0C2 Exercise Sheet 4 – Further integration SOLUTIONS

---

1. Use the substitution $x = \sin(u)$ and a trigonometric identity to calculate:

$$\int \sqrt{1-x^2}\, dx\,.$$

Solution. Using substitution $x = \sin(u)$, that is, $u = \sin^{-1}(x)$ with $\frac{-\pi}{2} \le u \le \frac{\pi}{2}$ we obtain using $\sin^2 u + \cos^2 u = 1$ that

$$\begin{aligned}\int \sqrt{1-x^2}\, dx &= \int \sqrt{1-\sin^2 u}\cos u\, du \\ &= \int \cos u \cos u\, du \\ &= \int \cos^2 u\, du.\end{aligned}$$

By the trigonometric identity $2\cos^2 u - 1 = \cos(2u)$ we have

$$\int \cos^2 u\, du = \frac{1}{2}\int \cos(2u) + 1\, du = \frac{1}{2}\Big(\frac{1}{2}\sin(2u) + u\Big) + C$$

By the trigonometric identity $\sin(2A) = 2\sin(A)\cos(A)$ we have

$$\frac{1}{2}\Big(\frac{1}{2}\sin(2u) + u\Big) = \frac{1}{2}\sin(u)\cos(u) + \frac{1}{2}u.$$

Substitute $x = \sin(u)$, which means $\cos(u) = \sqrt{1-x^2}$ and $u = \sin^{-1}(x)$ gives us

$$\frac{1}{2}\sin(u)\cos(u) + \frac{1}{2}u = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x).$$

Thus the final answer is

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + C.$$

2. Use the formula for integration by parts ($\int u\, dv = uv - \int v\, du$) setting $u = x$ and $dv = e^x\, dx$ to calculate

$$\int xe^x\, dx\,.$$

Solution. We have

$$du = 1\, dx, \quad v = e^x$$

Hence

$$\int xe^x\, dx = \int u\, dv = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - e^x + C.$$

3. Use the formula for integration by parts with $u = x^2$ and $dv = xe^{x^2}\,dx$ to calculate

$$\int x^3 e^{x^2}\,dx\,.$$

Solution. We have

$$du = 2x\,dx, \quad v = \frac{1}{2}e^{x^2}$$

Hence

$$\int x^3 e^{x^2}\,dx = \int u\,dv = uv - \int v\,du = x^2 \cdot \frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}\,2x\,dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

Note: if you did not spot that $\frac{d}{dx}(\frac{1}{2}e^{x^2}) = xe^{x^2}$ *which gives*

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

we can use substitution instead: letting $w = x^2$ *then* $dw = 2x\,dx$ *so*

$$\int xe^{x^2}\,dx = \int e^w \cdot \frac{1}{2}\,dw = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{x^2} + C.$$

4. Use integration by parts to find the following indefinite integral:

$$\int_1^e x\,\ln(x)\,dx$$

Solution. With $u = \ln(x)$ and $dv = x\,dx$, we have $du = \frac{dx}{x}$ and $v = \frac{1}{2}x^2$. Hence

$$\int x\,\ln(x)\,dx = uv - \int v\,du = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx,$$

which equals to

$$\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

Thus the definite integral is

$$\int_1^e x\,\ln(x)\,dx = \left[\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right]_{x=1}^{x=e} = \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{e^2+1}{4}.$$

5. Find the improper integral

$$\int_1^\infty e^{-x}\,dx$$

Solution. Set $b > 1$. Then

$$\int_1^b e^{-x}\,dx = [-e^{-x}]_{x=1}^{x=b} = -e^{-b} - (-e^{-1}) = e^{-1} - e^{-b} = \frac{1}{e} - \frac{1}{e^b}$$

As $b \to \infty$ we see that

$$\frac{1}{e} - \frac{1}{e^b} \to \frac{1}{e}.$$

Hence the improper integral is

$$\int_1^\infty e^{-x}\,dx = \frac{1}{e}.$$

6. Find the improper integral

$$\int_{-\infty}^{-1} \frac{1}{x^2}\,dx.$$

Solution. Let $a < -1$. Then

$$\int_a^{-1} \frac{1}{x^2}\,dx = \int_a^{-1} x^{-2}\,dx = [-x^{-1}]_{x=a}^{x=-1} = -(-1)^{-1} - (-a^{-1}) = 1 + \frac{1}{a}.$$

As $a \to -\infty$, we see that

$$1 + \frac{1}{a} \to 1.$$

Hence the improper integral is

$$\int_{-\infty}^{-1} \frac{1}{x^2}\,dx = 1.$$