OC2 – Exam 2017 – Solutions

Statement on "bookwork" etc.

* The questions on the paper are similar in style to questions from exercise sheets and previous exam papers.

* Students may use formula tables, so there are no "bookwork" type questions.

* Students may use a calculator provided it cannot perform symbolic calculations (e.g. multiplying out brackets, calculating derivatives or indefinite integrals, performing numerical integration etc.)

* The paper is 2 hours = roughly 15 minutes/question (some may take longer than others).

① (a) $a_9 = a + 8d = 1$ ——— ①

$S_9 = \frac{9}{2}(2a + 8d) = 45 \Rightarrow 9a + 36d = 45$ ——— ②

Multiplying ① by 9 gives $9a + 72d = 9$

and subtracting ② yields $9a + 36d = 45$

$36d = -36$

3 marks

So $\boxed{d = -1}$ 1 mark.

Now $a + 8d = 1 \Rightarrow a - 8 = 1 \Rightarrow \boxed{a = 9}$ 1 mark.

We want $a_{100} = a + 99d = 9 - 99 = -90$ 1 mark.

(b) First term $a = 1{,}000{,}000$, common ratio $r = \frac{1}{10}$.

$\boxed{S_\infty = \frac{a}{1-r} = \frac{1{,}000{,}000}{9/10}}$ ← 1 mark.

$= \frac{10{,}000{,}000}{9}$

$= 1{,}111{,}111\frac{1}{9}$.

↑ 1 mark.

1 mark →

$\boxed{S_8 = \frac{a(1-r^8)}{1-r} = \frac{1{,}000{,}000(1-10^{-8})}{9/10}}$

$= 1{,}111{,}111 \cdot 1$

↑ 1 mark.

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② (a) $(2x-2)^5 = (2(x-1))^5 = 2^5 (x-1)^5$.

$= 2^5[x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1]$

$= 32x^5 - 160x^4 + 320x^3 - 320x^2 + 160x - 32$.

3 marks for correct coefficients

1 marks for alternating signs

1 mark for answer

(b)

$$
\begin{array}{r}
32x^3 - 160x^2 + 288x - 160 \\
x^2+1 \,\big)\, 32x^5 - 160x^4 + 320x^3 - 320x^2 + 160x - 32 \\
32x^5 \qquad\qquad + 32x^3 \qquad\qquad\qquad\qquad \\
\hline
-160x^4 + 288x^3 - 320x^2 + 160x - 32 \\
-160x^4 \qquad\qquad - 160x^2 \qquad\qquad\quad \\
\hline
288x^3 - 160x^2 + 160x - 32 \\
288x^3 \qquad\qquad + 288x \qquad \\
\hline
-160x^2 - 128x - 32 \\
-160x^2 \qquad\qquad - 160
\end{array}
$$

1 mark.

1 mark.

1 mark.

1 mark.

2(b) continued...

$$\frac{(2x-2)^5}{x^2+1} = 32x^3 - 160x^2 + 288x - 160 + \frac{128-128x}{x^2+1}$$

1 mark.

3(a) $$\frac{4x-1}{x^3+3x^2+2x} = \frac{4x-1}{x(x^2+3x+2)} = \underbrace{\frac{4x-1}{x(x+2)(x+1)}}_{\text{1 mark for factorising}} = \underbrace{\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}}_{\text{1 mark for form.}}$$

$4x-1 = A(x+2)(x+1) + Bx(x+1) + Cx(x+2)$ 1 mark for multiplying out.

Set $x=0$: $-1 = 2A$, so $A = -\frac{1}{2}$.
Set $x=-1$: $-5 = -C$, so $C = 5$.
Set $x=-2$: $-9 = 2B$, so $B = -9/2$.

3 marks for coefficients.

$$\frac{4x-1}{x^3+3x^2+2x} = \frac{-1}{2x} - \frac{9}{2(x+2)} + \frac{5}{x+1}.$$

(b) $$\int \frac{4x-1}{x^3+3x^2+2x}\,dx = \int \frac{-1}{2x}\,dx + \int \frac{-9}{2(x+2)}\,dx + \int \frac{5}{x+1}\,dx.$$

$$= -\frac{1}{2}\int \frac{1}{x}\,dx - \frac{9}{2}\int \frac{1}{x+2}\,dx + 5\int \frac{1}{x+1}\,dx.$$

$$= \underbrace{-\frac{1}{2}\ln|x|}_{\text{1 mark}} \underbrace{- \frac{9}{2}\ln|x+2|}_{\text{1 mark}} \underbrace{+ 5\ln|x+1|}_{\text{1 mark}} \underbrace{+ C}_{\text{1 mark}}.$$

4(a) $\sin^3(2x) = \sin(2x)\cdot\sin^2(2x)$

$= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right]$

$= \frac{1}{2}\sin(2x) - \frac{1}{2}\sin(2x)\cos(4x)$

2 marks for applying an appropriate identity

There are a few different ways to do this using the identities provided.

$= \frac{1}{2}\sin(2x) - \frac{1}{4}\left[2\sin(2x)\cos(4x)\right]$

$= \frac{1}{2}\sin(2x) - \frac{1}{4}(\sin(6x) - \sin(2x))$

2 marks for applying an appropriate identity

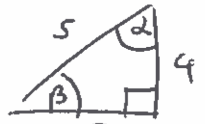
$= \frac{1}{2}\sin(2x) - \frac{1}{4}\sin(6x)$.

$= \underbrace{\frac{3}{4}\sin(2x)}_{\text{1 mark}} \underbrace{- \frac{1}{4}\sin(6x)}_{\text{1 mark.}}$

(b) $\int \sin^3(2x)\,dx = \frac{3}{4}\int \sin(2x)\,dx - \frac{1}{4}\int \sin(6x)\,dx.$ ← 1 mark.

$$= \underbrace{\frac{-3}{8}\cos(2x)}_{\text{1 mark}} \underbrace{+ \frac{1}{24}\cos(6x)}_{\text{1 mark}} \underbrace{+ C}_{\text{1 mark}}$$

⑤ (a)



1 mark.

one angle is $\pi/2$ (≈ 1.5708 radians.)

The remaining angles are

2 marks → $\alpha = \sin^{-1}(4/5) \approx 0.9273$ radians

$\beta = \sin^{-1}(3/5) \approx 0.6435$ radians

b) $\sin^3(2x) = 2\sin(2x)$

$\sin^3(2x) - 2\sin(2x) = 0$

$\sin(2x)\left[\sin^2(2x) - 2\right] = 0$ So either $\sin(2x) = 0$ or $\sin(2x) = \pm\sqrt{2}$ 1 mark.

No solutions, since $-1 \le \sin(2x) \le 1$. 1 mark.

So $\sin(2x) = 0$

giving $2x = \text{Arcsin}(0)$ 1 mark.

$2x = 0 + 2k\pi,\ \pi + 2k\pi,\ k \in \mathbb{Z}$. 2 marks

$x = 0 + k\pi,\ \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

$x = 0 + \frac{k\pi}{2},\ k \in \mathbb{Z}$. 1 mark.

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⑥ (a) $y = \sec(x)\cos(x^2)$

1 mark for using product rule.
1 mark for using chain rule
1 mark for differentiating $\sec(x)$.

$$dy = \Big(\big(\sec(x)\tan(x)\big)\cos(x^2) + \sec(x).(-\sin(x^2)).2x\Big)\,dx.$$

$$= \big(\sec(x)\tan(x)\cos(x^2) - 2x\sec(x)\sin(x^2)\big)\,dx.$$

1 mark for final answer.

(b) $y = x^{x^3 - x^2}$.

$\ln(y) = \ln\left(x^{x^3-x^2}\right) = (x^3 - x^2)\ln(x)$. ← 2 marks.

$\frac{1}{y}\frac{dy}{dx} = (3x^2 - 2x)\ln(x) + (x^3 - x^2).\frac{1}{x}$ ← 3 marks

$\frac{dy}{dx} = y\left[(3x^2 - 2x)\ln(x) + (x^2 - x)\right]$

$= x^{x^3-x^2}\left[(3x^2 - 2x)\ln(x) + x^2 - x\right]$ } 1 mark.

$= x^{x^3-x^2+1}\left[(3x - 2)\ln(x) + x - 1\right]$

(7) (a) $f(x) = \sin(x)$. | $f(\pi/2) = 1$ ← 1 mark.

$f'(x) = \cos(x)$ | $f'(\pi/2) = 0$ 2 marks

$f''(x) = -\sin(x)$ | $f''(\pi/2) = -1$ 2 marks

$f'''(x) = -\cos(x)$ | $f'''(\pi/2) = 0$. 2 marks

As $x \to \pi/2$

$$\sin(x) \approx 1 + 0\cdot(x-\pi/2) - \frac{1}{2!}(x-\pi/2)^2 + \frac{0}{3!}\cdot(x-\pi/2)^3 + O((x-\pi/2)^4)$$

$$\approx 1 - \frac{1}{2}(x-\pi/2)^2 + O((x-\pi/2)^4).$$

1 mark.

(b)
$$\lim_{x\to\pi/2}\left(\frac{\sin(x)-1}{2x-\pi}\right) = \lim_{x\to\pi/2}\left(\frac{(1-\frac{1}{2}(x-\pi/2)^2+\cdots)-1}{2(x-\pi/2)}\right)$$

$$= \lim_{x\to\pi/2}\left(\frac{-\frac{1}{2}(x-\pi/2)^2+\cdots}{2(x-\pi/2)}\right)$$

1 mark.

$$= \lim_{x\to\pi/2}\left(-\frac{1}{4}(x-\pi/2) + O((x-\pi/2)^3)\right)$$

$$= 0$$ 1 mark.

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(8) (a) $I = \int e^x(5x-2)\,dx$. 2 marks → | $u = 5x-2$ $dv = e^x$. | $du = 5\,dx$ $v = e^x$.

$I = uv - \int v\,du$

$= (5x-2)e^x - \int e^x . 5\,dx$ ← 1 mark

$= (5x-2)e^x - 5e^x + C.$ $(= e^x(5x-7)+C)$ ← 1 mark.

$$\int_0^1 e^x(5x-2)\,dx = \left[(5x-2)e^x - 5e^x\right]_0^1 = (3.e-5.e)-(-2-5)$$

$= 7-2e.$ ← 1 mark.

(8) (b) Note that setting $u = \cos^2(x) + \sin(x)$ ← 1 mark.

gives $\frac{du}{dx} = -2\cos(x).\sin(x) + \cos(x)$ ← 1 mark.

$= \cos(x)[1 - 2\sin(x)]$

So $\int \frac{\cos(x)(1-2\sin(x))}{\cos^2(x)+\sin(x)} dx = \int \frac{du}{u} = \ln|u| + C$ ← 1 mark.

$= \ln|\cos^2(x) + \sin(x)| + C.$

So $\int_0^\pi \frac{\cos(x)(1-2\sin(x))}{\cos^2(x)+\sin(x)} dx = \left[\ln|\cos^2(x)+\sin(x)|\right]_0^\pi$ ← 1 mark.

$= \ln|1+0| - \ln|1+0|$

$= \underline{\underline{0}}.$ 1 mark.