbers or trigonometry. Find

$$\frac{dy}{dx}.$$

Solution: Using the parametric differentiation, we first differentiate both parametrisations with respect to t:

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = \cos t.$$

Therefore

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = \frac{1}{-\tan t}$$

for those t when $\tan t \neq 0$.

Here is an example using polynomial functions:

Example

Use parametric differentiation to find $\frac{dy}{dx}$ when $x = t^3 + 2t + 1$ and $y = t^7 - 8$.

Solution: Differentiate both parametrisations with respect to t:

$$\frac{dx}{dt} = 3t^2 + 2 \quad \text{and} \quad \frac{dy}{dt} = 7t^6.$$

Therefore

$$\frac{dy}{dx} = \frac{7t^6}{3t^2 + 2}.$$

3.5 Taylor and MacLaurin series

Write $f^{(r)}(x)$ to denote the function obtained from $f(x)$ after differentiating r times.

Example

Find $f^{(3)}(x)$ for $f(x) = x^2 + \sin(x)$.
Solution:

$$f^{(1)}(x) = 2x + \cos(x), \quad f^{(2)}(x) = 2 - \sin(x), \quad f^{(3)}(x) = -\cos(x)$$

The main statement of this section is as follows.

Theorem 7 (Taylor series about the point $x = a$)

Let $f(x)$ be a function which can be differentiated infinitely many times at $x = a$.

$$\begin{aligned}\text{For } x \to a, \quad f(x) &= \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!}(x-a)^r \\ &= f(a) + f^{(1)}(a)\,(x-a) \;+ \frac{1}{2}\,f^{(2)}(a)\,(x-a)^2 + \frac{1}{3!}\,f^{(3)}(a)\,(x-a)^3 \\ &\quad + \cdots \quad + \frac{1}{n!}\,f^{(n)}(a)\,(x-a)^n \;+ \cdots\end{aligned}$$

Taylor series about the point $x = 0$ (that is setting $a = 0$ in the above), are sometimes called **Maclaurin series**.

Example

Let α be a real number. Find the Taylor series of $f(x) = (1+x)^\alpha$ at zero.

Solution: We need to evaluate the function and its derivatives at $x = 0$.

$f(x) = (1+x)^\alpha,$	$f(0) = 1$
$f^{(1)}(x) = \alpha(1+x)^{\alpha-1},$	$f^{(1)}(0) = \alpha$
$f^{(2)}(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$	$f^{(2)}(0) = \alpha(\alpha-1)$
$\vdots$	$\vdots$
$f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}$	$f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)$

Substituting these values into the Taylor formula we find that for $x \to 0$

$$\begin{aligned}(1+x)^\alpha \;&=\; 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\,x^3 + \cdots \\ &\quad + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\,x^n + \cdots\end{aligned}$$

Note the similarity between the formula above and the expansion of $(1+x)^n$ given by the **Binomial theorem**, where n is a positive integer.

Example

Find the Taylor series about $x = 0$ of $f(x) = (1+x)^{-1}$.

Solution: Setting $\alpha = -1$ in the previous example we see that the coefficient of x^n will be $(-1)(-2)\dots(-1-n+1)/n! = (-1)(-2)\dots(-n)/n! = (-1)^n$. Therefore for $x \to 0$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{k=0}^{\infty}(-x)^n$$

Notice that the terms on the right-hand side can be viewed as a geometric progression with first term 1 and common ratio $-x$. Using the formula for the sum of a geometric series (which holds for all $|x| < 1$) we obtain the left hand side.

Notice that when x is very close to a, or in other words, when the difference $x - a$ is very small, the powers $(x-a)^2$, $(x-a)^3$, etc. get smaller and smaller and smaller. If we are interested in finding an *approximation* to the function $f(x)$ in an interval around the point $x = a$, we can therefore do so by ignoring all terms in the expansion involving powers $(x-a)^{n+1}$ and higher.

Example

If we ignore all terms involving powers $(x-a)^2$ and higher we obtain

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \to a$$

(Writing $x = a + h$, we recover the definition of the derivative at a.)

In general by ignoring all terms in the expansion involving $(x-a)^{n+1}$ and higher powers, the Taylor formula allows us to approximate a function by a polynomial of degree n.

Definition (Taylor expansion)

The n**th order Taylor expansion** of a function $f(x)$ at a is written as

$$\text{For } x \to a, \quad f(x) \approx f(a) + f^{(1)}(a)\,(x-a) + \frac{1}{2}\,f^{(2)}(a)\,(x-a)^2 + \cdots + \frac{1}{n!}\,f^{(n)}(a)\,(x-a)^n.$$

Example

Find the third order Taylor expansion of $f(x) = \sqrt{1+x}$ at zero.

Solution: This is another special case of $(1+x)^\alpha$, where $\alpha = \frac{1}{2}$. Note that this time we were only asked to find the expansion up to and including the x^3 term. Setting $\alpha = \frac{1}{2}$ in the Taylor expansion of $(1+x)^\alpha$ we find that for $x \to 0$,

$$\begin{aligned}\sqrt{1+x} &\approx 1 + \frac{1}{2}\,x + \frac{1}{2!}\,\frac{1}{2}\left(\frac{1}{2}-1\right)\,x^2 + \frac{1}{3!}\,\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\,x^3 \\ &\approx 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3\end{aligned}$$

Example

Find the Taylor series about $x = 0$ for the function $f(x) = e^x$.

Solution: We have $f^{(1)}(x) = e^x$, $f^{(2)}(x) = e^x$, etc.
Evaluating at $x = 0$ gives, $f(0) = f^{(1)}(0) = f^{(2)}(0) = \cdots = 1$. Therefore for $x \to 0$

$$e^x = 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \frac{1}{4!}\,x^4 + \cdots + \frac{1}{n!}\,x^n + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

Example

Find the Taylor series about $x = 0$ for the function $f(x) = \sin(x)$.

Solution: We have
$f^{(1)}(x) = \cos(x)$, $f^{(2)}(x) = -\sin(x)$, $f^{(3)}(x) = -\cos(x)$ and $f^{(4)}(x) = \sin(x) = f(x)$.
Evaluating at $x = 0$ gives $f(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$, etc. Notice that we get a value of 0 for all derivatives of even orders, whilst the values for the derivatives of odd orders alternate between $+1$ and -1. Therefore for $x \to 0$

$$\sin(x) = x - \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 - \frac{1}{7!}\,x^7 + \cdots + (-1)^n \frac{1}{(2n+1)!}\,x^{2n+1} + \cdots$$

Example

Find the Taylor series about $x = 0$ for the function $f(x) = \cos(x)$.

Solution: This is very similar to the previous example.
We have $f^{(1)}(x) = -\sin(x)$, $f^{(2)}(x) = -\cos(x)$, $f^{(3)}(x) = \sin x$, etc.
Evaluating at $x = 0$ gives $f(0) = 1$; we get a value of 0 for all derivatives of odd orders and values alternating between $+1$ and -1 for the derivatives of even orders. Therefore using the Taylor formula we find:

$$\cos(x) = 1 - \frac{1}{2!}\,x^2 + \frac{1}{4!}\,x^4 - \frac{1}{6!}\,x^6 + \ldots + (-1)^n \frac{1}{(2n)!}\,x^{2n} + \cdots.$$

Example

Find the Taylor series about $x = 0$ for the function $f(x) = \ln(1 + x)$.
Solution:

$f(x) = \ln(1+x)$	$f(0) = \ln(1) = 0$
$f^{(1)}(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f^{(1)}(0) = 1$
$f^{(2)}(x) = (-1)(1+x)^{-2}$	$f^{(2)}(0) = -1 \times 1!$
$f^{(3)}(x) = (-1)(-2)(1+x)^{-3}$	$f^{(3)}(0) = (-1)^2 \times 2!$
$\vdots$	$\vdots$
$f^{(n)}(x) = (-1)(2)\ldots(-(n-1))(1+x)^{-n}$	$f^{(n)}(0) = (-1)^{n-1}\,(n-1)!$

and hence

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots + (-1)^{n-1}\,\frac{1}{n}\,x^n + \cdots$$

Taylor formula has many applications. One such application is to finding limits.

Example

Calculate the limit:
$$\lim_{x\to 0} \frac{\sqrt{1+2x}-1}{x} \,.$$

Solution: Notice that it is not possible to simply substitute $x = 0$ into the top and bottom of the fraction, because we cannot evaluate " $\frac{0}{0}$ ".
Using the first order Taylor expansion $\sqrt{1+2x}$ for $x \to 0$ we have

$$\sqrt{1+2x} \;=\; 1 + \frac{1}{2}\,2x + \cdots = 1 + x + \text{ terms involving higher powers of } x$$

and hence

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sqrt{1+2x}-1}{x}\right) &= \lim_{x\to 0}\left(\frac{x + \text{ terms involving higher powers of } x}{x}\right)\\ &= \lim_{x\to 0}\Big(1 + \text{ terms involving powers of } x\Big) = 1\,.\end{aligned}$$

Chapter 4

Further integration

4.1 Reminder of basic integration

Roughly speaking, **integration** can be thought of as the *reverse process* to differentiation. Last semester you learned how to find the derivative or *rate of change* of a given function. Let's turn this problem on its head. Suppose that you know the rate of change of a function. Can you determine what the function is?

Exercise

If $f'(x) = 3x^2 + 2$, what could $f(x)$ be?

$\to\to\to\to\to\to\to\to\to\to\to\to$ **Reversing differentiation** $\to\to\to\to\to\to\to\to\to\to\to\to$	
$f'(x) = 3x^2 + 2$	$f(x) =$

There is more than one correct answer! (See below for an explanation.)

Definition (Antiderivative)

An **antiderivative** of a function $f(x)$ is a function $F(x)$ whose derivative is equal to $f(x)$.

Example

The function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because

$$F'(x) = 2x = f(x).$$

The function $G(x) = x^2 - 12$ is another antiderivative of $f(x) = 2x$, because

$$G'(x) = 2x = f(x).$$

In fact, it is easy to see that if C is any constant then the function $H(x) = x^2 + C$ will also be an antiderivative of $f(x) = 2x$.

Note: The example above illustrates that once one antiderivative is known, infinitely many can be found by adding on different constant terms. In fact, two antiderivatives of the same function can *only* differ by addition of a constant term, as we shall now show.

Theorem 8

If $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then $F(x) = G(x) + C$ for some constant C.

Reason: If $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then $F'(x) = f(x)$ and $G'(x) = f(x)$, which gives

$$F'(x) - G'(x) = f(x) - f(x) = 0, \text{ and hence } \frac{d}{dx}(F(x) - G(x)) = 0.$$

Since only constants differentiate to give zero, it follows that $F(x) - G(x) = C$ for some constant C. Rearranging this gives $F(x) = G(x) + C$.

Note: It follows from the above that if we can find just one antiderivative, then we can describe all the others by adding on constants.

Exercise

For each of the following functions $f(x)$, find *all* antiderivatives:

→→→→→→→→→→→→ **Reversing differentiation** →→→→→→→→→→→→	
$f(x) = \dfrac{1}{x}$ is the derivative of $\boxed{\ln(x)}$	Every antiderivative of $f(x) = \dfrac{1}{x}$ has the form $F(x) = \boxed{\ln(x) + C}$, where C is a constant.
$f(x) = e^x$ is the derivative of	Every antiderivative of $f(x) = e^x$ has the form $F(x) =$
$f(x) = \cos(x)$ is the derivative of	Every antiderivative of $f(x) = \cos(x)$ has the form $F(x) =$
$f(x) = -\sin(x)$ is the derivative of	Every antiderivative of $f(x) = -\sin(x)$ has the form $F(x) =$
$f(x) = \sec^2(x)$ is the derivative of	Every antiderivative of $f(x) = \sec^2(x)$ has the form $F(x) =$
$f(x) = nx^{n-1}$ is the derivative of	Every antiderivative of $f(x) = nx^{n-1}$ has the form $F(x) =$
$f(x) = m$, a constant is the derivative of	Every antiderivative of $f(x) = m$ has the form $F(x) =$

Note: Sometimes we may wish to find a *particular* antiderivative, satisfying an extra condition.

Example

Find the antiderivative $F(x)$ of $f(x) = 2x$, satisfying $F(1) = 2$.

Solution:

- $f(x) = 2x$ is the derivative of $\boxed{x^2}$.
- Every antiderivative of $f(x)$ has the form $F(x) = \boxed{x^2 + C}$ where C is a constant.
- To find the correct value of C, substitute $x = 1$ and rearrange.
  We obtain $\boxed{F(1) = 1^2 + C}$. Since $F(1) = 2$ this gives $\boxed{2 = 1 + C}$ and hence $\boxed{C = 1}$.
- The required antiderivative is therefore $\boxed{F(x) = x^2 + 1}$

Exercise

Find the antiderivative $F(x)$ of $f(x) = \cos(2x)$ satisfying $F\left(\frac{\pi}{4}\right) = 0$.

Solution:

- $f(x) = \cos(2x)$ is the derivative of
- Every antiderivative of $f(x)$ has the form $F(x) =$, where C is a constant.
- To find the correct value of C, substitute $x = \frac{\pi}{4}$ and rearrange.
  We obtain $F(\frac{\pi}{4}) =$

  Since $F\left(\frac{\pi}{4}\right) = 0$ this gives and hence $C =$.
- The required antiderivative is therefore $F(x) =$

Definition (Indefinite integral)

The **indefinite integral** $\int f(x)\ dx$ represents all possible antiderivatives of $f(x)$. In other words, if $F(x)$ is some particular antiderivative for $f(x)$, then

$$\int f(x)\ dx = F(x) + C\,,$$

where C is an *undetermined* constant term.

Note: The mathematical expression above is read aloud as
The (indefinite) integral of the function "f of x" with respect to x is equal to "capital F of x " plus a constant term, C.

Example

Find $\int 3x^2\ dx$.
Solution: We have seen that any function of the form x^3+C (where C is a constant) is an antiderivative of $3x^2$. Hence,

$$\int 3x^2\ dx = x^3 + C.$$

You can apply **your knowledge of derivatives from last semester** to find indefinite integrals. By looking at (or 'inspecting') the function $f(x)$ you can try to recognise it as the derivative of some other function. This process is called **integration by inspection**. We have already seen some basic examples of this:

Function	Indefinite integral
m (a constant)	$\int m\ dx = mx + C$
nx^{n-1}	$\int nx^{n-1}\ dx = x^n + C$
e^x	$\int e^x\ dx = e^x + C$
$\frac{1}{x}$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\sin(x)$	$\int \sin(x)\ dx = -\cos(x) + C$
$\cos(x)$	$\int \cos(x)\ dx = \sin(x) + C$
$\sec^2(x)$	$\int \sec^2(x)\ dx = \tan(x) + C$

When asked to integrate a function, the general strategy is to try to *manipulate* the function into a well-known form (such as the forms in the table above) which can be easily integrated. These forms are called "standard forms" or "table integrals".

Keep in mind that integration by inspection may involve a bit of guess work on your part at first - as the course progresses you will learn some more techniques from basic algebra and trigonometry to help, as well as more advanced methods of integration. In general a calculation may contain several steps. Thus when finding indefinite integrals, we shall agree to add a constant only at the end of the calculation.

Example

Let n be an integer with $n \neq -1$. Find $\int x^n \; dx$.
Solution: We know that

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n \,.$$

Since $n \neq -1$ *we can divide both sides by* $n+1$ to find that

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = x^n \,.$$

Thus

$$\int x^n \; dx = \frac{1}{n+1}x^{n+1} + C \,.$$

Example

Let a, b be constants with $a \neq 0$. Find $\int e^{ax+b} \; dx$.
Solution: We know that

$$\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b} \,.$$

Since $a \neq 0$ *we can divide both sides by* a to find that

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = e^{ax+b} \,.$$

Thus

$$\int e^{ax+b} \; dx = \frac{1}{a}e^{ax+b} + C \,.$$

Example

Find $\int 3x^2 + 4x \ dx$.
Solution: We note that

$$3x^2 + 4x = \frac{d}{dx}(x^3) + 2 \times \frac{d}{dx}(x^2) = \frac{d}{dx}(x^3 + 2x^2)\,.$$

Thus

$$\int 3x^2 + 4x \ dx = x^3 + 2x^2 \ + C.$$

The previous examples illustrate some general principles:

Theorem 9

Let $f(x)$, $g(x)$ be functions of x, and let λ be a constant. Then

$$\int [f(x) + g(x)] \ dx = \int f(x)\, dx + \int g(x)\, dx + C$$

$$\int \lambda f(x)\, dx = \lambda \int f(x)\, dx + C$$

Example

Find $I = \int (x + \frac{1}{x})^2 \ dx$.
Solution: By expanding the brackets we find

$$(x + \frac{1}{x})^2 = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) = \left(x^2 + 1 + 1 + \frac{1}{x^2}\right) = x^2 + 2 + x^{-2}$$

Thus

$$\begin{aligned} I &= \int x^2\, dx + \int 2\, dx + \int x^{-2}\, dx \\ &= \frac{1}{3}x^3 + 2x - \frac{1}{x} \ + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 2x - x^{-1} \ + C\right) = x^2 + 2 + x^{-2}$$

Example

Find the indefinite integral
$$\int \frac{1}{1+2x}\,dx.$$

Solution: Recalling that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$, we decide to inspect the function $\ln|1+2x|$. By the chain rule we have

$$\frac{d}{dx}\left(\ln|1+2x|\right) = \frac{1}{1+2x} \times 2 = \frac{2}{1+2x}\,.$$

Therefore
$$\int \frac{1}{1+2x}\,dx = \frac{1}{2} \times \int \frac{2}{1+2x}\,dx = \frac{1}{2}\ln|1+2x| + C\,.$$

Check:
$$\frac{d}{dx}\left(\frac{1}{2}\ln|1+2x| + C\right) = \frac{1}{2} \times \frac{1}{1+2x} \times 2 = \frac{1}{1+2x}$$

Remark: All of the examples we have considered so far concern functions of a variable x. There is nothing special about the letter x. Depending on the nature of the problem, we may consider, for example, functions $f(t)$ of an argument denoted by t. (The letter t is a traditional notation for the time variable.)

Definition (Definite integral)

For constants a and b, the **definite integral** $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.
(The result is **independent** of choice of antiderivative $F(x)$.)

Example

Find $\int_2^3 3x^2 \ dx$.
Solution: Since $F(x) = x^3$ is an antiderivative of $3x^2$ we find that

$$\int_2^3 3x^2 \ dx = F(3) - F(2) = (3^3) - (2^3) = 27 - 8 = 19.$$

Notice that if C is a constant then $F(x) = x^3 + C$ is another antiderivative of $3x^2$, also giving

$$\int_2^3 3x^2 \ dx = F(3) - F(2) = (3^3 + C) - (2^3 + C) = 27 - 8 = 19.$$

Notation: It will often be convenient to write $[F(x)]_a^b$ to denote $F(b) - F(a)$ in our calculations.

Example

Find $\int_0^1 \sqrt{x} \ dx$.
Solution:

$$\int_0^1 \sqrt{x} \ dx = \int_0^1 x^{1/2} \ dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \left[\frac{2}{3} x^{3/2}\right]_0^1 = (\frac{2}{3} 1^{3/2}) - (\frac{2}{3} 0^{3/2}) = \frac{2}{3}\,.$$

The following properties can be easily deduced from the above definition:

Theorem 10

Let $f(x)$, $g(x)$ be functions of x, and let λ be a constant. Then

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx\,.$$

$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx\,.$$

Example

Find $\displaystyle\int_{-3}^{-1} \frac{1}{x} + x \; dx$.
Solution:

$$\begin{aligned}
\int_{-3}^{-1} \frac{1}{x} + x \; dx &= \int_{-3}^{-1} \frac{1}{x} \; dx + \int_{-3}^{-1} x \; dx \\
&= [\ln|x| \;]_{-3}^{-1} + \left[\frac{1}{2}x^2 \;\right]_{-3}^{-1} \\
&= (\ln(1) - \ln(3)) + (\frac{1}{2}(-1)^2) - (\frac{1}{2}(-3)^2) \\
&= \ln(1) - \ln(3) + \frac{1}{2} - \frac{9}{2} = -\ln(3) - 4 \approx -5.10.
\end{aligned}$$

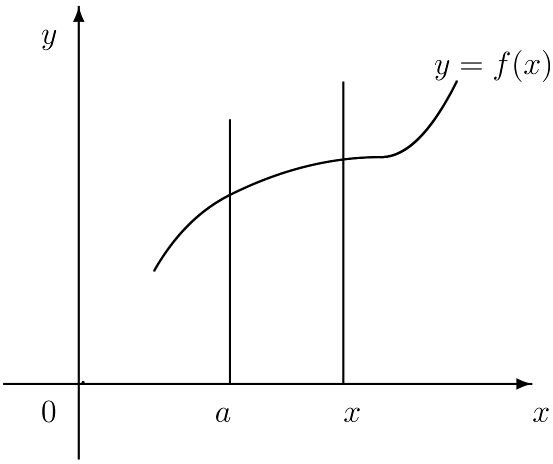
Example

Find $\displaystyle\int_{0}^{\pi/2} 5\sin(x) \; dx$.
Solution:

$$\begin{aligned}
\int_{0}^{\pi/2} 5\sin(x) \; dx &= 5\int_{0}^{\pi/2} \sin(x) \; dx \\
&= 5\,[-\cos(x)]_0^{\pi/2} \\
&= 5((-\cos(\pi/2)) - (-\cos(0))) \\
&= 5(0+1) = 5.
\end{aligned}$$

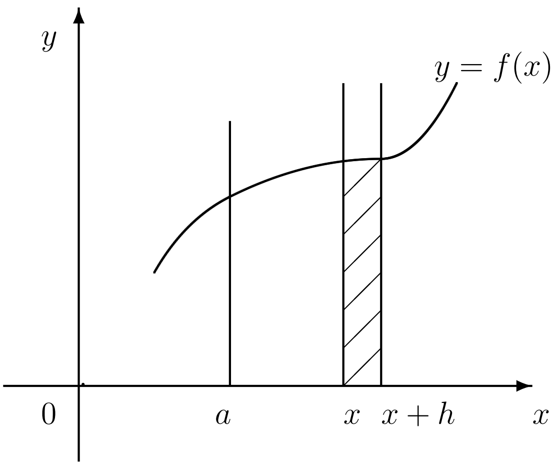
So far we have concentrated on *how to calculate* the value of a given definite integral, without considering *the meaning* behind this value. This turns out to be closely related to the the area under a curve, as we shall now explain.

Consider the area under a curve $y = f(x)$ between a fixed point a and a variable point x:



Note that this area depends on both the lower limit (which we have fixed) and the variable upper limit, x. In other words it is a *function of* x, which we shall denote by $A(x)$. We shall show that $A(x)$ is an *antiderivative* of $f(x)$.

If we change the upper limit by a small amount h...



...the change in area $A(x+h) - A(x)$ can be approximated by the area of the thin rectangle of width h and height $f(x)$:

$$A(x+h) - A(x) \approx f(x)h.$$

Note that rearranging gives:

$$f(x) \approx \frac{A(x+h) - A(x)}{h}.$$

Key idea: The thinner the rectangle, the better the approximation.

Thus as h tends to zero we obtain

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x),$$

where the second equality follows directly from the **definition of the derivative**. Notice that this means that $A(x)$ is an **antiderivative** of $f(x)$.

Since the area between the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ is equal to $A(b) - A(a)$, we note that this is given by the definite integral $\int_a^b f(x)\,dx$.

The relation between the two seemingly unrelated geometric problems, finding the area under the graph and finding the tangent, which links differentiation and integration, was Newton's extraordinary insight.

Theorem 11

If the curve $y = f(x)$ lies entirely above the x-axis between $x = a$ and $x = b$, then the area between the curve, the x-axis and the two vertical limits $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.

Exercise

Consider the constant function $f(x) = 3$. Make a sketch of the function $y = f(x)$.

For any constants a and b, it is clear that the area *between* the graph, the x-axis and the vertical lines $x = a$ and $x = b$ is equal to $(b - a) \times 3 = 3b - 3a$, since this is the area of a rectangle of width $b - a$ and height 3, agreeing with the formula above:

$$\int_a^b 3 \cdot\, dx = [3x]_a^b = 3b - 3a\,.$$

If the curve $y = f(x)$ lies entirely *below* the x-axis between $x = a$ and $x = b$, then $\int_a^b f(x)\,dx$ instead yields a negative value equal to minus 1 times the area between the curve, the x-axis and the two vertical limits $x = a$ and $x = b$.

Exercise

Consider the constant function $f(x) = -3$. Make a sketch of the function $y = f(x)$.

For any constants a and b, it is clear that the area *between* the graph, the $x - axis$ and the vertical lines $x = a$ and $x = b$ is again equal to $3b - 3a$, whilst the definite integral is equal to:

$$\int_a^b -3 \cdot dx = [-3x]_a^b = (-3b) - (-3a) = -3b + 3a = -(3b - 3a)\,.$$

Exercise

Calculate the integral $\int_{-2}^{5} x\,dx$ and interpret your answer in terms of the areas between the curve and the x-axis.

Solution: It is straightforward to calculate that

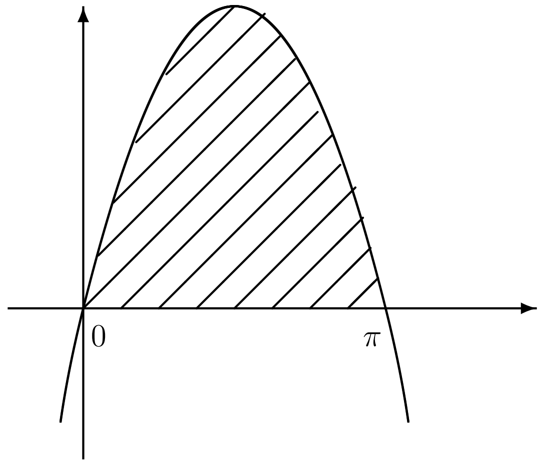
$$\int_{-2}^{5} x\,dx = \left[\frac{1}{2}x^2\right]_{-2}^{5} = \frac{1}{2}(5^2 - (-2)^2) = 10.5$$

Make a sketch of the curve $y = x$.

The graph $y = x$ is a straight line with slope 1. It lies below the x-axis for values of x between -2 and 0. It lies above the $x - axis$ for values of x between 0 and 5. The given indefinite integral therefore represents the difference of the area beneath the line $y = x$ between $x = 0$ and $x = 5$, and the area above the line $y = x$ between $x = -2$ and $x = 0$. Notice that this area can also be found (using your sketch) as the difference of the areas of the two right-angled triangles of heights 5 and 2 respectively and of the same bases, again giving $\frac{1}{2}(5^2) - \frac{1}{2}(2^2) = 10.5$.

Example

Find the area under the curve $y = \sin(x)$ between $x = 0$ and $x = \pi$.
Solution: The curve lies above the x-axis between $x = 0$ and $x = \pi$



Thus the area can be found as follows:

$$\int_0^{\pi} \sin(x)\ dx = [-\cos(x)\ ]_0^{\pi} = -\cos(\pi) + \cos(0) = 1 + 1 = 2,$$

showing that the area is 2.

Example

Find the two places where the curve $y = 2x - x^2$ crosses the x-axis, and hence calculate the enclosed area.
Solution: First make a sketch of $y = 2x - x^2$. Solving $2x - x^2 = 0$ we find that the curve crosses the x-axis at $x = 0$ and $x = 2$. Note that when $x = 1$, $y = 1 > 0$, so the graph *peaks* somewhere between $x = 0$ and $x = 2$.

Now integrating between these values:

$$\int_0^2 (2x - x^2)\ dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = (4 - \frac{8}{3}) - (0 - 0) = \frac{4}{3}$$

The following properties (which can be easily deduced from the definition) may be useful when calculating areas.

Theorem 12

For constants a, b and c,

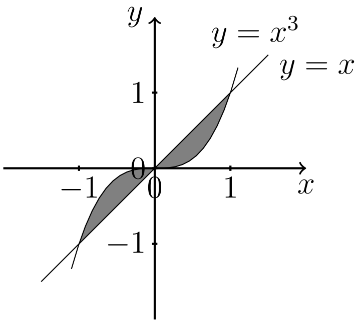
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx\,.$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

$$\int_a^a f(x)\,dx = 0\,.$$

Example

Find the area of the shaded region.



Solution: The shaded area can be divided into two regions:
(1) the region above the x-axis;
(2) the region below the x-axis.
Notice that area of the first region can be found by computing

$$\int_0^1 x\ dx - \int_0^1 x^3\ dx = \left[\frac{1}{2}x^2\right]_0^1 - \left[\frac{1}{4}x^4\right]_0^1 = (\frac{1}{2}-0)-(\frac{1}{4}-0) = \frac{1}{4}.$$

The area of the second region is also equal to $\frac{1}{4}$ (by symmetry). It can be found by computing

$$\int_0^{-1} x\ dx - \int_0^{-1} x^3\ dx = \left[\frac{1}{2}x^2\right]_0^{-1} - \left[\frac{1}{4}x^4\right]_0^{-1} = (\frac{1}{2}-0)-(\frac{1}{4}-0) = \frac{1}{4}.$$

Revision from last semester..

Recall the relationship between integration and differentiation that we established at the beginning of the course. For $F'(x) = f(x)$, we defined

Indefinite integrals: $\int f(x)\ dx = F(x) + C$ where C is a constant.
Definite integrals: $\int_a^b f(x)\ dx = F(b) - F(a)$.

Since the derivative and differential of a function $F(x)$ are related by

$$dF = F'(x)\ dx\,,$$

the above formulas can be conveniently expressed in terms of differentials as

$$\boxed{\int dF = F(x) + C \quad \text{and} \quad \int_a^b dF = F(b) - F(a)}$$

In this way the integral can easily be seen as the 'inverse operator' to the differential. By combining our knowledge of derivatives with other techniques we've learned so far, we can now integrate many functions.

Revision from last semester..

For example, we can use the table of differentials found in the previous chapter to add to our table of standard indefinite integrals:

Function	Indefinite integral	Notes
m	$\int m\ dx = mx + C$	m any constant
nx^{n-1}	$\int nx^{n-1}\ dx = x^n + C$	n any (non-zero) real number
e^x	$\int e^x\ dx = e^x + C$	
$\frac{1}{x}$	$\int \frac{1}{x}\ dx = \ln\|x\| + C$	
$\sin(x)$	$\int \sin(x)\ dx = -\cos(x) + C$	for x in radians
$\cos(x)$	$\int \cos(x)\ dx = \sin(x) + C$	for x in radians
$\sec^2(x)$	$\int \sec^2(x)\ dx = \tan(x) + C$	for x in radians
$\operatorname{cosec}^2(x)$	$\int \frac{dx}{\sin^2(x)} = -\cot(x) + C$	for x in radians
$\frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$	
$\frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1}x + C$	
x^α	$\int x^\alpha\ dx = \frac{x^{\alpha+1}}{\alpha+1} + C$	for $a \neq -1$
a^x	$\int a^x\ dx = \frac{a^x}{\ln a} + C$	for $a \neq 1$

4.2 Integration by substitution

If an integral appears difficult on first inspection, progress can often be made using a *substitution*. This method of solving integrals is known as the *change of variable* method or **integration by substitution** and in a sense can be thought of as the 'reverse' process to differentiating using the chain rule. The key idea is to choose a substitution so that the new integral (in terms of a new variable u) is simpler than the original integral. After solving it and obtaining a function of u, we can then back-substitute the expression for u (given as a function of x) in order to obtain the final answer as a function of x. This is best understood by means of a few examples.

Example

To find $\int e^{7x}\ dx$ we can make the substitution $u = 7x$. Then we have $\frac{du}{dx} = 7$, giving $dx = \frac{1}{7}\, du$ and hence

$$\int e^{7x}\ dx = \int e^u \frac{1}{7}\ du = \frac{1}{7}\int e^u\ du = \frac{1}{7} e^u\ + C = \frac{1}{7} e^{7x}\ + C\,.$$

Example

To find $\int \sin(3x+5)\ dx$ we can make the substitution $u = 3x+5$. Then we have $\frac{du}{dx} = 3$ or $dx = \frac{1}{3}\, du$ and hence

$$\begin{aligned}\int \sin(3x+5)\ dx = \int \sin(u)\frac{1}{3}\ du &= \frac{1}{3}\int \sin(u)\ du \\ &= -\frac{1}{3}\cos(u)\ + C \\ &= -\frac{1}{3}\cos(3x+5)\ + C\,.\end{aligned}$$

The previous two examples illustrate a general principle:

Theorem 13

Let $f(x)$ be a function of x and let $u = ax+b$ be a (non-constant) linear function of x. Then

$$\int f(ax+b)\ dx = \frac{1}{a}\int f(u) du.$$

(This can be seen by using the substitution $u = ax+b$.)

Example

Find $\int \tan(x)\ dx$.
Solution: By definition of the tangent function we have:

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx\,.$$

Consider the substitution $u = \cos(x)$. Then $\frac{du}{d=} - \sin(x)$, giving $\sin(x)\,dx = -du$ and so

$$\int \tan(x)\ dx = \int \frac{\sin(x)\ dx}{\cos(x)} = \int \frac{-du}{u} = -\int \frac{1}{u}\,du = -\ln|u|+C = -\ln|\cos(x)|+C\,.$$

Example

Find $\displaystyle\int \frac{2x}{1+x^2}\ dx$.
Solution: Let $u = 1 + x^2$. Then $\frac{du}{dx} = 2x$ giving $du = 2xdx$ and so

$$\int \frac{2x}{1+x^2}\ dx = \int \frac{2xdx}{1+x^2} = \int \frac{du}{u} = \ln|u|\ + C = \ln(1+x^2)\ + C\,.$$

The previous two examples illustrate another general principle:

Theorem 14

Let $f(x)$ be a function of x. Then

$$\int \frac{f'(x)}{f(x)}\ dx = \int \frac{f'(x)dx}{f(x)} = \ln|f(x)|\ + C\,.$$

(This can be seen by using the substitution $u = f(x)$.)

Example

To find $\int 2xe^{x^2}\ dx$ we can make the substitution $u = x^2$. Then $\frac{du}{dx} = 2x$ giving $du = 2x\ dx$ and hence

$$\int 2xe^{x^2}\ dx = \int e^{x^2} 2x\,dx = \int e^u\ du = e^u\ + C = e^{x^2}\ + C\,.$$

Example

Find

$$\int \frac{x}{\sqrt{1+x^2}}\ dx.$$

Solution: Let $u = 1 + x^2$, then $\frac{du}{dx} = 2x$ giving $du = 2x\ dx$ and so

$$\begin{aligned}
\int \frac{x}{\sqrt{1+x^2}}\ dx &= \int \frac{x\ dx}{\sqrt{1+x^2}} = \int \frac{\frac{1}{2}\,du}{\sqrt{u}} \\
&= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\frac{1}{(1/2)}u^{1/2} + C \\
&= u^{1/2} + C = \sqrt{1+x^2} + C\,.
\end{aligned}$$

Warning: When we have to evaluate a definite integral such as $\int_a^b f(x)\ dx$ by using a substitution, we need to take care of the limits of integration. Two approaches are possible:

(1) First solve the indefinite integral working as before (by substitution and then returning to the original variable x), obtaining

$$\int f(x)\ dx = F(x) + C\,,$$

and then evaluate the definite integral as usual

$$\int_{x=a}^{x=b} f(x)\ dx = F(b) - F(a).\ ;$$

(2) Work with the definite integral directly and *change the limits of integration accordingly* when we introduce the new variable u. In the second approach, the formula for a change of variable has the form

$$\boxed{\int_{x=a}^{x=b} f(x)\ dx = \int_{u=\alpha}^{u=\beta} f(x(u))\,\frac{dx}{du}\,du \quad \text{where} \quad \alpha = u(a), \beta = u(b)\,.}$$

(To avoid any confusion, we can write the limits showing the variable explicitly, as $x = a$ and $x = b$ in the original integral, and $u = \alpha$ and $u = \beta$ after the substitution.)

Example

Find

$$\int_1^5 (2x+3)^3 \; dx.$$

Solution: Let $u = 2x+3$. Then $\frac{du}{dx} = 2$, so $dx = \frac{1}{2}\, du$. Finding the indefinite integral and eliminating u in favor of x gives

$$\int (2x+3)^3 \; dx = \frac{1}{2}\int u^3 \; du = \frac{1}{8}u^4 \; + C = \frac{1}{8}(2x+3)^4 \; + C.$$

Evaluating using the limits of x now gives

$$\int_{x=1}^{x=5} (2x+3)^3 \; dx = \left[\frac{1}{8}(2x+3)^4\right]_1^5 = \frac{1}{8}(13^4 - 5^4) = 3492$$

Alternatively, in making the substitution and changing the variable of integration, we must *change the limits* accordingly. If $x = 1$, then $u = 5$, and if $x = 5$, then $u = 13$. The integral becomes

$$\int_{x=1}^{x=5} (2x+3)^3 \; dx = \frac{1}{2}\int_{u=5}^{u=13} u^3 \; du = \left[\frac{1}{8}\,u^4\right]_5^{13} = \frac{1}{8}\,(28561 - 625) = 3492\,.$$

giving the same answer.

Example

Use the substitution $x = 2\sin(v)$ to calculate:

$$\int_0^2 \sqrt{4 - x^2}\ dx\,.$$

Solution: Let $x = 2\sin(v)$, then $dx = 2\cos(v)\,dv$. The new limits are as follows: If $x = 0$, then $v = 0$, and if $x = 2$, then $v = \frac{\pi}{2}$. Hence

$$\begin{aligned}
\int_0^2 \sqrt{4 - x^2}\ dx &= \int_0^{\frac{\pi}{2}} \sqrt{4 - (2\,\sin(v))^2}\, 2\cos(v)\ dv \\
&= \int_0^{\frac{\pi}{2}} \sqrt{4(1 - \ \sin^2(v))}\, 2\cos(v)\ dv \\
&= \int_0^{\frac{\pi}{2}} \sqrt{4\cos^2(v)}\, 2\cos(v)\ dv
\end{aligned}$$

Since $\cos(v)$ is **positive** when $0 \le v \le \frac{\pi}{2}$, we note that $\sqrt{\cos^2(v)} = \cos(v)$:

$$\begin{aligned}
\int_0^2 \sqrt{4 - x^2}\ dx &= \int_0^{\frac{\pi}{2}} (2\cos(v))(2\cos(v))\ dv \\
&= \int_0^{\frac{\pi}{2}} 4\,\cos^2(v)\ dv \\
&= \int_0^{\frac{\pi}{2}} 2(1 + \cos(2v))\ dv \\
&= [2v + \ \sin(2v)]_0^{\frac{\pi}{2}} = (\pi + 0) - (0 + 0) = \pi\,.
\end{aligned}$$

Integration by substitution can be thought of as the “reverse” process to differentiation using the chain rule.

4.3 Integration by parts

The product rule for differentiating the product of functions $u = u(x)$ and $v = v(x)$ is:

$$d(uv) = du \cdot v \,+\, u \cdot dv\,.$$

Integrating both sides, we obtain:

$$\int d(uv) = \int du\, v + \int u\, dv\,.$$

Since integrating the differential, gives back the original function (in this case uv) we obtain:

$$uv = \int du\, v + \int u\, dv\,.$$

Rearranging, we arrive at the following **integration by parts** formula:

$$\boxed{\int u\, dv = uv - \int v\, du\,.}$$

This method is called 'integration by parts' because it requires us to re-write $f(x)\,dx$ in the form $u\,dv$, which has two 'parts'.

- The first part $u(x)$ must be a factor of $f(x)dx$ which we feel able to differentiate (since to apply the formula we have to first find du).
- The second part dv will be what remains of $f(x)dx$ once we have factored out $u(x)$. Note that to apply the formula we must be able to integrate dv.

There is the corresponding integration by parts formula for definite integrals:

$$\boxed{\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du\,.}$$

Here the limits refer to the independent variable x, where $u = u(x)$ and $v = v(x)$.

Integration by parts reduces the original integral to another integral $\int v\, du$, which may or may not be easier to calculate than the original.
The method is successful if this second integral is easier.

Example

Find $\int x\sin(x)\ dx$.
Solution: Let $u = x$ and $dv = \sin(x)\ dx$. Differentiating u, gives $du = dx$. Integrating dv gives $v = -\cos(x)$. Now using the integration by parts formula:

$$\begin{aligned}\int x\sin(x)\ dx &= -x\cos(x) - \int(-\cos(x))\ dx\\ &= -x\cos(x) + \sin(x)\ + C\,.\end{aligned}$$

(The second integral $\int(-\cos(x))\ \ dx$ is easier than the original integral so, in this case, the method works successfully.)

Notice that whilst we already know how to differentiate $\ln(x)$, we have not yet learned how to integrate this function. We can use integration by parts to do so:

Example

Find $\int \ln(x)\ dx$.
Solution: We take $u = \ln(x)$ and $dv = dx$ so that $du = \dfrac{1}{x}dx$ and $v = x$. Applying the integration by parts formula gives

$$\int \ln(x)\ dx = x\ln(x) - \int x\,\frac{dx}{x} = x\ln(x) - \int\ dx = x\ln(x) - x\ + C\,.$$

Example

Find $\int \sqrt{x}\ \ln x\ dx$.
Solution: Setting $u = \ln(x)$ gives $du = \dfrac{dx}{x}$. Then $dv = \sqrt{x}\ dx = x^{\frac{1}{2}}\ dx$, giving $v = \frac{2}{3}x^{\frac{3}{2}}$. Now, integrating by parts

$$\begin{aligned}\int \sqrt{x}\ \ln(x)\ dx &= \frac{2}{3}\,x^{\frac{3}{2}}\ln(x) - \int \frac{2}{3}\,x^{\frac{3}{2}}\,\frac{dx}{x} = \frac{2}{3}\,x^{\frac{3}{2}}\ln(x) - \frac{2}{3}\int x^{\frac{1}{2}}\ \ dx\\ &= \frac{2}{3}\,x^{\frac{3}{2}}\ln(x) - \frac{4}{9}\,x^{\frac{3}{2}}\ + C\\ &= \frac{2}{3}\,x^{\frac{3}{2}}\left(\ln(x) - \frac{2}{3}\right)\ + C\,.\end{aligned}$$

It may be necessary to use integration by parts more than once, as in the following example.

Example

Find $\int x^2e^x \ dx$.
Solution: Let $u = x^2$ and $dv = e^x \ dx$. Then $du = 2x \ dx$ and $v = e^x$. So

$$\int x^2e^x \ dx = x^2e^x - \int 2xe^x \ dx\,.$$

The second integral may be found by applying integration by parts *again*.
Let $u = 2x$ and let $dv = e^x \ dx$ so that $du = 2 \ dx$ and $v = e^x$. Then

$$\int 2xe^x \ dx = 2xe^x - \int 2e^x \ dx \ = 2xe^x - 2e^x + C$$

and so

$$\int x^2e^x \ dx = x^2e^x - 2xe^x + 2e^x \ + C = \left(x^2 - 2x + 2\right) e^x \ + C\,.$$

4.4 Improper integrals

So far in integration we have considered integrals over an intervals $[a, b]$ in the real line. However, in practise, we would like to often understand "asymptotic" behaviour of a phenomenon. This could for example be the long-time asymptotics of stock markets, weather systems, planetary motions, and for this purpose we might need to consider integration over "infinite length" intervals. Improper integrals give an answer to these and they are expressions of the form

$$\int_a^\infty f(x)\,dx, \quad \text{or} \quad \int_{-\infty}^b f(x)\,dx.$$

However, the risk here is that when we increase the size of the integration domain to infinity, the integral's value may explode or more formally "diverge".

Definition (Improper integrals)

Formally we will define the **improper integrals** of $f(x)$ as follows:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

provided that these limits exist. If the limits do not exist (e.g. when they grow to infinity), we say that the improper integrals **diverge**.

Let us first consider a typical example of an improper integral:

Example

Find the improper integral

$$\int_1^\infty x^{-2}\,dx.$$

Solution: Fix a real number $b > 0$. The antiderivative of x^{-2} if $-x^{-1}$ so

$$\int_1^b x^{-2}\,dx = \Big[-x^{-1}\Big]_{x=1}^{x=b} = -b^{-1} - (-1^{-1}) = 1 - \frac{1}{b}.$$

Now if we increase $b \to \infty$, we see that

$$1 - \frac{1}{b} \to 1.$$

Therefore, the improper integral exists and

$$\int_1^\infty x^{-2}\,dx = 1.$$

Now what about a situation where the improper integral diverges

Example

Show that the improper integral

$$\int_1^\infty x\,dx$$

diverges.

Solution: We need to show that

$$\int_1^b x\,dx$$

does not converge when $b \to \infty$. We see that $\frac{1}{2}x^2$ is the antiderivative of x so

$$\int_1^b x\,dx = \left[\frac{1}{2}x^2\right]_{x=1}^{x=b} = \frac{1}{2}b^2 - \frac{1}{2}1^2 = \frac{b^2-1}{2}.$$

Thus when $b \to \infty$ we see that also

$$\frac{b^2-1}{2} \to \infty.$$

Hence the improper integral

$$\int_1^b x\,dx$$

does not exist.

Example

Show that the improper integral

$$\int_{-\infty}^{0} e^x \, dx = 1.$$

Solution: Fix a real number $a < 0$. The antiderivative of e^x if e^x so

$$\int_a^0 e^x \, dx = \Big[e^x\Big]_{x=a}^{x=0} = e^0 - e^a = 1 - e^a.$$

Thus when $a \to -\infty$ we see that also

$$1 - e^a \to 1.$$

Hence the improper integral

$$\int_{-\infty}^{0} e^x \, dx = 1.$$

does not exist.

Example

Show that the improper integral

$$\int_1^{\infty} x^{-1} \, dx$$

diverges.

Solution: Fix a real number $b > 0$. The antiderivative of x^{-1} if $\ln x$ when $x \geq 1$ so

$$\int_1^b x^{-1} \, dx = \Big[\ln x\Big]_{x=1}^{x=b} = \ln b - \ln 1 = \ln b.$$

Thus when $b \to \infty$ we see that also

$$\ln b \to \infty.$$

Hence the improper integral

$$\int_1^{\infty} x^{-1} \, dx$$

diverges.

Chapter 5

Rational functions and Partial Fractions

Revision from last semester..

Fractions:

$$\frac{p}{q} + \frac{n}{m} = \frac{pm + nq}{qm}$$

$$\frac{p}{q} - \frac{n}{m} = \frac{pm - nq}{qm}$$

$$\frac{p}{q} \times \frac{n}{m} = \frac{pn}{qm}$$

$$\frac{p}{q} \div \frac{n}{m} = \frac{pm}{qn}$$

The quadratic formula:
The quadratic equation

$$ax^2 + bx + c = 0,$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5.1 Polynomials and long division

Definition (Polynomial)

A **polynomial** $P(x)$ is a function of x that may be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer and $a_n, a_{n-1}, ..., a_1, a_0$ are given real numbers called **coefficients**.

The **degree** of the polynomial, denoted $\deg(P)$, is the exponent of the highest power of x occurring. If $a_n \neq 0$ in the above expression, then $\deg(P) = n$.

We say that $a_n x^n$ is the **leading term** of this polynomial and that a_0 is the **constant term** of this polynomial.

Example

We give a few examples of these definitions below:

Polynomial	Degree	Leading term	Constant term
$4x^3 + 3x^2 - 2$	3	$4x^3$	-2
$(4x+1)(x+3)$	2	$4x^2$	3
$17x^2 + \frac{1}{2}x^6$	6	$\frac{1}{2}x^6$	0
3	0	3	3

Definition (Rational function)

A **rational function** $R(x)$ is the ratio of two polynomials $N(x)$, $D(x)$

$$R(x) = \frac{N(x)}{D(x)}$$

$R(x)$ is called **proper** if $\deg(N) < \deg(D)$ and **improper** if $\deg(N) \geq \deg(D)$.

Example

Rational function	Proper or Improper?
$\dfrac{x+1}{x^2+5x+6}$	Proper, since $\deg(x+1) < \deg(x^2+5x+6)$.
$\dfrac{x^3+x^2+1}{(x-1)(x-1)}$	Improper, since $\deg(x^3+x^2+1) \geq \deg((x-1)(x-1))$
$\dfrac{x^5-1}{x^5+1}$	Improper, since $\deg(x^5-1) \geq \deg(x^5+1)$

Theorem 15

An improper rational function, $R(x)$, can always be expressed as the sum of a polynomial $Q(x)$ and a proper rational function:

$$\frac{N(x)}{D(x)} = Q(x) + \frac{M(x)}{D(x)}$$

where $\deg(M) < \deg(D)$ and $\deg(Q) = \deg(N) - \deg(D)$

The above result tells us that there will be polynomials $Q(x)$ and $M(x)$ which satisfy this equation, but it does not tell us how to find them. (More on that later!)

Exercise

By multiplying out, check that

$$\frac{x^6-2x^4+x^2-2}{x^2-x-2} = x^4+x^3+x^2+3x+6+\frac{12x+10}{x^2-x-2}$$

Example

It is easy to see that

$$\frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}$$

Theorem 16 (A version of the previous one)

For every two polynomials, $N(x)$ and $D(x)$, it is possible to write

$$N(x) = Q(x)D(x) + M(x)$$

where $Q(x)$ and $D(x)$ are polynomials and $\deg(M) < \deg(D)$.

This is called "division with remainder". The polynomial $Q(x)$ is called the **quotient** and the polynomial $M(x)$ is called the **remainder**. If $M(x) = 0$, that is if there is no remainder, then $N(x) = Q(x)D(x)$; in this case we say that $D(x)$ is a factor of $N(x)$. If the degree of $D(x)$ exceeds the degree of $N(x)$, then we may write $N(x) = 0 \cdot D(x) + N(x)$, so that in this case $N(x)$ itself is the "remainder".

An important special case occurs when the divisor D is linear, $D(x) = x - a$. Then

$$N(x) = Q(x)(x-a) + M$$

where M is a polynomial of degree less than 1, or in other words, a constant.

Theorem 17 (The Remainder Theorem)

For any polynomial $N(x)$, the remainder after division by $x - a$ is the value of $N(x)$ at a:

$$N(x) = Q(x)(x-a) + N(a)\,.$$

Proof

By the previous remark we know that $N(x) = Q(x)(x-a) + M$ where M is a constant. Substituting $x = a$ into this equation yields $N(a) = Q(a) \cdot 0 + M = M$. □

Corollary 18

The linear polynomial $x-a$ is a factor of the polynomial $N(x)$ if and only if $N(a) = 0$.

The practical process of finding the polynomials $Q(x)$ and $M(x)$ is by **polynomial long division**, which is similar to usual long division of numbers. This is best illustrated by means of some examples.

Example

Use polynomial long division to write the following improper rational function as the sum of a polynomial and a proper rational function:

$$\frac{N(x)}{D(x)} = \frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2}.$$

Solution: We perform polynomial long division:

$$\begin{array}{rl} & x - 2 \qquad\qquad\qquad = Q(x) \text{ quotient} \\ x^3 + 4x^2 + 5x + 2 \,\big) & x^4 + 2x^3 - 2x^2 - x + 4 \\ & x^4 + 4x^3 + 5x^2 + 2x \\ & \quad -2x^3 - 7x^2 - 3x + 4 \\ & \quad -2x^3 - 8x^2 - 10x - 4 \\ & \qquad\qquad x^2 + 7x + 8 \qquad = M(x) \text{ remainder} \end{array}$$

Hence

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2} = x - 2 + \frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2}.$$

Notice that $\dfrac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2}$ is proper, as required.

Example

Show that the cubic polynomial

$$P(x) = x^3 - 3x^2 + 7x - 5$$

is divisible by $x - 1$ and factorise it.

Solution: Substituting $x = 1$ into $P(x)$ gives $P(1) = 1 - 3 + 7 - 5 = 0$, therefore $x - 1$ is a factor of $P(x)$. Now using polynomial long division we obtain

$$\begin{array}{rl} & x^2 - 2x + 5 \qquad\qquad = Q(x) \text{ quotient} \\ x - 1 \big) & x^3 - 3x^2 + 7x - 5 \\ & x^3 - x^2 \\ & \quad -2x^2 + 7x - 5 \\ & \quad -2x^2 + 2x \\ & \qquad\qquad 5x - 5 \\ & \qquad\qquad 5x - 5 \\ & \qquad\qquad\quad 0 \qquad\qquad = M(x) \text{ remainder} \end{array}$$

and hence $P(x) = (x - 1)(x^2 - 2x + 5)$. Since $x^2 - 2x + 5 = (x - 1)^2 + 4 > 0$ this quadratic polynomial does not have real roots, and so it does not factorise further.

5.2 Introducing partial fractions

Two rational functions can be added together to give another rational function. This involves bringing them to a common denominator, the simplest method being cross-multiplication.

Example

$$\frac{1}{x-1}+\frac{2}{x+2}=\frac{1(x+2)}{(x-1)(x+2)}+\frac{2(x-1)}{(x+2)(x-1)}=\frac{1(x+2)+2(x-1)}{(x-1)(x+2)}=\frac{3x}{x^2+x-2}\,.$$

Note that the denominator of the sum factorises as the product of the original denominators. It is sometimes necessary (e.g., for integration) to carry out the reverse procedure, i.e., to split a rational function into the sum of two (or more) 'simpler' ones. To begin with, we note that factorising the denominator may give us a clue as to how to proceed.

Example

Consider the rational function

$$\frac{x+7}{x^2-x-6}\,.$$

By factorising the denominator we see that $x^2-x-6=(x-3)(x+2)$. This factorisation suggests that it may be possible to write our rational function as the sum of a proper rational function with denominator $x-3$ and a proper rational function with denominator $x+2$. That is,

$$\frac{x+7}{x^2-x-6}=\frac{A}{x+2}+\frac{B}{x-3}\,,$$

where A and B are constants (or in other words, polynomials of degree 1) to be determined. We can check (by cross-multiplying) that setting $A=-1$ and $B=2$ gives a solution to the above equation. Indeed,

$$\frac{-1}{x+2}+\frac{2}{x-3}=\frac{-(x-3)+2(x+2)}{(x+2)(x-3)}=\frac{-x+3+2x+4}{(x+2)(x-3)}=\frac{x+7}{(x+2)(x-3)}\,.$$

So the desired decomposition is:

$$\frac{x+7}{x^2-x-6}=\frac{-1}{x+2}+\frac{2}{x-3}\,.$$

Notice that the right hand side can be integrated!

In order to decompose a proper rational function into a sum of the 'simplest possible' proper rational functions, we begin by factorising the denominator as much as possible.

Definition

An **irreducible** polynomial is a polynomial $P(x)$ that does not have any non-trivial factors of degree d where $1 \leq d < \deg(P)$.

Essentially, an irreducible polynomial is one which cannot be factorised further. Obviously any linear (degree 1) polynomial $ax + b$ is irreducible. A quadratic (degree 2) polynomial $ax^2 + bx + c$ is irreducible if it has no linear (degree 1) factors, or in other words, if the equation $ax^2 + bx + c = 0$ has no real solutions.

Example

The quadratic polynomial $x^2 + 1$ is irreducible.

Using the quadratic formula, we see that there are no real solutions to the equation $x^2 + 1 = 0$:

$$\frac{-0 \pm \sqrt{0^2 - 4}}{2} = \pm i$$

(Here i denotes the square root of -1, which is not a real number.)

Theorem 19

Every polynomial can be written as a product of linear factors ($ax + b$, with $a \neq 0$) and irreducible quadratic factors ($ax^2 + bx + c$, with $a \neq 0$ and $b^2 - 4ac < 0$).

Example

Note that irreducible factors such as linear factors or irreducible quadratics may repeat in the factorisation of a polynomial:

$$P(x) = (x - 1)^2(x + 5)(x^2 - x + 9)^3 \,.$$

Definition

Expressing a rational function $\dfrac{N(x)}{D(x)}$ as the sum of rational functions

$$\frac{N(x)}{D(x)} = \frac{N_1(x)}{D_1(x)} + \ldots + \frac{N_k(x)}{D_k(x)}$$

whose denominators $D_1(x)$, …, $D_k(x)$ are irreducible polynomials (or their powers) is called decomposition into **partial fractions**.

The **"partial fractions"** are the terms in this sum.

5.3 The method of partial fractions

To split a given rational function into partial fractions:

Step 1: Use long division to obtain a proper rational function.
Step 2: Factorize the denominator as much as possible.
Step 3: Identify the type of partial fractions involved.
Step 4: Multiply out.
Step 5: Determine the constants.

Steps 1 and 2 have already been discussed in the previous sections. To identify the type of partial fractions involved, we look at the factorisation of the denominator.

Denominator contains	Partial fraction decomposition contains
Exactly one irreducible factor $ax+b$	$\frac{A}{ax+b}$, where A is a constant to be determined.
Exactly n irreducible factors $ax+b$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$, where $A_1, A_2, \ldots, A_n$ are constants to be determined.
Exactly one irreducible factor ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$, where A and B are a constants to be determined.
Exactly n irreducible factors ax^2+bx+c	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$, where $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are constants to be determined.

Example

Let $f(x)$ be any polynomial such that each of the given rational functions is proper. Then we have the following expansions into partial fractions:

$$\begin{aligned}
\frac{f(x)}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2}\,, \\
\frac{f(x)}{(x+1)^3(x-2)} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-2}\,, \\
\frac{f(x)}{(x^2+1)(x-2)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x-2}\,, \\
\frac{f(x)}{(x^2+1)^2(x-2)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-2}\,.
\end{aligned}$$

Example

Split the rational function $\frac{5x-1}{x^2-x-2}$ into partial fractions.

Step 1: Note that we already have a proper rational function, so we do not need to perform long division.

Step 2: Factorising the denominator gives

$$\frac{5x-1}{x^2-x-2} = \frac{5x-1}{(x+1)(x-2)}\,,$$

Step 3: Since the denominator has two distinct linear factors, we see that

$$\frac{5x-1}{x^2-x-2} = \frac{5x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}\,,$$

Step 4: Multiplying both sides of this expression by x^2-x-2 yields

$$5x-1 = A(x-2) + B(x+1)\,.$$

Step 5: By setting $x=-1$, we see that $-6=-3A$ and hence $A=2$.
By setting $x=2$, we see that $9=3B$ and hence $B=3$.

Conclusion:

$$\frac{5x-1}{x^2-x-2} = \frac{2}{x+1} + \frac{3}{x-2}.$$

Notice that you can check your answer by combining the partial fractions you have found back into a single rational function:

$$\frac{2}{x+1} + \frac{3}{x-2} = \frac{2(x-2)+3(x+1)}{(x+1)(x-2)} = \frac{5x-1}{(x+1)(x-2)}.$$

Example

Split the rational function $\dfrac{1}{x^2(x-1)}$ into partial fractions.

Steps 1 and 2: Since we are given a proper rational function whose denominator has already been factorised as much as possible, there is nothing to do.

Step 3: Since the denominator factorises into three linear factors (one is repeated!) we have

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\,.$$

Step 4: Multiplying both sides by $x^2(x-1)$, we obtain $Ax(x-1)+B(x-1)+Cx^2 = 1$.

Step 5: By setting $x = 0$, we see that $-B = 1$ or in other words $B = -1$.

By setting $x = 1$, we see that $C = 1$.

Now by comparing the coefficients of x^2, we find that $A + C = 0$, so $A = -1$.

Conclusion:

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\,.$$

Example

Split the improper rational function

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2}$$

into partial fractions.
Step 1: Using polynomial long division (see calculation in section 2.1) we find that

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2} = (x - 2) \ + \frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2}\,.$$

Step 2: To factorise the cubic denominator $D(x) = x^3 + 4x^2 + 5x + 2$, we begin by trying to spot a linear factor by checking the possible factors of the constant term. Since $D(-1) = 0$, we note that by the remainder theorem $x + 1$ is a factor. Hence

$$\begin{aligned} x^3 + 4x^2 + 5x + 2 &= (x+1)(x^2 + ax + b) \\ &= (x+1)(x^2 + 3x + 2) \\ &= (x+1)(x+1)(x+2) \\ &= (x+1)^2(x+2)\,. \end{aligned}$$

Step 3: Using the factorisation of the denominator we find

$$\frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2} = \frac{x^2 + 7x + 8}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}\,.$$

Step 4: Multiplying both sides of this last equation by $x^3 + 4x^2 + 5x + 2$ gives

$$\begin{aligned} x^2 + 7x + 8 &= \frac{A(x+1)^2(x+2)}{(x+1)} + \frac{B(x+1)^2(x+2)}{(x+1)^2} + \frac{C(x+1)^2(x+2)}{(x+2)} \\ &= A(x+1)(x+2) + B(x+2) + C(x+1)^2\,. \end{aligned}$$

Step 5: Setting $x = -1$ gives $2 = B$. Setting $x = -2$ gives $-2 = C$.
Comparing the coefficients of x^2 gives $1 = A + C$, and hence $A = 3$.
Thus

$$\frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2} = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

Conclusion:

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2} = x - 2 \ + \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

Example

Split the rational function $\dfrac{1}{(x+1)^2(x^2+1)}$ into partial fractions.

Steps 1 and 2: Since we have been given a proper rational function whose denominator has already been factorised as much as possible, there is nothing to do.

Step 3: Using the factorisation of the denominator we obtain

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}\,.$$

Step 4: By multiplying both sides by $(x+1)^2(x^2+1)$ we arrive at

$$\begin{aligned} 1 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \\ &= A(x^3+x^2+x+1) + B(x^2+1) + C(x^3+2x^2+x) + D(x^2+2x+1). \end{aligned}$$

Step 5: Collecting terms with the same power of x gives

$$1 = (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + A+B+D\,.$$

By comparing the coefficients on each side of this equation we obtain a system of four linear equations for four variables A, B, C, D:

$$\begin{aligned} &\text{coefficient of } x^3: && A+C && =0 \\ &\text{coefficient of } x^2: && A+B+2C+D && =0 \\ &\text{coefficient of } x^1: && A+C+2D && =0 \\ &\text{coefficient of } x^0: && A+B+D && =1 \end{aligned}$$

By subtracting the first equations from the others, we can eliminate A from them:

$$\begin{aligned} A+C &= 0 \\ B+C+D &= 0 \\ 2D &= 0 \\ B-C+D &= 1 \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} A+C &= 0 \\ B+C &= 0 \\ B-C &= 1 \\ D &= 0 \end{aligned}$$

By back substitution, we find $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$, and $D = 0$.

Conclusion:

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}\,.$$

Example

Split the rational function $\dfrac{x^2+1}{x^3+7x}$ into partial fractions.
Step 1: Note that we already have a proper rational function, so we do not need to perform long division.
Step 2: Factorising the denominator gives

$$\frac{x^2+1}{x^3+7x} = \frac{x^2+1}{x(x^2+7)}\,,$$

Step 3: Since the denominator has one linear factor and one irreducible quadratic factor, we see that

$$\frac{x^2+1}{x^3+7x} = \frac{x^2+1}{x(x^2+7)} = \frac{A}{x} + \frac{Bx+C}{(x^2+7)}\,.$$

Step 4: Multiplying both sides of this expression by the denominator x^3+7x yields

$$x^2+1 = A(x^2+7) + (Bx+C)x\,.$$

Step 5: Comparing the constant terms, we have $1 = 7A$, so $A = \frac{1}{7}$.
Comparing coefficients of x^2 gives $1 = A + B$. Hence $B = \frac{6}{7}$.
Comparing coefficients of x gives $0 = C$, i.e., $C = 0$.
Conclusion:

$$\frac{x^2+1}{x^3+7x} = \frac{1/7}{x} + \frac{6x/7}{x^2+7} = \frac{1}{7x} + \frac{6x}{7(x^2+7)}\,.$$

5.4 Integration by partial fractions

Fractions with a quadratic denominator

Consider an integral of the following form:

$$\int \frac{dx}{ax^2 + bx + c}$$

assuming that $a \neq 0$ (otherwise the denominator is not quadratic). To solve the integral, we transform the quadratic polynomial $f(x) = ax^2 + bx + c$ as follows:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right].$$

(This is called **completing the square**.) Now there are two cases:

1. $b^2 - 4ac \geq 0$, so there are real roots, $f(x) = a(x - x_1)(x - x_2)$; or
2. $b^2 - 4ac < 0$, so the quadratic is irreducible.

In the first case, the fraction we started with decomposes into fractions with linear denominators, which we know how to integrate; the answer will contain logarithms. In the second case, by a substitution, the integral can be transformed into a table integral and the answer will involve $\tan^{-1}$. We consider some examples to illustrate this point.

Example

Solve

$$\int \frac{dx}{x^2 + x - 2}.$$

Solution: Here we see that

$$\begin{aligned} x^2 + x - 2 &= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4} = \left(x + \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 \\ &= \left(\left(x + \frac{1}{2}\right) - \frac{3}{2}\right)\left(\left(x + \frac{1}{2}\right) + \frac{3}{2}\right) = (x - 1)(x + 2) \end{aligned}$$

(so completing the square gives the roots of the quadratic).
Using the method of partial fractions we now find that:

$$\frac{1}{x^2 + x - 2} = \frac{1}{3}\left(\frac{1}{x - 1} - \frac{1}{x + 2}\right),$$

and hence

$$\begin{aligned} \int \frac{dx}{x^2 + x - 2} &= \frac{1}{3}\left(\int \frac{dx}{x - 1} - \int \frac{dx}{x + 2}\right) \\ &= \frac{1}{3}\ln|x - 1| - \frac{1}{3}\ln|x + 2| + C = \frac{1}{3}\ln\left|\frac{x - 1}{x + 2}\right| + C. \end{aligned}$$

Example

Find
$$\int \frac{dx}{x^2+2x+2}\,.$$
Solution: Completing the square: $x^2+2x+2 = (x+1)^2+2-1 = (x+1)^2+1$. This time we find that the denominator is an irreducible quadratic. Now making the substitution $u = x+1$, gives $du = dx$ and so the integral becomes:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \int \frac{du}{1+u^2} = \tan^{-1}(u) \ +C = \tan^{-1}(x+1) \ +C\,.$$

Example

Find
$$\int \frac{5\,dx}{x^2+4x+13}\,.$$
Solution: Completing the square, $x^2+4x+13 = (x+2)^2+9 = (x+2)^2+3^2$ allows us to rewrite the integral as follows:

$$\int \frac{5\,dx}{x^2+4x+13} = \int \frac{5\,dx}{(x+2)^2+9} = \frac{5}{9}\int \frac{dx}{\left(\frac{(x+2)}{3}\right)^2+1}\,.$$

By making the substitution $u = \frac{1}{3}(x+2)$ the denominator can be reduced to (u^2+1). We have $du = \frac{1}{3}dx$ or $dx = 3du$ and so

$$\begin{aligned}
\frac{5}{9}\int \frac{dx}{\left(\frac{(x+2)}{3}\right)^2+1} &= \frac{5}{9}\int \frac{\cdot 3\,du}{u^2+1} \\
&= \frac{5}{3}\int \frac{du}{u^2+1} \\
&= \frac{5}{3}\tan^{-1}(u) \ +C \\
&= \frac{5}{3}\tan^{-1}\left(\frac{1}{3}(x+2)\right) \ +C\,.
\end{aligned}$$

Example

Find

$$\int \frac{2x+9}{x^2+4x+13}\ dx$$

Solution: In this case the numerator depends on x. Notice that: $d(x^2+4x+13) = 2x+4$ and we may write

$$\int \frac{2x+9}{x^2+4x+13}\ dx = \int \frac{2x+4}{x^2+4x+13}\ dx + \int \frac{5}{x^2+4x+13}\ dx$$

Now, the first integral gives rise to a ln term, since the numerator is the derivative of the denominator, while the second integral is as in the previous example. Thus,

$$\int \frac{2x+9}{x^2+4x+13}\ dx = \ln\left(x^2+4x+13\right) + \frac{5}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\ + C\,.$$

In general when we integrate a proper rational function with an irreducible quadratic as the denominator, we should expect a $\ln$ term plus a $\tan^{-1}$ term.

Integration using partial fractions

The preceding examples hint that any rational function can be integrated by splitting it into partial fractions.

Example

Find the integral

$$\int \frac{5x-1}{x^2-x-2}\ dx\,.$$

Solution: Using partial fractions we find that

$$\frac{5x-1}{x^2-x-2} = \frac{2}{x+1} + \frac{3}{x-2}\,.$$

Integrating then gives

$$\int \frac{5x-1}{x^2-x-2}\ dx = \int \frac{2\ dx}{x+1} + \int \frac{3\ dx}{x-2} = 2\ \ln|x+1| + 3\ \ln|x-2|\ + C\,.$$

Example

Find the integral

$$\int \frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2}\ dx\,.$$

Solution: We met this rational function before (in chapter 2) and obtained that

$$\frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2} = x - 2\ + \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2}\,.$$

Altogether:

$$\begin{aligned}
\int \frac{x^4 + 2x^3 - 2x^2 - x + 4}{x^3 + 4x^2 + 5x + 2}\ dx &= \int \left(x - 2 + \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x+2} \right) dx \\
&= \int (x-2)\ dx + 3 \int \frac{dx}{(x+1)} + 2 \int (x+1)^{-2}\ dx \\
&\quad -2 \int \frac{dx}{(x+2)} \\
&= \frac{1}{2}x^2 - 2x + 3\ \ln|x+1| - \frac{2}{(x+1)} \\
&\quad -2\ \ln|x+2|\ + C\,.
\end{aligned}$$

Recall that when we expand a rational function into partial fractions we may also obtain powers of irreducible quadratics in the denominator. The general strategy in this case is to transform the function we want to integrate so to reduce the power of the denominator. Integration by parts can be useful here...

Example

Consider the integral

$$\int \frac{dx}{1+x^2}$$

(which as we know is $\tan^{-1}(x)+C$) and perform integration by parts taking $u = \frac{1}{x^2+1}$ and $dv = dx$. We obtain

$$\begin{aligned}
\int \frac{1}{1+x^2}\ dx &= \frac{x}{1+x^2} - \int x\ d\left(\frac{1}{1+x^2}\right) \\
&= \frac{x}{1+x^2} + \int x\ \frac{2x\ dx}{(1+x^2)^2} \\
&= \frac{x}{1+x^2} + \int \frac{2x^2\ dx}{(1+x^2)^2} \\
&= \frac{x}{1+x^2} + \int \frac{(2x^2+2-2)\ dx}{(1+x^2)^2} \\
&= \frac{x}{1+x^2} + 2\int \frac{dx}{1+x^2} - 2\int \frac{dx}{(1+x^2)^2}\,,
\end{aligned}$$

which gives a relation between the known integral $\int \frac{dx}{1+x^2}$ and the integral $\int \frac{dx}{(1+x^2)^2}$ that we want to find:

$$2\int \frac{dx}{(1+x^2)^2} = \frac{x}{1+x^2} + \int \frac{dx}{1+x^2}\,.$$

This gives the answer:

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\tan^{-1}x\ + C.$$