Exercise Sheet 5 – Solutions

1 (i)

$$
\begin{array}{r|l}
 & x^3 - x^2 - 2x - 1 \\
\hline
x-2 & x^4 - 3x^3 + 0x^2 + 3x - 1 \\
 & x^4 - 2x^3 \\
\hline
 & \quad -x^3 + 0x^2 + 3x - 1 \\
 & \quad -x^3 + 2x^2 \\
\hline
 & \qquad -2x^2 + 3x - 1 \\
 & \qquad -2x^2 + 4x \\
\hline
 & \qquad\quad -x - 1 \\
 & \qquad\quad -x + 2 \\
\hline
 & \qquad\qquad -3
\end{array}
$$

$$\frac{x^4 - 3x^3 + 3x - 1}{x-2} = (x^3 - x^2 - 2x - 1) + \left(\frac{-3}{x-2}\right)$$

$$= (x^3 - x^2 - 2x - 1) - \left(\frac{3}{x-2}\right)$$

(ii)

$$
\begin{array}{r|l}
 & x^3 - 4x^2 + 4x - 1 \\
\hline
x+1 & x^4 - 3x^3 + 0x^2 + 3x - 1 \\
 & x^4 + x^3 \\
\hline
 & \quad -4x^3 + 0x^2 + 3x - 1 \\
 & \quad -4x^3 - 4x^2 \\
\hline
 & \qquad 4x^2 + 3x - 1 \\
 & \qquad 4x^2 + 4x \\
\hline
 & \qquad\quad -x - 1 \\
 & \qquad\quad -x - 1 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$\frac{x^4 - 4x^2 + 4x - 1}{x+1} = (x^3 - 4x^2 + 4x - 1) + \left(\frac{0}{x+1}\right)$$

$$= x^3 - 4x^2 + 4x - 1.$$

2. $(x^2-x-2)\left((x^4+x^3+x^2+3x+6)+\left(\dfrac{12x+10}{x^2-x-2}\right)\right)$

$= (x^2-x-2)(x^4+x^3+x^2+3x+6) + (12x+10)$

$= \begin{pmatrix} x^6 + x^5 + x^4 + 3x^3 + 6x^2 \\ -x^5 - x^4 - x^3 - 3x^2 - 6x \\ -2x^4 - 2x^3 - 2x^2 - 6x - 12 \end{pmatrix} + \cancel{12x} + 10$

$= x^6 - 2x^4 + x^2 - 2.$

3(i) We can "spot" that setting $x = 2$ gives

$2\cdot(2^3) + (2^2) - 13(2) + 6 = 16 + 4 - 26 + 6 = 0$

from which it follows that $x-2$ must be a factor.

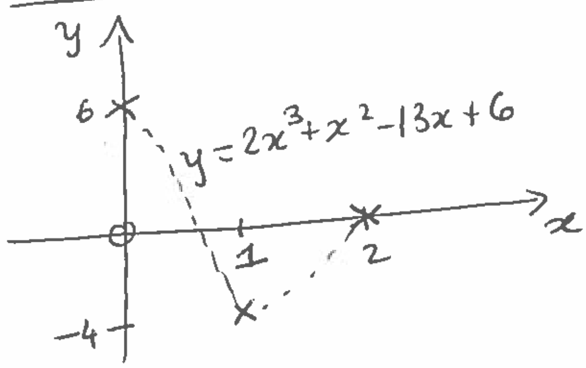
If you didn't spot this straightaway, you could try a few values to see if you can identify a root.

e.g.

x	-2	-1	0	1	2
$2x^3+x^2-13x+6$	20	18	6	-4	0

either get lucky and notice how 2 yields 0

or note that there must be a root between $x=0$ and $x=1$ since the graph goes from being positive to being negative.



Now factorise (can use long division ...)

$$\begin{array}{r} 2x^2 + 5x - 3 \\ x-2\,\overline{)\,2x^3 + x^2 - 13x + 6} \\ 2x^3 - 4x^2 \\ \hline 5x^2 - 13x + 6 \\ 5x^2 - 10x \\ \hline -3x + 6 \\ -3x + 6 \end{array}$$

$2x^3 + x^2 - 13x + 6$

$= (x-2)(2x^2+5x-3)$

$= (x-2)(2x-1)(x+3)$

$x = \frac{1}{2}$ is the root in the picture above!

$x=-3$ gives a third root.

3(i) $x^3 - 2x^2 - 5x + 6$.

Notice that the coefficients sum to 0, so setting $x = 1$ gives

$$(1^3) - 2(1^2) - 5(1) + 6 = 1 - 2 - 5 + 6 = 0.$$

from which we conclude that $x-1$ must be a factor.

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x-1 & x^3 - 2x^2 - 5x + 6 \\ & x^3 - x^2 \\ \hline & \quad -x^2 - 5x + 6 \\ & \quad -x^2 + x \\ \hline & \qquad -6x + 6 \\ & \qquad -6x + 6 \\ \hline & \qquad\qquad 0 \end{array}$$

$$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$$
$$= \underline{\underline{(x-1)(x-3)(x+2)}}$$

(ii) Again we can "spot" that $x = 2$ gives

$$2(2^3) + (2^2) - 8(2) - 4 = 16 + 4 - 16 - 4 = 0$$

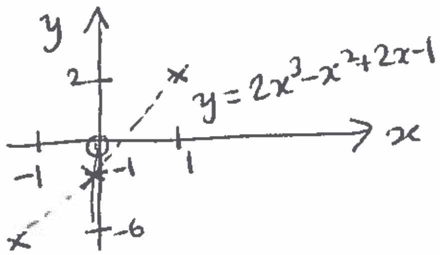
So $x-2$ must be a factor. [Again, if you didn't spot this, try some values.]

$$\begin{array}{r|l} & 2x^2 + 5x + 2 \\ \hline x-2 & 2x^3 + x^2 - 8x - 4 \\ & 2x^3 - 4x^2 \\ \hline & \quad 5x^2 - 8x - 4 \\ & \quad 5x^2 - 10x \\ \hline & \qquad 2x - 4 \\ & \qquad 2x - 4 \\ \hline & \qquad\quad 0 \end{array}$$

$$2x^3 + x^2 - 8x + 4$$
$$= (x-2)(2x^2 + 5x + 2)$$
$$= \underline{\underline{(x-2)(2x+1)(x+2)}}$$

(iv) Try some values...

x	-1	0	1
$2x^3 - x^2 + 2x - 1$	-6	$\boxed{-1}$	$\boxed{2}$



* since the graph goes from a negative value to a positive value, we know that it must cross the x-axis somewhere between $x = 0$ and $x = 1$ ...

3(iv) Try $x = \frac{1}{2}$

$$2\left(\tfrac{1}{2}\right)^3 - \left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right) - 1 = \frac{2}{8} - \frac{1}{4} + 1 - 1 = 0.$$

[Note: If we didn't guess correctly first time, we can continue to use trial + error to attempt to find a root! Everytime we have $p(x_1) > 0$ and $p(x_2) < 0$, we can look for a root between x_1 and x_2.]

This tells us that $x - \frac{1}{2}$ is a factor, or if we prefer, $(2x-1)$ is a factor.

$$\begin{array}{r|l} & x^2 + 1 \\ \hline 2x-1 & 2x^3 - x^2 + 2x - 1 \\ & 2x^3 - x^2 \\ \hline & \quad\quad 2x - 1 \\ & \quad\quad 2x - 1 \\ \hline & \quad\quad\quad 0. \end{array}$$

So

$$2x^3 - x^2 + 2x - 1 = (2x-1)(x^2+1)$$

this is irreducible (ie doesn't factorise further).
since $x^2+1 > 0$ for all values of x.

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4 (i) $$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

$$1 = \left(\frac{A}{x+2} + \frac{B}{x+3}\right) \cdot (x+2)(x+3)$$

$$1 = A(x+3) + B(x+2).$$

Set $x=-3$: $1 = A(-3+3) + B(-3+2)$

$1 = -B$

Set $x=-2$: $1 = A(-2+3) + B(-2+2)$

$1 = A$

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} + \frac{-1}{x+3}$$

$$= \frac{1}{x+2} - \frac{1}{x+3}.$$

4(i) $$\frac{2x-1}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

$$2x-1 = \left(\frac{A}{x-1} + \frac{B}{x+5}\right)(x-1)(x+5)$$

$$2x-1 = A(x+5) + B(x-1).$$

Set $x=-5$: $2(-5)-1 = A(-5+5) + B(-5-1)$

$-11 = -6B$, so $B = \frac{11}{6}$.

Set $x=1$: $2(1)-1 = A(1+5) + B(1-1)$

$1 = 6A$, so $A = \frac{1}{6}$.

$$\frac{2x-1}{(x-1)(x+5)} = \frac{\frac{1}{6}}{x-1} + \frac{\frac{11}{6}}{x+5} = \frac{1}{6(x-1)} + \frac{11}{6(x+5)}$$

(ii) $$\frac{x^2-11x-6}{(2x-1)(x-2)(x+2)} = \frac{A}{2x-1} + \frac{B}{x-2} + \frac{C}{x+2}.$$

$$x^2-11x-6 = \left(\frac{A}{2x-1} + \frac{B}{x-2} + \frac{C}{x+2}\right)\cdot(2x-1)(x-2)(x+2)$$

$$x^2-11x-6 = A(x-2)(x+2) + B(2x-1)(x+2) + C(2x-1)(x-2)$$

Set $x=2$: $2^2-11(2)-6 = A(2-2)(2+2) + B(2(2)-1)(2+2) + C(2(2)-1)(2-2)$

$-24 = 12B$, so $B=-2$.

Set $x=-2$: $(-2)^2-11(-2)-6 = A(-2-2)(-2+2) + B(2(-2)-1)(-2+2) + C(2(-2)-1)(-2-2)$

$20 = 20C$, so $C=1$

Set $x=\frac{1}{2}$: $(\frac{1}{2})^2-11(\frac{1}{2})-6 = A(\frac{1}{2}-2)(\frac{1}{2}+2) + B(2(\frac{1}{2})-1)(\frac{1}{2}+2) + C(2(\frac{1}{2})-1)(\frac{1}{2}+2)$

$\frac{-45}{4} = \frac{-15}{4}A$, so $A=3$.

4(iii)

$$\frac{x^2-11x-6}{(2x-1)(x-2)(x+2)} = \frac{3}{2x-1} - \frac{2}{x-2} + \frac{1}{x+2}.$$

5(i) $\int \frac{1}{(x+2)(x+3)}\,dx = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx.$

$= \int \frac{1}{x+2}\,dx - \int \frac{1}{x+3}\,dx.$

$= \ln|x+2| - \ln|x+3| + C.$

(ii) $\int \frac{2x-1}{(x-1)(x+5)}\,dx = \int \frac{\frac{1}{6}}{x-1} + \frac{\frac{11}{6}}{x+5}\,dx.$

$= \frac{1}{6}\int \frac{1}{x-1}\,dx + \frac{11}{6}\int \frac{1}{x+5}\,dx.$

$= \frac{1}{6}\ln|x-1| + \frac{11}{6}\ln|x+5| + C.$

(iii) $\int \frac{x^2-11x-6}{(2x-1)(x-2)(x+2)}\,dx = \int \frac{3}{2x-1} - \frac{2}{x-2} + \frac{1}{x+2}\,dx.$

$= \frac{3}{2}\int \frac{2}{2x-1}\,dx - 2\int \frac{1}{x-2}\,dx + \int \frac{1}{x+2}\,dx.$

$= \frac{3}{2}\ln|2x-1| - 2\ln|x-2| + \ln|x+2| + C$

6.(i) Note that $\deg(x^2+x+1) = \deg(x^2+3x+2)$ so this is an improper rational function.

$$\begin{array}{r} 1 \\ x^2+3x+2 \,\big)\, \overline{x^2+x+1} \\ \underline{x^2+3x+2} \\ -2x-1 \end{array}$$

$\frac{x^2+x+1}{x^2+3x+2} = 1 + \frac{-2x-1}{x^2+3x+2}$

$= 1 - \frac{(2x+1)}{x^2+3x+2}.$

(continued...)

6 (i) continued. Now we need to decompose $\frac{2x+1}{x^2+3x+2}$ into partial fractions.

$$\frac{2x+1}{x^2+3x+2} = \frac{2x+1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}.$$

$$2x+1 = \left(\frac{A}{x+2} + \frac{B}{x+1}\right)(x+2)(x+1).$$

$$2x+1 = A(x+1) + B(x+2)$$

Set $x=-1$: $-2+1 = A(-1+1) + B(-1+2)$.

$-1 = B$. , so $\underline{B=-1}$

Set $x=-2$: $-4+1 = A(-2+1) + B(-2+2)$

$-3 = -A$, so $\underline{A=3}$.

So $\frac{2x+1}{x^2+3x+2} = \frac{3}{x+2} - \frac{1}{x+1}$

Putting all this together...

$$\frac{x^2+x+1}{x^2+3x+2} = 1 - \left(\frac{2x+1}{x^2+3x+2}\right) = 1 - \left(\frac{3}{x+2} - \frac{1}{x+1}\right)$$

$$= 1 - \frac{3}{x+2} + \frac{1}{x+1}.$$

(ii) Note that since $\deg(x^3) > \deg(x^2-x-6)$ this is an improper rational function.

$$\begin{array}{r}
x+1 \\
x^2-x-6 \overline{)\, x^3 \qquad\qquad} \\
x^3 - x^2 - 6x \\
\hline
x^2 + 6x \\
x^2 - x - 6 \\
\hline
7x + 6
\end{array}$$

So $\frac{x^3}{x^2-x-6} = x+1 + \frac{7x+6}{x^2-x-6}.$

Now we write $\frac{7x+6}{x^2-x-6}$ in terms of partial fractions...

(continued...)

6(ii) continued.

$$\frac{7x+6}{x^2-x-6} = \frac{7x+6}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}.$$

$$7x+6 = \left(\frac{A}{x-3} + \frac{B}{x+2}\right)(x-3)(x+2)$$

$$7x+6 = A(x+2) + B(x-3)$$

Set $x=-2$: $-14+6 = A(-2+2) + B(-2-3)$.
$-8 = -5B$, so $B = \frac{8}{5}$.

Set $x=3$: $21+6 = A(3+2) + B(3-3)$.
$27 = 5A$. , so $A = \frac{27}{5}$

So
$$\frac{7x+6}{x^2-x-6} = \frac{27/5}{x-3} + \frac{8/5}{x+2}$$
$$= \frac{27}{5(x-3)} + \frac{8}{5(x+2)}$$

Putting this all together gives

$$\frac{x^3}{x^2-x-6} = x+1 + \frac{27}{5(x-3)} + \frac{8}{5(x+2)}.$$

7(i) $$\int \frac{x^2+x+1}{x^2+3x+2}\,dx = \int 1 - \frac{3}{x+2} + \frac{1}{x+1}\,dx.$$
$$= \int 1\,dx - 3\int \frac{1}{x+2}\,dx + \int \frac{1}{x+1}\,dx.$$
$$= x - 3\ln|x+2| + \ln|x+1| + C.$$

(ii) $$\int \frac{x^3}{x^2-x-6}\,dx = \int x+1 + \frac{27}{5(x-3)} + \frac{8}{5(x+2)}\,dx.$$
$$= \int x+1\,dx + \frac{27}{5}\int \frac{1}{x-3}\,dx + \frac{8}{5}\int \frac{1}{x+2}\,dx.$$
$$= \frac{1}{2}x^2 + x + \frac{27}{5}\ln|x-3| + \frac{8}{5}\ln|x+2| + C.$$

8 (i) $$\frac{x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

$$x+3 = \left(\frac{A}{x-1} + \frac{B}{(x-1)^2}\right)(x-1)^2.$$

$$x+3 = A(x-1) + B.$$

Set $x=1$: $1+3 = A(1-1) + B$

$\underline{4 = B}$.

Note: there is not another "convenient value".
We may proceed by comparing coefficients

LHS: coefficient of x is 1
RHS coefficient of x is A
so $\underline{A=1}$

$$\frac{x+3}{(x-1)^2} = \frac{1}{x-1} + \frac{4}{(x-1)^2}.$$

(ii) $$\frac{2x-3}{x^2+8x+16} = \frac{2x-3}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2}.$$

$$2x-3 = \left(\frac{A}{x+4} + \frac{B}{(x+4)^2}\right).(x+4)^2.$$

$$2x-3 = A(x+4) + B.$$

Set $x=-4$: $-8-3 = A(-4+4) + B.$

$\underline{-11 = B}$

Note: there is not another "convenient value".

LHS: coefficient of x is 2
RHS: coefficient of x is A
so $\underline{A=2}$

$$\frac{2x-3}{x^2+8x+16} = \frac{2}{x+4} - \frac{11}{(x+4)^2}.$$

8 (iii) $\dfrac{x^2+x-1}{x^3-x^2} = \dfrac{x^2+x-1}{x^2(x-1)} = \dfrac{x^2+x-1}{\underbrace{(x-0)^2}(x-1)}$

↑ repeated linear factor!

$$\frac{x^2+x-1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$x^2+x-1 = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\right) x^2(x-1).$$

$$x^2+x-1 = Ax(x-1) + B(x-1) + Cx^2.$$

Set $x=0$: $-1 = -B$, so $\underline{B=1}$.

Set $x=1$: $1+1-1 = Ax(1-1) + B(1-1) + C.1^2.$

$\underline{1 = C}$.

Note: There is not another "convenient value".

LHS : coefficient of x^2 is 1

RHS : coefficient of x^2 is $A+C$

← so $A+C=1$. We have already found $C=1$, so must have $\underline{A=0}$.

$$\frac{x^2+x-1}{x^3-x^2} = \frac{0}{x} + \frac{1}{x^2} + \frac{1}{x-1}$$

$$= \underline{\underline{\frac{1}{x^2} + \frac{1}{x-1}.}}$$

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9 (i) $\displaystyle\int \frac{x+3}{(x-1)^2}\,dx = \int \frac{1}{x-1} + \frac{4}{(x-1)^2}\,dx.$

$$= \int \frac{1}{x-1}\,dx + 4\int (x-1)^{-2}\,dx.$$

$$= \ln|x-1| - 4(x-1)^{-1} + C.$$

$$= \underline{\underline{\ln|x-1| - \frac{4}{x-1} + C.}}$$

9(ii) $\int \frac{2x-3}{x^2+8x+16}\,dx = \int \frac{2}{x+4} - \frac{11}{(x+4)^2}\,dx.$

$= 2\int \frac{1}{x+4}\,dx - 11\int (x+4)^{-2}\,dx.$

$= 2\ln|x+4| + 11\,(x+4)^{-1} + C$

$= 2\ln|x+4| + \frac{11}{x+4} + C$

9(iii) $\int \frac{x^2+x-1}{x^3-x^2}\,dx = \int \frac{1}{x^2} + \frac{1}{x-1}\,dx.$

$= \int x^{-2}\,dx + \int \frac{1}{x-1}\,dx.$

$= -x^{-1} + \ln|x-1| + C.$

$= \ln|x-1| - \frac{1}{x} + C.$

10.(i) $\frac{2}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1}.$

irreducible

$2 = \left(\frac{Ax+B}{x^2+2} + \frac{C}{x+1}\right)(x^2+2)(x+1)$

$2 = (Ax+B)(x+1) + C(x^2+2).$

Set $x=-1$: $2 = 3C$, so $C = \frac{2}{3}$.

Multiply out and compare coefficients :-

$2 = Ax^2 + Ax + Bx + B + Cx^2 + 2C.$

$0x^2+0x+2 = (A+C)x^2 + (A+B)x + (B+2C)$

So $0 = A+C$. Since $C=2/3$, must have $A=-2/3$.

$0 = A+B$ Since $A=-2/3$, must have $B=2/3$.

$2 = B+2C.$ [Check: $\frac{2}{3} + 2\left(\frac{2}{3}\right) = \frac{6}{3} = 2$]

(continued...)

10(i) continued - - -

$$\frac{2}{(x^2+2)(x+1)} = \frac{-\frac{2}{3}x + \frac{2}{3}}{x^2+2} + \frac{\frac{2}{3}}{x+1}$$

$$= \frac{2}{3}\left(\frac{1-x}{x^2+2} + \frac{1}{x+1}\right).$$

0(ii) $$\frac{3x+2}{(x^2+x+1)(x-1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}.$$

irreducible

$$3x+2 = \left(\frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}\right)(x^2+x+1)(x-1)$$

$$3x+2 = (Ax+B)(x-1) + C(x^2+x+1).$$

Set $x=1$: $5 = 3C$, so $C = \frac{5}{3}$.

multiply out and compare coefficients :-

$$3x+2 = Ax^2 - Ax + Bx - B + Cx^2 + Cx + C.$$

$$0x^2+3x+2 = (A+C)x^2 + (B-A+C)x + (C-B).$$

So $0 = A+C$. Since $C = 5/3$, must have $A = -5/3$.

$3 = B - A + C$. Since $-A = C = 5/3$, have $3 = B + \frac{10}{3}$, so $B = -1/3$.

$2 = C - B$ [Check : $5/3 + 1/3 = 6/3 = 2$].

$$\frac{3x+2}{(x^2+x+1)(x-1)} = \frac{-5/3\, x - 1/3}{x^2+x+1} + \frac{5/3}{x-1}.$$

$$= \frac{1}{3}\left(\frac{5}{x-1} - \frac{(5x+1)}{x^2+x+1}\right)$$

10(iii) $\dfrac{x}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$.

$x = \left(\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}\right)(x+2)^3$.

$x = A(x+2)^2 + B(x+2) + C$.

Set $x=-2$: $\underline{-2 = C}$

multiply out and compare coefficients.

$x = A(x^2+4x+4) + B(x+2) + C$.

$x = Ax^2 + 4Ax + 4A + Bx + 2B + C$.

$x = Ax^2 + 4Ax + 4A + Bx + 2B + C$.

$x = Ax^2 + (4A+B)x + (4A+2B+C)$

$0x^2 + 1x + 0 = Ax^2 + (4A+B)x + (4A+2B+C)$

so $0 = A \Rightarrow \underline{A=0}$

$1 = 4A+B \Rightarrow \underline{B=1}$

$0 = 4A+2B+C$ [Check : $4\times 0 + 2\times 1 + -2 = 0$]

$\dfrac{x}{(x+2)^3} = \dfrac{0}{x+2} + \dfrac{1}{(x+2)^2} + \dfrac{-2}{(x+2)^3}$.

$\underline{\underline{\dfrac{x}{(x+2)^3} = \dfrac{1}{(x+2)^2} - \dfrac{2}{(x+2)^3}}}$.

11(i) $\displaystyle\int \frac{2}{(x^2+2)(x+1)}\,dx = \int \frac{2}{3}\left(\frac{1-x}{x^2+2} + \frac{1}{x+1}\right)dx$.

$\displaystyle = \frac{2}{3}\int \frac{1-x}{x^2+2}\,dx + \frac{2}{3}\int \frac{1}{x+1}\,dx$.

$\displaystyle = \frac{2}{3}\int \frac{1}{x^2+2}\,dx - \frac{1}{3}\int \frac{2x}{x^2+2}\,dx + \frac{2}{3}\int \frac{1}{x+1}\,dx$.

$= \frac{2}{3}\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right) - \frac{1}{3}\ln|x^2+2| + \frac{2}{3}\ln|x+1| + C$

$= \underline{\underline{\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{3}\ln|x^2+2| + \frac{2}{3}\ln|x+1| + C}}$.

11(ii) $\displaystyle\int \frac{3x+2}{(x^2+x+1)(x-1)}\,dx = \int \frac{1}{3}\left(\frac{5}{x-1} - \frac{(5x+1)}{x^2+x+1}\right) dx$

$$= \frac{5}{3}\int \frac{1}{x-1}\,dx - \frac{5}{3}\int \frac{x+1/5}{x^2+x+1}\,dx.$$

$$= \frac{5}{3}\ln|x-1| - \frac{5}{6}\int \frac{2x+2/5}{x^2+x+1}\,dx.$$

$$= \frac{5}{3}\ln|x-1| - \frac{5}{6}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{5}{6}\int \frac{-3/5}{x^2+x+1}\,dx.$$

$$= \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln|x^2+x+1| + \frac{15}{30}\int \frac{1}{(x+1/2)^2+3/4}\,dx.$$

$$= \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln|x^2+x+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + C$$

$$= \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

(iii) $\displaystyle\int \frac{x}{(x+2)^3}\,dx = \int \frac{1}{(x+2)^2} - \frac{2}{(x+2)^3}\,dx.$

$$= \int (x+2)^{-2} - 2(x+2)^{-3}\,dx.$$

$$= -(x+2)^{-1} + (x+2)^{-2} + C.$$

$$= -\frac{1}{x+2} + \frac{1}{(x+2)^2} + C$$

or

$$= \frac{-(x+2)+1}{(x+2)^2} + C$$

$$= -\frac{(x+1)}{(x+2)^2} + C$$

12. Need to factorise

$(x^2+2x+3)^2 \cdot (x^2+2x-3)^2$ as much as possible.

Quadratic formula

ax^2+bx+c factorises $\Leftrightarrow$ $b^2-4ac \geqslant 0$.

x^2+2x+3: $a=1, b=2, c=3$, so $b^2-4ac = 4-4\times3 < 0$

This polynomial does not factorise further!

x^2+2x-3: $a=1, b=2, c=-3$, so $b^2-4ac = 4+4\times3 > 0$

This can be factorised!

$x^2+2x-3 = (x-1)(x+3)$

So

$$\frac{1}{(x^2+2x+3)^2 \cdot (x^2+2x-3)^2} = \frac{1}{(x^2+2x+3)^2\,(x-1)^2 \cdot (x+3)^2}.$$

- we get one partial fraction for each of the six factors above
- The form of the partial fractions involved is as follows...

$$\frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{G}{x+3} + \frac{H}{(x+3)^2}.$$

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13 (i) $\dfrac{1}{(x+3)(2x+3)} = \dfrac{A}{x+3} + \dfrac{B}{2x+3}$. so $\boxed{1 = A(2x+3) + B(x+3)}$

Setting $x=-3$ gives $1 = -3A$, so $A = -1/3$.

Setting $x=-3/2$ gives $1 = 3/2\,B$, so $B = 2/3$.

$$\int \frac{1}{(x+3)(2x+3)}\,dx = \int \frac{-1/3}{x+3} + \frac{2/3}{2x+3}\,dx.$$

$$= -\tfrac{1}{3}\ln|x+3| + \tfrac{2}{3}\ln|2x+3| + C.$$

13 (ii)

$$\begin{array}{r} 4x \\ x^2-1 \overline{)\, 4x^3 \quad\quad} \\ 4x^3 - 4x \\ \hline 4x. \end{array}$$

So $\displaystyle\int \frac{4x^3}{x^2-1}\,dx = \int 4x + \frac{4x}{x^2-1}\,dx.$

$= 2x^2 + 2\ln|x^2-1| + C.$

(iii) $\displaystyle\frac{4x+3}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$

So $4x+3 = A(2x-1) + B.$

Setting $x=\frac{1}{2}$ gives $\underline{5 = B}$

Setting $x=0$ gives $3 = -A + B$, so $\underline{A} = B-3 = 5-3 = \underline{2}$

$\displaystyle\int \frac{4x+3}{(2x-1)^2}\,dx = \int \frac{2}{2x-1}\,dx + \int \frac{5}{(2x-1)^2}\,dx.$

$= \ln|2x-1| - \frac{5}{2}(2x-1)^{-1} + C.$

iv) $\displaystyle\frac{x+1}{x^3+x} = \frac{x+1}{(x+0)(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

So $x+1 = A(x^2+1) + (Bx+C)x.$

Setting $x=0$ gives $\underline{1 = A}.$

comparing coeffs of x^2 gives $0 = A + B$, so $\underline{B = -1.}$

comparing coeffs of x^1 gives $\underline{1 = C}$

$\displaystyle\int \frac{x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1-x}{x^2+1}\,dx.$

$\displaystyle= \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx.$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C.$