0C2 Exercise Sheet 4
Further calculus

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1. Complete the following sentences using the phrases: **change in x, change in $f(x)$, limit, linear, $df = f'(x)\delta x$, chord, $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$, tangent line, small.**

   The derivative $f'(x)$ of a continuous function $f(x)$ represents the gradient of the ____________ to f at the point x. We can make an approximation to this value by using the gradient of the ____________ between the point $(x, f(x))$ and the point $(x+\delta x, f(x+\delta x))$, where δx represents a ____________ change in x.

   The derivative $f'(x)$ is found by taking the ____________ of these gradients as δx tends to 0, as expressed by the formula ____________

   The differential df is a ____________ function of the ____________, that approximates the ____________, as expressed by the formula ____________.

2. (i) Calculate the derivative of $\frac{1}{x^3}$ from first principles.
   (ii) What is the differential of $\frac{1}{x^3}$?

3. (i) Use the binomial theorem to expand $(1+x)^4$ and then differentiate your answer.
   (ii) Express your answer to part (i) in terms of $(1+x)^3$.
   (iii) Use the chain rule to differentiate $(1+x)^4$.

4. (i) Use the binomial theorem to expand $(1+x^2)^4$ and then differentiate your answer.
   (ii) Express your answer to part (i) terms of $(1+x^2)^3$.
   (iii) Use the chain rule to differentiate $(1+x^2)^4$.

5. In each case find the derivative $\frac{dy}{dx}$ and the differential dy:

(i) $y = x^3 - 33x^2 + 216x$	(ii) $y = 2x^3 + x - \sin(x)$	(iii) $y = x^2 e^x$
(iv) $y = x\ln(x)$	(v) $y = e^x \cos(x)$	(vi) $y = (2x-1)^3 \sin(x) + \cos^2(x)$
(vii) $y = \frac{\sin(x)}{x}$	(viii) $y = \frac{x}{1+x^2}$	(ix) $y = \ln(x)(2x+1)^{-1}$
(x) $y = \sec(x)$	(xi) $y = \mathrm{cosec}(x)$	(xii) $y = \cot(x)$
(xiii) $y = \frac{e^x - e^{-x}}{2}$	(xiv) $y = \frac{e^x + e^{-x}}{2}$	(xv) $y = x^x$

6. Let $y = \cos^{-1}(x)$. Use implicit differentiation to find $\frac{dy}{dx}$ in terms of y, and then substitute to find $\frac{dy}{dx}$ as a function of x.

7. Calculate the derivative:

$$\frac{d}{dx}\tan^{-1}(e^x)\,.$$

8. Calculate the differentials:

(i) $d\sin^{-1}(x^2)$ (ii) $d\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$.

9. Use the substitution $x = \sin(u)$ and a trigonometric identity to calculate:

$$\int \sqrt{1-x^2}\,dx\,.$$

10. Use the formula for integration by parts ($\int u\,dv = uv - \int v\,du$) setting $u = x$ and $dv = e^x\,dx$ to calculate

$$\int xe^x\,dx\,.$$

11. Use the formula for integration by parts with $u = x^2$ and $dv = xe^{x^2}\,dx$ to calculate

$$\int x^3 e^{x^2}\,dx\,.$$

12. Use integration by parts to find the following indefinite integrals:

(i) $\int x\,\ln(x)\,dx$, (ii) $\int x^2\,\sin(x)\,dx$, (iii) $\int x\,\tan^2(x)\,dx$, (iv) $\int \tan^{-1}(x)\,dx$.

13. Expand $\tan(x)$ near $x = 0$ to the third power of x (inclusive).

14. Using the Taylor expansion for $\frac{1}{1+x^2}$, find the Taylor expansion for $\tan^{-1}(x)$.

15. Calculate the following limits by using Taylor expansions:

(i) $\lim_{x\to 0}\frac{\sqrt{1+2x^2}-1}{x^2}$

(ii) $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{\sqrt[5]{1-x}-1}$

(iii) $\lim_{x\to 0}\frac{\ln(1+x)-\sin x}{(e^x-e^{-x})^2}$