# 0C2 Exercise Sheet 5 - Rational Functions
# SOLUTIONS

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**1.** Use polynomial long division to divide the polynomial $P(x) = x^4 - 3x^3 + 3x - 1$ by

$$\text{(i) } D(x) = x - 2, \quad \text{(ii) } D(x) = x + 1, \quad \text{(iii) } D(x) = x^2 + 7x - 9$$

(In each case express your answer in the form $\frac{P}{D} = Q + \frac{M}{D}$.)

**Solution.** (i)

$$\begin{array}{rrrrrr}
 & & x^3 & -x^2 & -2x & -1 \\ \cline{2-6}
x-2 \big) & x^4 & -3x^3 & & +3x & -1 \\
 & -x^4 & +2x^3 & & & \\ \cline{2-3}
 & & -x^3 & & & \\
 & & x^3 & -2x^2 & & \\ \cline{3-4}
 & & & -2x^2 & +3x & \\
 & & & 2x^2 & -4x & \\ \cline{4-5}
 & & & & -x & -1 \\
 & & & & x & -2 \\ \cline{5-6}
 & & & & & -3
\end{array}$$

Hence

$$\frac{x^4 - 3x^3 + 3x - 1}{x - 2} = x^3 - x^2 - 2x - 1 + \frac{-3}{x - 2}$$

(ii)

$$\begin{array}{rrrrrr}
 & & x^3 & -4x^2 & +4x & -1 \\ \cline{2-6}
x+1 \big) & x^4 & -3x^3 & & +3x & -1 \\
 & -x^4 & -x^3 & & & \\ \cline{2-3}
 & & -4x^3 & & & \\
 & & 4x^3 & +4x^2 & & \\ \cline{3-4}
 & & & 4x^2 & +3x & \\
 & & & -4x^2 & -4x & \\ \cline{4-5}
 & & & & -x & -1 \\
 & & & & x & +1 \\ \cline{5-6}
 & & & & & 0
\end{array}$$

Hence

$$\frac{x^4 - 3x^3 + 3x - 1}{x + 1} = x^3 - 4x^2 + 4x - 1$$

(iii)

$$\begin{array}{rrrrrr}
 & & & x^2 & -10x & +79 \\ \cline{2-6}
x^2 + 7x - 9 \big) & x^4 & -3x^3 & & +3x & -1 \\
 & -x^4 & -7x^3 & +9x^2 & & \\ \cline{2-4}
 & & -10x^3 & +9x^2 & +3x & \\
 & & 10x^3 & +70x^2 & -90x & \\ \cline{3-5}
 & & & 79x^2 & -87x & -1 \\
 & & & -79x^2 & -553x & +711 \\ \cline{4-6}
 & & & & -640x & +710
\end{array}$$

Hence

$$\frac{x^4 - 3x^3 + 3x - 1}{x^2 + 7x - 9} = x^2 - 10x + 79 + \frac{-640x + 710}{x^2 + 7x - 9}$$

**2.** Express the following as partial fractions:

$$\frac{2x - 1}{(x-1)(x+5)}$$

**Solution.** We need to find $A$ and $B$ such that

$$\frac{2x - 1}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}.$$

Multiplying out:

$$2x - 1 = \left(\frac{A}{x-1} + \frac{B}{x+5}\right)(x-1)(x+5)$$

Thus

$$2x - 1 = A(x+5) + B(x-1).$$

Setting $x = -5$ gives us

$$2(-5) - 1 = A(-5+5) + B(-5-1)$$

Hence

$$-11 = -6B$$

so $B = \frac{11}{6}$. Moreover, setting $x = 1$ gives us

$$2 \cdot 1 - 1 = A(1+5) + B(1-1)$$

so

$$1 = 6A$$

giving us $A = \frac{1}{6}$. Hence the final answer is

$$\frac{2x - 1}{(x-1)(x+5)} = \frac{\frac{1}{6}}{x-1} + \frac{\frac{11}{6}}{x+5} = \frac{1}{6(x-1)} + \frac{11}{6(x+5)}.$$

**3.** Use your answer to question 2 to find the following indefinite integral:

$$\int \frac{2x-1}{(x-1)(x+5)}\,dx$$

**Solution.** The integral is

$$\int \frac{2x-1}{(x-1)(x+5)}\,dx = \int \frac{1}{6(x-1)} + \frac{11}{6(x+5)}\,dx = \frac{1}{6}\int \frac{dx}{x-1} + \frac{11}{6}\int \frac{dx}{x+5},$$

which equals to

$$\frac{1}{6}\ln|x-1| + \frac{11}{6}\ln|x+5| + C.$$

**4.** Express the following as partial fractions:

$$\frac{x^2+x+1}{x^2+3x+2}$$

**Solution.** Notice that the degree of the numerator $P(x) = x^2+x+1$ and denominator $D(x) = x^2+3x+2$ agree (they both are 2), so this is an improper rational function. Hence we first need to use polynomial long division to write it as a sum of a polynomial $Q(x)$ and a proper rational function $\frac{M(x)}{D(x)}$ (i.e. $\frac{P(x)}{D(x)} = Q(x) + \frac{M(x)}{D(x)}$, where the degree of $M(x)$ is strictly smaller than the degree of $D(x) = x^2+3x+2$).

Polynomial long division gives us

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+3x+2 & \;\; x^2 + x + 1 \\ & -x^2 - 3x - 2 \\ \hline & \quad -2x - 1 \end{array}$$

Hence $Q(x) = 1$ and $M(x) = -2x-1$, that is,

$$\frac{x^2+x+1}{x^2+3x+2} = 1 + \frac{-2x-1}{x^2+3x+2} = 1 - \frac{2x+1}{x^2+3x+2}.$$

Let us now decompose the proper rational function

$$\frac{2x+1}{x^2+3x+2}$$

into partial fractions. First of all, we see that the roots of $x^2+3x+2$ are $x=-2$ and $x=-1$. Thus

$$\frac{2x+1}{x^2+3x+2} = \frac{2x+1}{(x+2)(x+1)}.$$

Let us now find $A$ and $B$ such that

$$\frac{2x+1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}.$$

Multiplying out gives us

$$2x+1 = \Big(\frac{A}{x+2} + \frac{B}{x+1}\Big)(x+2)(x+1)$$

giving us

$$2x+1 = A(x+1) + B(x+2).$$

Set now $x=-1$. This gives us $-2+1 = A(-1+1)+B(-1+2)$, so $B=-1$.
Set now $x=-2$. This gives us $-4+1 = A(-2+1)+B(-2+2)$, so $A=3$.
Hence we have

$$\frac{2x+1}{x^2+3x+2} = \frac{2x+1}{(x+2)(x+1)} = \frac{3}{x+2} - \frac{1}{x+1}.$$

Putting all together gives us

$$\frac{x^2+x+1}{x^2+3x+2} = 1 - \frac{2x+1}{x^2+3x+2} = 1 - \frac{3}{x+2} + \frac{1}{x+1}.$$

**5.** Use your answer to question 4 to find the following indefinite integral:

$$\int \frac{x^2+x+1}{x^2+3x+2}\,dx$$

**Solution.** The integral is

$$\int \frac{x^2+x+1}{x^2+3x+2}\,dx = \int 1\,dx - 3\int \frac{dx}{x+2} + \int \frac{dx}{x+1} = x - 3\ln|x+2| + \ln|x+1| + C.$$

**6.** Prove

$$\frac{x^4+x^3-2x^2-x+5}{x^3+4x^2+5x+2} = x - 3 - \frac{2}{x+1} + \frac{4}{(x+1)^2} + \frac{7}{x+2}$$

by reducing $\dfrac{x^4+x^3-2x^2-x+5}{x^3+4x^2+5x+2}$ to proper fractions and then expressing as partial fractions.

**Solution. Step 1:** Polynomial long division gives us:

$$\begin{array}{r|rrrrr}
 & & & & x & -3 \\
\hline
x^3+4x^2+5x+2 & x^4 & +x^3 & -2x^2 & -x & +5 \\
 & -x^4 & -4x^3 & -5x^2 & -2x & \\
\hline
 & & -3x^3 & -7x^2 & -3x & +5 \\
 & & 3x^3 & +12x^2 & +15x & +6 \\
\hline
 & & & 5x^2 & +12x & +11
\end{array}$$

Hence

$$\frac{x^4+x^3-2x^2-x+5}{x^3+4x^2+5x+2} = (x-3) + \frac{5x^2+12x+11}{x^3+4x^2+5x+2}.$$

**Step 2**: To factorise the cubic denominator $D(x) = x^3+4x^2+5x+2$, we begin by trying to spot a linear factor by checking the possible factors of the constant term.
Since $D(-1)=0$, we note that by the remainder theorem $x+1$ is a factor. Hence for some $a, b$ we have

$$x^3+4x^2+5x+2 = (x+1)(x^2+ax+b)$$

Now multiplying out, the right-hand side of this becomes

$$x^3 + ax^2 + bx + x^2 + ax + b = x^3 + (a+1)x^2 + (a+b)x + b$$

so $a = 3$ and $b = 2$. Thus

$$x^3 + 4x^2 + 5x + 2 = (x+1)(x^2 + 3x + 2).$$

By solving the roots of $x^2 + 3x + 2$, that is, $x^2 + 3x + 2 = 0$, which are $x = -1$ and $x = -2$, we see that

$$x^2 + 3x + 2 = (x+1)(x+2)$$

Hence we have proved:

$$x^3 + 4x^2 + 5x + 2 = (x+1)(x+1)(x+2) = (x+1)^2(x+2).$$

**Step 3:** Using the factorisation of the denominator we find

$$\frac{5x^2 + 12x + 11}{x^3 + 4x^2 + 5x + 2} = \frac{5x^2 + 12x + 11}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}\,.$$

**Step 4:** Multiplying both sides of this last equation by $x^3 + 4x^2 + 5x + 2$ gives

$$\begin{aligned} 5x^2 + 12x + 11 &= \frac{A(x+1)^2(x+2)}{(x+1)} + \frac{B(x+1)^2(x+2)}{(x+1)^2} + \frac{C(x+1)^2(x+2)}{(x+2)} \\ &= A(x+1)(x+2) + B(x+2) + C(x+1)^2\,. \end{aligned}$$

**Step 5:** Setting here $x = -1$ gives

$$5 - 12 + 11 = 0 + B + 0,$$

that is $B = 4$. Moreover, setting $x = -2$ gives

$$5 \cdot 4 - 12 \cdot (-2) + 11 = C,$$

that is $C = 7$. Finally, opening up

$$A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

we see that the coefficient of $x^2$ is $A + C$. Hence comparing the coefficients of $x^2$ gives us $5 = A + C$ and as $C = 7$ we obtain $A = -2$.

Thus

$$\frac{x^2 + 7x + 8}{x^3 + 4x^2 + 5x + 2} = -\frac{2}{x+1} + \frac{4}{(x+1)^2} + \frac{7}{x+2}\,.$$

**Conclusion:**

$$\frac{x^4 + x^3 - 2x^2 - x + 5}{x^3 + 4x^2 + 5x + 2} = x - 3 - \frac{2}{x+1} + \frac{4}{(x+1)^2} + \frac{7}{x+2}$$

**7.** Use question 6 to find the indefinite integral $\displaystyle\int \frac{x^4 + x^3 - 2x^2 - x + 5}{x^3 + 4x^2 + 5x + 2}\, dx$.

**Solution.** We have

$$\begin{aligned}
\int \frac{x^4 + x^3 - 2x^2 - x + 5}{x^3 + 4x^2 + 5x + 2}\; dx &= \int \left( x - 3 - \frac{2}{x+1} + \frac{4}{(x+1)^2} + \frac{7}{x+2} \right) dx \\
&= \int (x-3)\; dx - 2 \int \frac{dx}{x+1} + 4 \int (x+1)^{-2}\; dx + 7 \int \frac{dx}{x+2} \\
&= \frac{1}{2}x^2 - 3x - 2\; \ln|x+1| + \frac{4}{x+1} + 7 \ln|x+2|\; + C\,.
\end{aligned}$$