# WAVES IN NONLINEAR ELASTIC MEDIA WITH INHOMOGENEOUS PRE-STRESS 

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy<br>in the Faculty of Engineering and Physical Sciences

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# The University of Manchester 

## Tom Shearer <br> Doctor of Philosophy <br> Waves in nonlinear elastic media with inhomogeneous pre-stress December 18, 2012

In this thesis, the effect of inhomogeneous pre-stress on elastic wave propagation and scattering in nonlinear elastic materials is investigated. Four main problems are considered: 1. torsional wave propagation in a pre-stressed annular cylinder, 2. the scattering of horizontally polarised shear waves from a cylindrical cavity in a pre-stressed, infinite, nonlinear elastic material, 3. the use of pre-stress to cloak cylindrical cavities from incoming horizontally polarised shear waves, and 4. the scattering of shear waves from a spherical cavity in a pre-stressed, infinite, nonlinear elastic material.

It is observed that waves in a hyperelastic material are significantly affected by pre-stress, and different results are obtained from those which would be obtained if the underlying stress was neglected and only geometrical changes were considered.

In Chapter 3 we show that the dispersion curves for torsional waves propagating in an annular cylinder are strongly dependent on the pre-stress applied. A greater pressure on the inner surface than the outer causes the roots of the dispersion curves to be spaced further apart, whereas a greater pressure on the outer surface than the inner causes them to be spaced closer together. We also show that a longitudinal stretch causes the cut-on frequencies to move closer together and decreases the gradient of the dispersion curves, whilst a longitudinal compression causes the cut-on frequencies to move further apart and increases the gradient of the dispersion curves.

In Chapter 4 we observe that pre-stress affects the scattering coefficients for shear waves scattered from a cylindrical cavity. It is shown that, for certain parameter values, the scattering coefficients obtained in a pre-stressed medium are closer to those that would be obtained in the undeformed configuration than those that would be obtained in the deformed configuration if the pre-stress were neglected. This result is utilised in Chapter 5 where the cloaking of a cylindrical cavity from horizontally polarised shear waves is examined. It is shown that neo-Hookean materials are optimal for this type of cloaking. A stonger dependence of the strain energy function on the second strain invariant leads to a less efficient cloak.

We observe that, for a Mooney-Rivlin material, as $S_{1}$ tends from 1 towards 0 (in other words, as a material becomes less dependent on the first strain invariant, and more dependent on the second strain invariant), there is more scattering from the cloaking region. For materials which are strongly dependent on the second strain invariant the pre-stress actually increases the scattering cross-section relative to the scattering cross-section for an unstressed material, hence these materials are unsuitable for pre-stress cloaking.

Finally, in Chapter 6 we study the effect of pressure applied to the inner surface of a spherical cavity and at infinity on the propagation and scattering of shear waves in an unbounded medium. It is shown that the scattering coefficients and cross-sections for this problem are strongly dependent on the pre-stress considered. We observe that a region of inhomogeneous pre-stress can lead to some counterintuitive relationships between cavity size and scattering cross-sections and coefficients.

## Declaration

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Dedicated to the memory of
my mother, Janette
$5^{\text {th }}$ July 1959-10 ${ }^{\text {th }}$ November 2011
my aunty, Gill
$27^{\text {th }}$ November 1958-21 ${ }^{\text {st }}$ October 2010
and my grandfather, Bill
31 ${ }^{\text {st }}$ May 1933-14 ${ }^{\text {th }}$ January 2010

## Chapter 1

## Introduction

Waves commonly occur in pre-stressed nonlinear hyperelastic materials and many of these materials have a complex, heterogeneous structure. The application of concern in this thesis is the propagation and scattering of waves in the various materials used in the underwater engineering industry.

Hyperelasticity is a constitutive model for ideally elastic materials in which there is no dissipation of energy so that all the energy resulting from the deformation of such a material is stored as elastic potential energy. In this thesis, we shall be concerned with hyperelastic materials which have a nonlinear stress-strain relationship and are capable of undergoing large deformations.

A good example of the occurance of waves in pre-stressed materials is the use of ultrasound as a non-invasive sensor. Biological soft tissue is an extremely complex material which is naturally stressed as the body moves and grows. It is apparent that the magnitude of the displacements that these tissues are subjected to are large relative to the magnitude of the waves used in ultrasonic detection and it is this disparity in magnitudes that is crucial to the theory of small-on-large, whereby small perturbations to a large initial displacement are considered by performing a linearisation.

The classic example of a nonlinear elastic material is rubber and much of the work in this thesis is based on modelling the propagation and scattering of waves in heterogeneous rubberlike materials, such as the one in Figure 1.1. This is an


Figure 1.1: A heterogeneous rubberlike material used in underwater engineering - Xray micrograph produced by Fonseca, McDonald and Withers (School of Materials, University of Manchester).
imaged slice of a (roughly) 2 mm by 2 mm sample of a material used in underwater engineering applications. The grey coloured part of the image is the rubberlike host phase of the material and the white and black areas are inclusions with different properties. The white flecks are barytes, which have a higher atomic number than the surrounding material and are used to tune the density of the material. The black discs are microspheres made of a stiff (linear elastic) gas-filled shell which buckles under pressure. As can be seen, the microstructure of this material is very complex, and so modelling the propagation and scattering of waves through it is a difficult task. Great pressures are exerted on this material by the ocean and the residual stress throughout the material caused by this pressure adds an extra layer of complexity.

In order to make progress on a problem such as this it is necessary to make some simplifications. Firstly, we can neglect the barytes and assume that the host material is homogeneous. This is a reasonable assumption as the contrast in material properties between the barytes and host material is small compared to the contrast


Figure 1.2: A schematic representation of waves propagating through a composite material containing spherical cavities.
in material properties between the microspheres and host material. Secondly, it is known that the bulk modulus of rubber is much greater than its shear modulus, and so it seems reasonable to model rubber-like materials as incompressible.

It is known that when the material we are modelling is put under a significant magnitude of pressure, the microspheres in the material buckle and debond from the host material. So, if now we assume that we can model the microspheres in their buckled state as cavities, then we can obtain a schematic representation of the problem as shown in Figure 1.2. In this figure, $p_{\infty}$ is the hydrostatic pressure applied at infinity.

This is still a difficult problem; however, we expect that, in order to make progress on such a problem, it will be of importance to first consider some canonical models involving wave scattering from single voids in pre-stressed materials. We will investigate how pre-stress affects the scattering of waves from a single spherical void in a pre-stressed, incompressible, nonlinear hyperelastic material. This specific problem is addressed in the final chapter of this thesis, with the other chapters being dedicated
to problems in simpler geometries in order to familarise the reader with some of the techniques which are employed in the final chapter. The structure of the thesis is as follows.

In Chapter 2 we shall discuss background material relevant to this thesis and will give a broad overview of the theory of waves, nonlinear elasticity and related topics. The reader with experience in these areas can, if they wish, skip this chapter and move immediately to Section 2.6 where we describe the structure of the thesis in more detail. The reader less experienced in waves and nonlinear elasticity should certainly read this chapter however, since it provides a review of the literature and, in particular, it presents some classical problems relevant to those considered in this thesis

In Chapter 3 we discuss torsional wave propagation in a pre-stressed annular circular cylinder. In Chapter 4 we consider the scattering of antiplane waves from a cylindrical cavity in a pre-stressed, infinite host medium. In Chapter 5 we build on the work of Chapter 4 and discuss how the results may be used to construct a theoretical cloak. Finally, in Chapter 6, we consider the problem of the scattering of waves from a spherical cavity in a pre-stressed, infinite host medium and we conclude in Chapter 7, indicating areas for further work.

## Chapter 2

## Background and literature review

In this chapter, a brief review of the background material relevant to this thesis is provided. General aspects of continuum mechanics and wave theory are covered with particular emphasis given to the literature on waves in elastic solids. We shall discuss the history of nonlinear elasticity, the mathematical modelling of rubber, and the theory of small-on-large.

It is assumed that the reader has some prior knowledge of continuum mechanics, but for a good introduction to the subject, see Spencer [96] and the references therein, or see the opening chapters of Ogden [73]. For an advanced text on waves in linear elastic solids, see Graff [36] and for further texts on linear elasticity, see Love [58] and Sokolnikoff [94].

For detailed texts on nonlinear elasticity and the theory of small-on-large, see Fu and Ogden [34], Green and Zerna [38] and Ogden [73] and for a text concentrating more specifically on waves in pre-stressed nonlinear elastic materials, see Destrade and Saccomandi [26].

In this thesis we are concerned with the behaviour of elastic materials, and we neglect any thermal or microscopic effects so that the material we are considering can be modelled as an isothermal continuum. We shall first discuss the theory of nonlinear elasticity and small-on-large, with a detailed analysis of strain energy functions and the incompressible limit, and then will look at the reduction of the nonlinear theory to that of linear elasticity. In the following section we shall discuss linear wave
scattering and propagation in both linear and pre-stressed nonlinear elastic solids and will review some of the papers in these areas which are relevant to this thesis.

### 2.1 Definitions

We begin with some definitions.

## Contraction of tensors

We define the single contraction of two tensors, $\boldsymbol{A}$, and $\boldsymbol{B}$ by

$$
\begin{equation*}
(\boldsymbol{A} \cdot \boldsymbol{B})_{a b \ldots i k \ldots z}=(\boldsymbol{A} \boldsymbol{B})_{a b \ldots i k . . z}=A_{a b \ldots i j} B_{j k \ldots z}, \tag{2.1}
\end{equation*}
$$

where in the above we have used suffix notation with the Einstein summation convention, whereby repeated suffices imply summation over those suffices. We adopt this convention throughout this thesis unless otherwise stated. In the above, the suffices $a, b, \ldots j$ are associated with the tensor $\boldsymbol{A}$, the suffices $j, k, \ldots z$ are associated with the tensor $\boldsymbol{B}$ and $j$ is the dummy suffix to be summed over.

The symbol : represents the double contraction of two tensors, defined by

$$
\begin{equation*}
(\boldsymbol{A}: \boldsymbol{B})_{a b \ldots i l \ldots \zeta}=A_{a b \ldots i j k} B_{j k l \ldots z} . \tag{2.2}
\end{equation*}
$$

In the above, the suffices $a, b, \ldots k$ are associated with the tensor $\boldsymbol{A}$, the Greek suffices $j, k, \ldots z$ are associated with the tensor $\boldsymbol{B}$ and $j$ and $k$ are the dummy suffices to be summed over.

## The Kronecker delta

The Kronecker delta, $\delta_{i j}$, is defined as follows:

$$
\delta_{i j}=\left\{\begin{array}{lll}
1, & \text { if } \quad i=j,  \tag{2.3}\\
0, & \text { if } \quad i \neq j
\end{array}\right.
$$

## Covariant and contravariant basis vectors

Given arbitrary curvilinear coordinates $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$, which can be expressed in terms of Cartesian coordinates $\left(x_{1}, x_{2}, x_{3}\right)$, so that

$$
\begin{equation*}
\theta_{1}=\theta_{1}\left(x_{1}, x_{2}, x_{3}\right), \quad \theta_{2}=\theta_{2}\left(x_{1}, x_{2}, x_{3}\right), \quad \theta_{3}=\theta_{3}\left(x_{1}, x_{2}, x_{3}\right), \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{1}=x_{1}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \quad x_{2}=x_{2}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \quad x_{3}=x_{3}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \tag{2.5}
\end{equation*}
$$

we define covariant base vectors $\boldsymbol{g}_{1}, \boldsymbol{g}_{2}, \boldsymbol{g}_{3}$ in terms of the Cartesian base vectors $\boldsymbol{e}_{1}$, $e_{2}, e_{3}$ by

$$
\begin{equation*}
\boldsymbol{g}_{i}=\frac{\partial x_{j}}{\partial \theta_{i}} \boldsymbol{e}_{j} \tag{2.6}
\end{equation*}
$$

and contravariant base vectors $\boldsymbol{g}^{1}, \boldsymbol{g}^{2}, \boldsymbol{g}^{3}$ by

$$
\begin{equation*}
\boldsymbol{g}^{i}=\frac{\partial \theta_{i}}{\partial x_{j}} \boldsymbol{e}_{j} \tag{2.7}
\end{equation*}
$$

so that

$$
\begin{equation*}
\boldsymbol{g}^{i} \cdot \boldsymbol{g}_{j}=\delta_{i j} \tag{2.8}
\end{equation*}
$$

In cylindrical coordinates,

$$
\begin{array}{ll}
\boldsymbol{g}_{r}=\boldsymbol{e}_{r}, & \boldsymbol{g}_{\theta}=r \boldsymbol{e}_{\theta}, \quad \boldsymbol{g}_{z}=\boldsymbol{e}_{z}, \\
\boldsymbol{g}^{r}=\boldsymbol{e}_{r}, & \boldsymbol{g}^{\theta}=\frac{\boldsymbol{e}_{\theta}}{r}, \quad \boldsymbol{g}^{z}=\boldsymbol{e}_{z}, \tag{2.10}
\end{array}
$$

and in spherical coordinates

$$
\begin{gather*}
\boldsymbol{g}_{r}=\boldsymbol{e}_{r}, \quad \boldsymbol{g}_{\theta}=r \boldsymbol{e}_{\theta}, \quad \boldsymbol{g}_{\phi}=r \sin \theta \boldsymbol{e}_{\phi},  \tag{2.11}\\
\boldsymbol{g}^{r}=\boldsymbol{e}_{r}, \quad \boldsymbol{g}^{\theta}=\frac{\boldsymbol{e}_{\theta}}{r}, \quad \boldsymbol{g}^{\phi}=\frac{\boldsymbol{e}_{\phi}}{r \sin \theta} \tag{2.12}
\end{gather*}
$$

## Dyadic product

We define the dyadic product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}, \boldsymbol{a} \otimes \boldsymbol{b}$, via its action on a third, arbitrary vector $\boldsymbol{c}$, as

$$
\begin{equation*}
(a \otimes b) \cdot c=a(b \cdot c) \tag{2.13}
\end{equation*}
$$

By evaluating the above expression in index notation, we can see that the dyadic product of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ forms a second order tensor which can be evaluated componentwise as

$$
\begin{equation*}
(\boldsymbol{a} \otimes \boldsymbol{b})_{i j}=a_{i} b_{j} . \tag{2.14}
\end{equation*}
$$

## Gradient of a scalar

We define the gradient of a scalar $f$ with respect to the coordinates $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$ as

$$
\begin{equation*}
\operatorname{grad} f=\frac{\partial f}{\partial \theta_{i}} \boldsymbol{g}^{i} . \tag{2.15}
\end{equation*}
$$

In cylindrical coordinates the gradient of the scalar $f(r, \theta, z)$ is given by

$$
\begin{equation*}
\nabla f=\operatorname{grad} \phi=\frac{\partial f}{\partial r} \boldsymbol{e}_{r}+\frac{1}{r} \frac{\partial f}{\partial \theta} \boldsymbol{e}_{\theta}+\frac{\partial f}{\partial z} \boldsymbol{e}_{z} \tag{2.16}
\end{equation*}
$$

and in spherical coordinates the gradient of the scalar $f(r, \theta, \phi)$ is given by

$$
\begin{equation*}
\nabla f=\operatorname{grad} f=\frac{\partial f}{\partial r} \boldsymbol{e}_{r}+\frac{1}{r} \frac{\partial f}{\partial \theta} \boldsymbol{e}_{\theta}+\frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \boldsymbol{e}_{\phi} . \tag{2.17}
\end{equation*}
$$

## Gradient of a vector

We define the gradient of a vector $\boldsymbol{a}$ with respect to the coordinates $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$ as

$$
\begin{equation*}
\operatorname{grad} \boldsymbol{a}=\frac{\partial \boldsymbol{a}}{\partial \theta_{i}} \otimes \boldsymbol{g}^{i} . \tag{2.18}
\end{equation*}
$$

In cylindrical coordinates the gradient of the vector $\boldsymbol{a}=a_{r} \boldsymbol{e}_{r}+a_{\theta} \boldsymbol{e}_{\theta}+a_{z} \boldsymbol{e}_{z}$ is given by

$$
\begin{align*}
\operatorname{grad} \boldsymbol{a} & =\frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_{r}+\frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_{\theta}}{r}+\frac{\partial \boldsymbol{a}}{\partial z} \otimes \boldsymbol{e}_{z} \\
& =\frac{\partial a_{r}}{\partial r} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{r}+\frac{1}{r}\left(\frac{\partial a_{r}}{\partial \theta}-a_{\theta}\right) \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{\theta}+\frac{\partial a_{r}}{\partial z} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{z} \\
& +\frac{\partial a_{\theta}}{\partial r} \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{r}+\frac{1}{r}\left(\frac{\partial a_{\theta}}{\partial \theta}+a_{r}\right) \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{\theta}+\frac{\partial a_{\theta}}{\partial z} \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{z} \\
& +\frac{\partial a_{z}}{\partial r} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{r}+\frac{1}{r} \frac{\partial a_{z}}{\partial \theta} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{\theta}+\frac{\partial a_{z}}{\partial z} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{z} . \tag{2.19}
\end{align*}
$$

and in spherical coordinates the gradient of the vector $\boldsymbol{a}=a_{r} \boldsymbol{e}_{r}+a_{\theta} \boldsymbol{e}_{\theta}+a_{\phi} \boldsymbol{e}_{\phi}$ is given by

$$
\begin{align*}
\operatorname{grad} \boldsymbol{a} & =\frac{\partial \boldsymbol{a}}{\partial r} \otimes \boldsymbol{e}_{r}+\frac{\partial \boldsymbol{a}}{\partial \theta} \otimes \frac{\boldsymbol{e}_{\theta}}{r}+\frac{\partial \boldsymbol{a}}{\partial \phi} \otimes \frac{\boldsymbol{e}_{\phi}}{r \sin \theta} \\
& =\frac{\partial a_{r}}{\partial r} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{r}+\frac{1}{r}\left(\frac{\partial a_{r}}{\partial \theta}-a_{\theta}\right) \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{\theta}+\left(\frac{1}{r \sin \theta} \frac{\partial a_{r}}{\partial \phi}-\frac{a_{\phi}}{r}\right) \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{\phi} \\
& +\frac{\partial a_{\theta}}{\partial r} \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{r}+\frac{1}{r}\left(\frac{\partial a_{\theta}}{\partial \theta}+a_{r}\right) \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{\theta}+\left(\frac{1}{r \sin \theta} \frac{\partial a_{\theta}}{\partial \phi}-\frac{a_{\phi} \cot \theta}{r}\right) \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{\phi} \\
& +\frac{\partial a_{\phi}}{\partial r} \boldsymbol{e}_{\phi} \otimes \boldsymbol{e}_{r}+\frac{1}{r} \frac{\partial a_{\phi}}{\partial \theta} \boldsymbol{e}_{\phi} \otimes \boldsymbol{e}_{\theta}+\left(\frac{a_{r}}{r}+\frac{a_{\theta} \cot \theta}{r}+\frac{1}{r \sin \theta} \frac{\partial a_{\phi}}{\partial \phi}\right) \boldsymbol{e}_{\phi} \otimes \boldsymbol{e}_{\phi} \tag{2.20}
\end{align*}
$$

## Divergence of a vector

We define the divergence of vector $\boldsymbol{a}$ with respect to the coordinates $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$ as

$$
\begin{equation*}
\operatorname{div} \boldsymbol{a}=\boldsymbol{g}^{i} \cdot \frac{\partial \boldsymbol{a}}{\partial \theta_{i}} \tag{2.21}
\end{equation*}
$$

In cylindrical polar coordinates, the divergence of the vector $\boldsymbol{a}=a_{r} \boldsymbol{e}_{r}+a_{\theta} \boldsymbol{e}_{\theta}+a_{z} \boldsymbol{e}_{z}$ is given by

$$
\begin{equation*}
\nabla \cdot \boldsymbol{a}=\operatorname{div} \boldsymbol{a}=\frac{1}{r} \frac{\partial}{\partial r}\left(r a_{r}\right)+\frac{1}{r} \frac{\partial a_{\theta}}{\partial \theta}+\frac{\partial a_{z}}{\partial z}, \tag{2.22}
\end{equation*}
$$

and in spherical coordinates, the divergence of the vector $\boldsymbol{a}=a_{r} \boldsymbol{e}_{r}+a_{\theta} \boldsymbol{e}_{\theta}+a_{\phi} \boldsymbol{e}_{\phi}$ is given by

$$
\begin{equation*}
\nabla \cdot \boldsymbol{a}=\operatorname{div} \boldsymbol{a}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} a_{r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta a_{\theta}\right)+\frac{1}{r \sin \theta} \frac{\partial a_{\phi}}{\partial \phi} \tag{2.23}
\end{equation*}
$$

## Divergence of a second order tensor

We define the divergence of a second order tensor $\boldsymbol{A}$ with respect to the coordinates $\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$ as

$$
\begin{equation*}
\operatorname{div} \boldsymbol{A}=\boldsymbol{g}^{i} \cdot \frac{\partial \boldsymbol{A}}{\partial \theta_{i}} \tag{2.24}
\end{equation*}
$$

In cylindrical polar coordinates, the divergence of a tensor $\boldsymbol{A}=A_{i j} \boldsymbol{e}_{i} \otimes \boldsymbol{e}_{j}$ is given by

$$
\begin{align*}
\operatorname{div} \boldsymbol{A} & =\boldsymbol{e}_{r} \cdot \frac{\partial \boldsymbol{A}}{\partial r}+\frac{\boldsymbol{e}_{\theta}}{r} \cdot \frac{\partial \boldsymbol{A}}{\partial \theta}+\boldsymbol{e}_{z} \cdot \frac{\partial \boldsymbol{A}}{\partial z} \\
& =\left(\frac{\partial A_{r r}}{\partial r}+\frac{1}{r}\left(\frac{\partial A_{\theta r}}{\partial \theta}+A_{r r}-A_{\theta \theta}\right)+\frac{\partial A_{z r}}{\partial z}\right) \boldsymbol{e}_{r} \\
& +\left(\frac{\partial A_{r \theta}}{\partial r}+\frac{1}{r}\left(\frac{\partial A_{\theta \theta}}{\partial \theta}+A_{r \theta}+A_{\theta r}\right)+\frac{\partial A_{z \theta}}{\partial z}\right) \mathbf{e}_{\theta}  \tag{2.25}\\
& +\left(\frac{\partial A_{r z}}{\partial r}+\frac{1}{r}\left(\frac{\partial A_{\theta z}}{\partial \theta}+A_{r z}\right)+\frac{\partial A_{z z}}{\partial z}\right) \mathbf{e}_{z}
\end{align*}
$$

and in spherical coordinates, the divergence of a tensor $\boldsymbol{A}=A_{i j} \boldsymbol{e}_{i} \otimes \boldsymbol{e}_{j}$ is given by

$$
\begin{align*}
\operatorname{div} \boldsymbol{A} & =\boldsymbol{e}_{r} \cdot \frac{\partial \boldsymbol{A}}{\partial r}+\frac{\boldsymbol{e}_{\theta}}{r} \cdot \frac{\partial \boldsymbol{A}}{\partial \theta}+\frac{\boldsymbol{e}_{\phi}}{r \sin \theta} \cdot \frac{\partial \boldsymbol{A}}{\partial \phi} \\
& =\left(\frac{\partial A_{r r}}{\partial r}+\frac{1}{r} \frac{\partial A_{\theta r}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial A_{\phi r}}{\partial \phi}+\frac{1}{r}\left(2 A_{r r}-A_{\theta \theta}-A_{\phi \phi}\right)+\frac{\cot \theta}{r} A_{\theta r}\right) \boldsymbol{e}_{r} \\
& +\left(\frac{\partial A_{r \theta}}{\partial r}+\frac{1}{r} \frac{\partial A_{\theta \theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial A_{\phi \theta}}{\partial \phi}+\frac{1}{r}\left(2 A_{r \theta}+A_{\theta r}\right)+\frac{\cot \theta}{r}\left(A_{\theta \theta}-A_{\phi \phi}\right)\right) \boldsymbol{e}_{\theta} \\
& +\left(\frac{\partial A_{r \phi}}{\partial r}+\frac{1}{r} \frac{\partial A_{\theta \phi}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial A_{\phi \phi}}{\partial \phi}+\frac{1}{r}\left(2 A_{r \phi}+A_{\phi r}\right)+\frac{\cot \theta}{r}\left(A_{\theta \phi}+A_{\phi \theta}\right)\right) \boldsymbol{e}_{\phi} . \tag{2.26}
\end{align*}
$$

## Invariants of tensors

The principal invariants of a tensor $\boldsymbol{A}$ are the coefficients of its characteristic polynomial:

$$
\begin{equation*}
p(\lambda)=\operatorname{det}(\boldsymbol{A}-\lambda \boldsymbol{I}), \tag{2.27}
\end{equation*}
$$

where $\boldsymbol{I}$ is the identity tensor.
The invariants of a tensor do not change with rotation of the coordinate system (i.e. they are objective) and any function of the invariants only is also objective.

## Partial differentiation of a scalar with respect to a second order tensor

We define partial differentation of a scalar $\phi$ with respect to a second order tensor $\boldsymbol{A}$ as follows:

$$
\begin{equation*}
\left(\frac{\partial \phi}{\partial \boldsymbol{A}}\right)_{i j}=\frac{\partial \phi}{\partial A_{j i}} . \tag{2.28}
\end{equation*}
$$

### 2.2 Nonlinear elasticity and the theory of small-on-large

In this section we introduce the notation used throughout this thesis. As previously mentioned, good texts on this subject include Fu and Ogden [34], Green and Zerna [38] and Ogden [73] as well as the paper by Spencer, published in 1970 [95]. There are also two papers by Haughton and Ogden which concentrate on specific aspects of this theory, [41], [42]. Note that the notation used in this thesis differs from these other works and has been defined for consistency and uniformity.

### 2.2.1 Static deformation

## Position vectors and displacement

Consider an elastic body, $B$, with bounding surface $S$, at rest. A general point, $R$, in $B$, is described by the position vector

$$
\begin{equation*}
\boldsymbol{X}=\boldsymbol{X}\left(\Theta_{1}, \Theta_{2}, \Theta_{3}\right), \tag{2.29}
\end{equation*}
$$

where $\Theta_{1}, \Theta_{2}$ and $\Theta_{3}$ are general curvilinear coordinates in this configuration referred to an origin $O$. We refer to this configuration as the undeformed configuration. In Cartesian coordinates, for example, the point $R$ is defined by $\boldsymbol{X}\left(X_{1}, X_{2}, X_{3}\right)$.

We assume that $B$ undergoes some deformation so that the point $R$ is moved to the point $r$, described by the position vector

$$
\begin{equation*}
\boldsymbol{x}=\boldsymbol{x}\left(\theta_{1}, \theta_{2}, \theta_{3}\right), \tag{2.30}
\end{equation*}
$$

where $\theta_{1}, \theta_{2}$ and $\theta_{3}$ are general curvilinear coordinates in this configuration referred to the same origin, $O$. We refer to the deformed body as $b$, with bounding surface $s$, and this configuration as the deformed configuration. This is illustrated in Figure 2.1.

We define the displacement associated with this deformation as

$$
\begin{equation*}
\boldsymbol{U}=\boldsymbol{x}-\boldsymbol{X} \tag{2.31}
\end{equation*}
$$



Figure 2.1: The static deformation of an elastic body, $B$.

## Deformation gradient tensor

Now consider the differentials $\boldsymbol{d} \boldsymbol{X}$ and $\boldsymbol{d} \boldsymbol{x}$ in the undeformed, and deformed, configurations, respectively. We assume that there exists a tensor $\boldsymbol{F}$ that maps $\boldsymbol{d} \boldsymbol{X}$ to $\boldsymbol{d} \boldsymbol{x}$, so that

$$
\begin{equation*}
d x=F d X \tag{2.32}
\end{equation*}
$$

We refer to the tensor, $\boldsymbol{F}$ as the deformation gradient tensor.
In Cartesian coordinates, we have

$$
\begin{equation*}
d x_{i}=\left(\frac{\partial x_{i}}{\partial X_{j}}\right) d X_{j} \tag{2.33}
\end{equation*}
$$

and so

$$
\begin{equation*}
F_{i j}=\frac{\partial x_{i}}{\partial X_{j}} . \tag{2.34}
\end{equation*}
$$

In a general curvilinear coordinate system, the deformation gradient tensor is given by

$$
\begin{equation*}
\boldsymbol{F}=\operatorname{Grad} \boldsymbol{x} \tag{2.35}
\end{equation*}
$$

where Grad represents the gradient operator in the undeformed configuration.
In cylindrical coordinates, for example, a position vector in the deformed configuration is given by

$$
\begin{equation*}
\boldsymbol{x}=r \boldsymbol{e}_{r}(\theta)+z \boldsymbol{e}_{z}, \tag{2.36}
\end{equation*}
$$

and the gradient operator in the undeformed configuration applied to $\boldsymbol{x}$ is given by

$$
\begin{equation*}
\operatorname{Grad} \boldsymbol{x}=\frac{\partial \boldsymbol{x}}{\partial R} \otimes \boldsymbol{e}_{R}+\frac{\partial \boldsymbol{x}}{\partial \theta} \otimes \frac{\boldsymbol{e}_{\Theta}}{R}+\frac{\partial \boldsymbol{x}}{\partial Z} \otimes \boldsymbol{e}_{z} \tag{2.37}
\end{equation*}
$$

and, therefore,

$$
\begin{align*}
\boldsymbol{F} & =\frac{\partial r}{\partial R} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{R}+\frac{1}{R} \frac{\partial r}{\partial \Theta} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{\Theta}+\frac{\partial r}{\partial Z} \boldsymbol{e}_{r} \otimes \boldsymbol{e}_{Z}+\frac{r}{R} \boldsymbol{e}_{\theta} \otimes \boldsymbol{e}_{\Theta} \\
& +\frac{\partial z}{\partial R} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{R}+\frac{1}{R} \frac{\partial z}{\partial \Theta} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{\Theta}+\frac{\partial z}{\partial Z} \boldsymbol{e}_{z} \otimes \boldsymbol{e}_{Z} \tag{2.38}
\end{align*}
$$

This can be expressed in matrix notation as follows:

$$
\boldsymbol{F}=\left(\begin{array}{ccc}
\frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z}  \tag{2.39}\\
0 & \frac{r}{R} & 0 \\
\frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial Z}
\end{array}\right)
$$

Since $\boldsymbol{F}$ is a second order tensor, it has three invariants:

$$
\begin{gather*}
\operatorname{tr}(\boldsymbol{F})=F_{11}+F_{22}+F_{33}=F_{i i}  \tag{2.40}\\
\frac{1}{2}\left((\operatorname{tr}(\boldsymbol{F}))^{2}-\operatorname{tr}\left(\boldsymbol{F}^{2}\right)\right)=\frac{1}{2}\left(F_{i i} F_{j j}-F_{i j} F_{j i}\right), \tag{2.41}
\end{gather*}
$$

and

$$
\begin{equation*}
\operatorname{det}(\boldsymbol{F})=J \tag{2.42}
\end{equation*}
$$

For an incompressible material, we have

$$
\begin{equation*}
\operatorname{det}(\boldsymbol{F})=J=1 \tag{2.43}
\end{equation*}
$$

## Displacement gradient tensor

We define the displacement gradient tensor to be

$$
\begin{equation*}
\boldsymbol{D}=\operatorname{Grad} \boldsymbol{U}=\boldsymbol{F}-\boldsymbol{I} \tag{2.44}
\end{equation*}
$$

## Stretch and strain tensors

Note that, upon applying the polar decomposition theorem, we may decompose $\boldsymbol{F}$ equivalently in the form

$$
\begin{equation*}
\boldsymbol{F}=\boldsymbol{V} \boldsymbol{R} \tag{2.45}
\end{equation*}
$$

or in the form

$$
\begin{equation*}
\boldsymbol{F}=\boldsymbol{R} \boldsymbol{W}, \tag{2.46}
\end{equation*}
$$

where $\boldsymbol{R}$ is a proper, orthogonal tensor which we call the rotation tensor which describes the local rigid body rotation of a material element, $\boldsymbol{V}$ is the left stretch tensor and $\boldsymbol{W}$ is the right stretch tensor. $\boldsymbol{V}$ and $\boldsymbol{W}$ characterise the local deformation of a material element. The left and right stretch tensors have the same invariants, $i_{1}$, $i_{2}$ and $i_{3}$ :

$$
\begin{align*}
& i_{1}=\operatorname{tr}(\boldsymbol{V})=\operatorname{tr}(\boldsymbol{W})=\lambda_{1}+\lambda_{2}+\lambda_{3},  \tag{2.47}\\
& i_{2}=\frac{1}{2}\left(i_{1}^{2}-\operatorname{tr}\left(\boldsymbol{V}^{2}\right)\right)=\frac{1}{2}\left(i_{1}^{2}-\operatorname{tr}\left(\boldsymbol{W}^{2}\right)\right)=\lambda_{1} \lambda_{2}+\lambda_{1} \lambda_{3}+\lambda_{2} \lambda_{3},  \tag{2.48}\\
& i_{3}=\operatorname{det} \boldsymbol{V}=\operatorname{det} \boldsymbol{W}=\lambda_{1} \lambda_{2} \lambda_{3}, \tag{2.49}
\end{align*}
$$

where $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ are the principal values of $\boldsymbol{V}$ and $\boldsymbol{W}$ and are refered to as the prinicpal stretches of the deformation.

Using the above decomposition of $\boldsymbol{F}$, we define the left and right Cauchy-Green strain tensors as, respectively,

$$
\begin{equation*}
\boldsymbol{B}=\boldsymbol{F} \boldsymbol{F}^{T}=\boldsymbol{V}^{2} \tag{2.50}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{C}=\boldsymbol{F}^{T} \boldsymbol{F}=\boldsymbol{W}^{2} \tag{2.51}
\end{equation*}
$$

The left and right Cauchy-Green strain tensors have the same invariants, $I_{1}, I_{2}$ and $I_{3}:$

$$
\begin{align*}
& I_{1}=\operatorname{tr}(\boldsymbol{B})=\operatorname{tr}(\boldsymbol{C})=\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2},  \tag{2.52}\\
& I_{2}=\frac{1}{2}\left(I_{1}^{2}-\operatorname{tr}\left(\boldsymbol{B}^{2}\right)\right)=\frac{1}{2}\left(I_{1}^{2}-\operatorname{tr}\left(\boldsymbol{C}^{2}\right)\right)=\lambda_{1}^{2} \lambda_{2}^{2}+\lambda_{1}^{2} \lambda_{3}^{2}+\lambda_{2}^{2} \lambda_{3}^{2},  \tag{2.53}\\
& I_{3}=\operatorname{det} \boldsymbol{B}=\operatorname{det} \boldsymbol{C}=\lambda_{1}^{2} \lambda_{2}^{2} \lambda_{3}^{2} . \tag{2.54}
\end{align*}
$$

We refer to $I_{1}, I_{2}$ and $I_{3}$ as strain invariants. $I_{1}, I_{2}$ and $I_{3}$ are connected with $i_{1}, i_{2}$ and $i_{3}$ by the relations

$$
\begin{equation*}
I_{1}=i_{1}^{2}-2 i_{2}, \quad I_{2}=i_{2}^{2}-2 i_{1} i_{3}, \quad I_{3}=i_{3}^{2} \tag{2.55}
\end{equation*}
$$

as can be seen by substituting the expressions for $I_{1}, I_{2}, I_{3}, i_{1}, i_{2}$ and $i_{3}$ in terms of the principal stretches into the above.

The Green (or Lagrangian) strain tensor is defined as

$$
\begin{equation*}
\boldsymbol{E}=\frac{1}{2}\left(\boldsymbol{F}^{T} \boldsymbol{F}-\boldsymbol{I}\right)=\frac{1}{2}\left(\boldsymbol{D}+\boldsymbol{D}^{T}+\boldsymbol{D}^{T} \boldsymbol{D}\right) . \tag{2.56}
\end{equation*}
$$

## Volumes

Consider the differential volumes $d V$ and $d v$ in the undeformed, and deformed, configurations, respectively. They are related by the expression

$$
\begin{equation*}
d v=\operatorname{det}(\boldsymbol{F}) d V=J d V \tag{2.57}
\end{equation*}
$$

## Areas

Consider the differential areas $d S$ and $d s$ in the undeformed, and deformed, configurations, respectively, with respective unit normals $\boldsymbol{N}$ and $\boldsymbol{n}$. These quantities satisfy the following relation

$$
\begin{equation*}
\boldsymbol{n} d s=J \boldsymbol{F}^{-T} \boldsymbol{N} d S \tag{2.58}
\end{equation*}
$$

Alternatively, by taking the transpose of (2.58), we obtain

$$
\begin{equation*}
\boldsymbol{n}^{T} d s=J \boldsymbol{N}^{T} \boldsymbol{F}^{-1} d S \tag{2.59}
\end{equation*}
$$

which can be written in index notation as

$$
\begin{equation*}
n_{i} d s=J N_{j} F_{j i}^{-1} d S \tag{2.60}
\end{equation*}
$$

The above are equivalent forms of Nanson's formula, which is derived in Ogden [73]. The derivation is reproduced below in the current notation:

Consider an infinitesimal vector element of material surface $\boldsymbol{d} \boldsymbol{S}$ in a neighbourhood of the point $\boldsymbol{X}$ in $B$ such that $\boldsymbol{d} \boldsymbol{S}=\boldsymbol{N} d S$, where $\boldsymbol{N}$ is the (positive) unit normal to the surface. Let $\boldsymbol{d} \boldsymbol{X}$ be an arbitrary material line element cutting the edge of $\boldsymbol{d} \boldsymbol{S}$ such that $\boldsymbol{d} \boldsymbol{X} \cdot \boldsymbol{d} \boldsymbol{S}>0$. Then the cylinder with base $d S$ and generators $\boldsymbol{d} \boldsymbol{X}$ has volume $d V=\boldsymbol{d} \boldsymbol{X} \cdot \boldsymbol{d} \boldsymbol{S}$. Suppose that $\boldsymbol{d} \boldsymbol{X}$ and $\boldsymbol{d} \boldsymbol{S}$ respectively become $\boldsymbol{d} \boldsymbol{x}$ and $\boldsymbol{d} s$ under the deformation, where $\boldsymbol{d} \boldsymbol{s}=\boldsymbol{n} d s$ and $\boldsymbol{n}$ is the (positive) unit normal to the
surface $d s$. The material of the volume $d V$ forms a cylinder of volume $d v=\boldsymbol{d} \boldsymbol{x} \cdot \boldsymbol{d} \boldsymbol{s}$ in the deformed configuration and so, by (2.57), we have

$$
\begin{equation*}
d x \cdot d s=J d X \cdot d S \tag{2.61}
\end{equation*}
$$

On use of (2.32) we obtain

$$
\begin{equation*}
\boldsymbol{F}^{T} d \boldsymbol{s}=J d \boldsymbol{S} \tag{2.62}
\end{equation*}
$$

after removal of the arbitrary $\boldsymbol{d} \boldsymbol{X}$. Hence by the definitions of $\boldsymbol{d} \boldsymbol{s}$ and $\boldsymbol{d} \boldsymbol{S}$, and upon mulitplying both sides of (2.62) by $\boldsymbol{F}^{-T}$, we obtain (2.58).

Note that to get from (2.61) to (2.62), the following intermediary step has been taken into account:

$$
\begin{equation*}
\boldsymbol{d} \boldsymbol{x} \cdot \boldsymbol{d} \boldsymbol{s}=(\boldsymbol{F} \cdot \boldsymbol{d} \boldsymbol{X}) \cdot \boldsymbol{d} \boldsymbol{s}=\left(\boldsymbol{d} \boldsymbol{X} \cdot \boldsymbol{F}^{T}\right) \cdot \boldsymbol{d} \boldsymbol{s}=\boldsymbol{d} \boldsymbol{X} \cdot\left(\boldsymbol{F}^{T} \cdot \boldsymbol{d} \boldsymbol{s}\right)=\boldsymbol{d} \boldsymbol{X} \cdot\left(\boldsymbol{F}^{T} \boldsymbol{d} \boldsymbol{s}\right) \tag{2.63}
\end{equation*}
$$

## Traction and Stress

Consider a deformable body subjected to some arbitrary loading in equilibrium. At any given point, we can imagine a plane slicing through the body (see Figure 2.2). If we now consider one of the surfaces created by this imaginary plane, then any small area element of this surface $\triangle S$, with outer unit normal $\boldsymbol{n}$, will have a resultant force $\triangle \mathcal{F}$ acting on it. The traction vector $\boldsymbol{t}$ is defined by

$$
\begin{equation*}
\boldsymbol{t}=\lim _{\triangle S \rightarrow 0} \frac{\triangle \mathcal{F}}{\triangle S}, \tag{2.64}
\end{equation*}
$$

and is dependent on the spacial position in the body and the outer unit normal to the plane under consideration.

## Cauchy stress

The relationship between the traction vector and the outer unit normal of the plane being considered can be expressed in terms of the Cauchy stress tensor $\boldsymbol{T}$ :

$$
\begin{equation*}
\boldsymbol{t}=\boldsymbol{T} \cdot \boldsymbol{n} . \tag{2.65}
\end{equation*}
$$

The components of the Cauchy stress tensor can be related to an infinitesimal block of material whose faces are parallel to the axes (see Figure 2.3. The element $T_{i j}$ of


Figure 2.2: The traction vector


Figure 2.3: The Cauchy stress tensor
the tensor $\boldsymbol{T}$ gives the component in the positive $i$-direction of the traction on the face $\theta_{j}=$ constant, with normal pointing in the positive $j$-direction.

## Conservation of energy and Cauchy stress

One of the fundamental assumptions of continuum mechanics is conservation of energy. Following making this assumption, it can be shown that [96]:

$$
\begin{equation*}
\rho \frac{D e}{D t}=\boldsymbol{T}: \operatorname{grad} \boldsymbol{v}-\operatorname{div} \boldsymbol{q} \tag{2.66}
\end{equation*}
$$

where $\rho$ is the density of the continuum under consideration in the deformed configuration,

$$
\begin{equation*}
\frac{D}{D t}=\frac{\partial}{\partial t}+\boldsymbol{v} \cdot \operatorname{grad} \tag{2.67}
\end{equation*}
$$

is the material time derivative, $e$ is the internal energy density of the continuum under consideration, $\boldsymbol{T}$ is the Cauchy stress,

$$
\begin{equation*}
\boldsymbol{v}=\frac{\partial \boldsymbol{U}}{\partial t}=\frac{\partial \boldsymbol{x}}{\partial t}, \tag{2.68}
\end{equation*}
$$

is the velocity vector of a particle at position $\boldsymbol{x}$, and $\boldsymbol{q}$ is the heat flux vector. We have neglected other forms of energy entering the system such as electromagnetic energy or energy caused by chemical changes, for example.

For an elastic material, we assume there is no heat flux $(\boldsymbol{q}=\mathbf{0})$ and so (2.66) becomes

$$
\begin{equation*}
\rho \frac{D e}{D t}=\boldsymbol{T}: \operatorname{grad} \boldsymbol{v} \tag{2.69}
\end{equation*}
$$

Now, if we define the strain energy function per unit volume in $B$ as

$$
\begin{equation*}
W=\rho_{0} e, \tag{2.70}
\end{equation*}
$$

where $\rho_{0}$ is the initial density of the material under consideration, then (2.69) becomes

$$
\begin{equation*}
\frac{\rho}{\rho_{0}} \frac{D W}{D t}=\boldsymbol{T}: \operatorname{grad} \boldsymbol{v} \tag{2.71}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\frac{\rho}{\rho_{0}}=J^{-1}, \tag{2.72}
\end{equation*}
$$

and, therefore,

$$
\begin{equation*}
J^{-1} \frac{D W}{D t}=\boldsymbol{T}: \operatorname{grad} \boldsymbol{v} \tag{2.73}
\end{equation*}
$$

For a hyperelastic material, it is assumed that the strain energy function depends only on the deformation gradient tensor, i.e. only on the current state of deformation, and so

$$
\begin{align*}
J^{-1} \frac{D W}{D t}= & J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}: \frac{D \boldsymbol{F}}{D t}=J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}: \frac{D}{D t}(\operatorname{Grad} \boldsymbol{x})=J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}:(\operatorname{Grad} \boldsymbol{v}) \\
& =J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}:(\operatorname{grad} \boldsymbol{v} \cdot \boldsymbol{F})=J^{-1}\left(\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}\right): \operatorname{grad} \boldsymbol{v} \tag{2.74}
\end{align*}
$$

where we define the derivative of $W$ with respect to $\boldsymbol{F}$ by

$$
\begin{equation*}
\left(\frac{\partial W}{\partial \boldsymbol{F}}\right)_{i j}=\frac{\partial W}{\partial F_{j i}}, \tag{2.75}
\end{equation*}
$$

as in (2.28).
Therefore, (2.73) can be rewritten as

$$
\begin{equation*}
J^{-1}\left(\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}\right): \operatorname{grad} \boldsymbol{v}=\boldsymbol{T}: \operatorname{grad} \boldsymbol{v} \tag{2.76}
\end{equation*}
$$

Since this must hold for all values of $\operatorname{grad} \boldsymbol{v}$, we have

$$
\begin{equation*}
\boldsymbol{T}=J^{-1} \boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}} . \tag{2.77}
\end{equation*}
$$

If the strain energy function is described in terms of the principal invariants $I_{j}$, then the principal Cauchy stresses are

$$
\begin{align*}
& T_{11}=\frac{2 \lambda_{1}^{2}}{J}\left(\frac{\partial W}{\partial I_{1}}+\left(\lambda_{2}^{2}+\lambda_{3}^{2}\right) \frac{\partial W}{\partial I_{2}}+\lambda_{2}^{2} \lambda_{3}^{2} \frac{\partial W}{\partial I_{3}}\right),  \tag{2.78}\\
& T_{22}=\frac{2 \lambda_{2}^{2}}{J}\left(\frac{\partial W}{\partial I_{1}}+\left(\lambda_{1}^{2}+\lambda_{3}^{2}\right) \frac{\partial W}{\partial I_{2}}+\lambda_{1}^{2} \lambda_{3}^{2} \frac{\partial W}{\partial I_{3}}\right),  \tag{2.79}\\
& T_{33}=\frac{2 \lambda_{3}^{2}}{J}\left(\frac{\partial W}{\partial I_{1}}+\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right) \frac{\partial W}{\partial I_{2}}+\lambda_{1}^{2} \lambda_{2}^{2} \frac{\partial W}{\partial I_{3}}\right) . \tag{2.80}
\end{align*}
$$

For an incompressible material, the Cauchy stress is given by

$$
\begin{equation*}
\boldsymbol{T}=\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}+Q \boldsymbol{I} \tag{2.81}
\end{equation*}
$$

where $Q$ is a Lagrange multiplier associated with the incompressibility constraint, and $\boldsymbol{I}$ is the second order identity tensor. For a justification of the introduction of the constant $Q$ see Ogden [73].

Note that the $W$ in (2.77) can depend on the strain invariants $I_{1}, I_{2}$ and $I_{3}$, whereas, the $W$ in (2.81) can only depend on $I_{1}$ and $I_{2}$, since $I_{3}=1$ for an incompressible material. We use the same symbol, $W$, for both, for simplicity. For a derivation of the theory of small-on-large with respect to strain invariants, see Green and Zerna [38].

The Cauchy stress can also be expressed as

$$
\begin{equation*}
\boldsymbol{T}=\beta_{0} \boldsymbol{I}+\beta_{1} \boldsymbol{B}+\beta_{-1} \boldsymbol{B}^{-1}, \tag{2.82}
\end{equation*}
$$

where

$$
\begin{gather*}
\beta_{0}=\frac{2}{\sqrt{I_{3}}}\left(I_{2} \frac{\partial W}{\partial I_{2}}+I_{3} \frac{\partial W}{\partial I_{3}}\right),  \tag{2.83}\\
\beta_{1}=\frac{2}{\sqrt{I_{3}}} \frac{\partial W}{\partial I_{1}}  \tag{2.84}\\
\beta_{-1}=-2 \sqrt{I_{3}} \frac{\partial W}{\partial I_{2}} \tag{2.85}
\end{gather*}
$$

In the above, $\beta_{0}, \beta_{1}$ and $\beta_{-1}$ are referred to as the elastic response functions. When the material under consideration is incompressible, $\beta_{0}$ is replaced by the Lagrange multiplier $Q$ in (2.82).

## Nominal stress

The nominal stress, $\boldsymbol{S}$ is defined by

$$
\begin{equation*}
\boldsymbol{S}=\frac{\partial W}{\partial \boldsymbol{F}} \tag{2.86}
\end{equation*}
$$

for a compressible material, and

$$
\begin{equation*}
\boldsymbol{S}=\frac{\partial W}{\partial \boldsymbol{F}}+Q \boldsymbol{F}^{-1} \tag{2.87}
\end{equation*}
$$

for an incompressible material.
Again, note that the $W$ in (2.86) can depend on the strain invariants $I_{1}, I_{2}$ and $I_{3}$, whereas, the $W$ in (2.87) can only depend on $I_{1}$ and $I_{2}$.

## Relationship between Cauchy and nominal stresses

We observe that

$$
\begin{equation*}
\boldsymbol{T}=J^{-1} \boldsymbol{F} \boldsymbol{S} \tag{2.88}
\end{equation*}
$$

## Static equations of equilbrium

Balance of linear momentum for a volume $v$, with closed surface $s$, gives

$$
\begin{equation*}
\iint_{s} \boldsymbol{n} \boldsymbol{T} d s+\iiint_{v} \rho \boldsymbol{B} d v=\iiint_{v} \rho \frac{\partial^{2} \boldsymbol{U}}{\partial t^{2}} d v \tag{2.89}
\end{equation*}
$$

where $\boldsymbol{n}$ is the outward normal to the surface $s, \rho$ is the density of the body $b$, and $\boldsymbol{B}$ is the body force acting on $b$.

Assuming that $s$ is at least piecewise smooth and that $\boldsymbol{T}$ is continuously differentiable and defined on a neighbourhood of $v$, we apply the divergence theorem to obtain

$$
\begin{equation*}
\iiint_{v} \operatorname{div} \boldsymbol{T} d v+\iiint_{v} \rho \boldsymbol{B} d v=\iiint_{v} \rho \frac{\partial^{2} \boldsymbol{U}}{\partial t^{2}} d v \tag{2.90}
\end{equation*}
$$

where div represents the divergence operator in the deformed configuration.
Rearranging the above, we obtain

$$
\begin{equation*}
\iiint_{v}\left(\operatorname{div} \boldsymbol{T}+\rho \boldsymbol{B}-\rho \frac{\partial^{2} \boldsymbol{U}}{\partial t^{2}}\right) d v=\mathbf{0} . \tag{2.91}
\end{equation*}
$$

If we now assume that every term in the integrand in the above equation is continously differentiable and defined on a neighbourhood of $v$, then the above equation must hold for all volumes $v$. This allows us to shrink $v$ down to a point and hence obtain the pointwise equation

$$
\begin{equation*}
\operatorname{div} \boldsymbol{T}+\rho \boldsymbol{B}-\rho \frac{\partial^{2} \boldsymbol{U}}{\partial t^{2}}=\mathbf{0} \tag{2.92}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\operatorname{div} \boldsymbol{T}+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{U}}{\partial t^{2}} \tag{2.93}
\end{equation*}
$$

The static equations of equilibrium in the absence of body forces are, therefore, given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{T}=\mathbf{0} \tag{2.94}
\end{equation*}
$$

For further details in the derivation of the balance of linear momentum, see [73], for example.

### 2.2.2 Incremental deformation

A great deal of this thesis relies upon the theory of small-on-large, whereby a linearisation is performed about a nonlinearly pre-stressed state. Here we give a detailed derivation of this theory with an original, consistent notation. There is an extensive literature on the theory of small-on-large, and the methodology detailed in this section is largely based on the text by Ogden, published in 1997 [73], although the notation presented by Ogden has been modified extensively here.

In general, the theory of small-on-large is most commonly used in problems involving waves or stability. For some applications of the theory of small-on-large see, for example, Ogden and Sotiropoulos [75], Chattopadhyay and Rogerson [16] or Baek et al. [6]. In [75], Ogden and Sotiropoulos discuss the effect of pre-stress on the propagation and reflection of plane waves in incompressible elastic solids; in [16], Chattopadhyay and Rogerson consider wave reflection in slightly compressible, finitely deformed elastic media; and, in [6], Baek et al. examine the potential use of the theory of small-on-large in computations of fluid-solid interactions in arteries.

## Position vectors and displacement

We now consider a separate deformation of $B$ into a deformed body $\bar{b}$ with bounding surface $\bar{s}$. We assume that the point $R$ in $B$ is now deformed to the point $\bar{r}$ in $\bar{b}$, where $\bar{b}$ is in some sense "close" to $b$ (see Figure 2.4). We assume that the point $\bar{r}$ can be described by the position vector

$$
\begin{equation*}
\overline{\boldsymbol{x}}=\overline{\boldsymbol{x}}\left(\bar{\theta}_{1}, \bar{\theta}_{2}, \bar{\theta}_{3}\right), \tag{2.95}
\end{equation*}
$$

where $\bar{\theta}_{1}, \bar{\theta}_{2}$ and $\bar{\theta}_{3}$ are general curvilinear coordinates in this configuration referred to the origin $O$. We refer to this configuration as the perturbed configuration (in the sense that it is close to $b$ ).

We will define the difference between position vectors in the perturbed and deformed configurations as

$$
\begin{equation*}
\boldsymbol{u}=\overline{\boldsymbol{x}}-\boldsymbol{x} \tag{2.96}
\end{equation*}
$$

and since we assume that the perturbed configuration is close to the deformed configuration,

$$
\begin{equation*}
|\bar{x}-\boldsymbol{x}|=|\boldsymbol{u}| \ll 1 . \tag{2.97}
\end{equation*}
$$

## Deformation gradient tensor

We define the tensor $\boldsymbol{f}$ by

$$
\begin{equation*}
\boldsymbol{f}=\operatorname{grad} \overline{\boldsymbol{x}}, \tag{2.98}
\end{equation*}
$$



Figure 2.4: The deformation of an elastic body, $B$, to two similar configurations.
where grad is the gradient operator with respect to $\boldsymbol{x}$.
We also define the tensor $\overline{\boldsymbol{F}}$ by

$$
\begin{equation*}
\overline{\boldsymbol{F}}=\operatorname{Grad} \overline{\boldsymbol{x}}=\boldsymbol{f} \boldsymbol{F}, \tag{2.99}
\end{equation*}
$$

and so,

$$
\begin{equation*}
\boldsymbol{\Gamma}=\operatorname{Grad} \boldsymbol{u}=\operatorname{Grad}(\overline{\boldsymbol{x}}-\boldsymbol{x})=\operatorname{Grad} \overline{\boldsymbol{x}}-\operatorname{Grad} \boldsymbol{x}=\boldsymbol{f} \boldsymbol{F}-\boldsymbol{F}=(\boldsymbol{f}-\boldsymbol{I}) \boldsymbol{F} . \tag{2.100}
\end{equation*}
$$

We also have

$$
\begin{equation*}
\boldsymbol{\gamma}=\operatorname{grad} \boldsymbol{u}=\operatorname{grad}(\overline{\boldsymbol{x}}-\boldsymbol{x})=\operatorname{grad} \overline{\boldsymbol{x}}-\operatorname{grad} \boldsymbol{x}=\boldsymbol{f}-\boldsymbol{I}, \tag{2.101}
\end{equation*}
$$

so that

$$
\begin{equation*}
f=I+\gamma \tag{2.102}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma=\gamma \boldsymbol{F} \tag{2.103}
\end{equation*}
$$

Note that

$$
\begin{equation*}
|\boldsymbol{\Gamma}| \ll 1, \tag{2.104}
\end{equation*}
$$

and

$$
\begin{equation*}
|\gamma| \ll 1, \tag{2.105}
\end{equation*}
$$

where we have defined $|\cdot|$ via its action on the tensor $\boldsymbol{A}$ as

$$
\begin{equation*}
|\boldsymbol{A}|=\left(\boldsymbol{A}: \boldsymbol{A}^{T}\right)^{\frac{1}{2}} . \tag{2.106}
\end{equation*}
$$

Also note that

$$
\begin{equation*}
f^{-1} \approx I-\gamma \tag{2.107}
\end{equation*}
$$

This can be seen by evaluating $\boldsymbol{f} \boldsymbol{f}^{-1}$ and $\boldsymbol{f}^{-1} \boldsymbol{f}$ :

$$
\begin{align*}
& \boldsymbol{f} \boldsymbol{f}^{-1}=(\boldsymbol{I}+\gamma)(\boldsymbol{I}-\gamma)=\boldsymbol{I}-\gamma \boldsymbol{\gamma}=\boldsymbol{I}+O\left(|\gamma|^{2}\right) \approx \boldsymbol{I}  \tag{2.108}\\
& \boldsymbol{f}^{-1} \boldsymbol{f}=(\boldsymbol{I}-\gamma)(\boldsymbol{I}+\boldsymbol{\gamma})=\boldsymbol{I}-\gamma \boldsymbol{\gamma}=\boldsymbol{I}+O\left(|\boldsymbol{\gamma}|^{2}\right) \approx \boldsymbol{I} . \tag{2.109}
\end{align*}
$$

We will define the quantity $\bar{j}$ in $\bar{b}$ as

$$
\begin{equation*}
\bar{j}=\operatorname{det} \boldsymbol{f}=\operatorname{det}(\boldsymbol{I}+\boldsymbol{\gamma}), \tag{2.110}
\end{equation*}
$$

Since $\boldsymbol{\gamma}=O(|\boldsymbol{u}|)$, we have $\operatorname{det}(\boldsymbol{I}+\boldsymbol{\gamma}) \approx 1+\operatorname{tr}(\boldsymbol{\gamma})$, and so

$$
\begin{equation*}
\bar{j} \approx 1+\operatorname{tr}(\boldsymbol{\gamma}) \tag{2.111}
\end{equation*}
$$

We will define the quantity $\bar{J}$ in $\bar{b}$ as

$$
\begin{equation*}
\bar{J}=\operatorname{det} \overline{\boldsymbol{F}}=\operatorname{det}(\boldsymbol{f} \boldsymbol{F})=\operatorname{det}(\boldsymbol{f}) \operatorname{det}(\boldsymbol{F})=\operatorname{det}(\boldsymbol{I}+\boldsymbol{\gamma}) J, \tag{2.112}
\end{equation*}
$$

hence, to first order in $|\boldsymbol{u}|$,

$$
\begin{equation*}
\bar{J}=(1+\operatorname{tr}(\boldsymbol{\gamma})) J . \tag{2.113}
\end{equation*}
$$

We will assume that the quantity $\bar{J}^{-1}$ can be expanded as

$$
\begin{equation*}
\bar{J}^{-1}=J^{-1}+j^{-1} \tag{2.114}
\end{equation*}
$$

where $j^{-1} \ll 1$.
Using (2.113), we have

$$
\begin{equation*}
j^{-1}=\frac{1}{(1+\operatorname{tr}(\gamma)) J}-\frac{1}{J} \approx \frac{1-\operatorname{tr}(\gamma)}{J}-\frac{1}{J}=-J^{-1} \operatorname{tr}(\gamma) . \tag{2.115}
\end{equation*}
$$

## Areas

Consider the differential areas $d s$ and $\bar{d} s$ in the deformed, and perturbed, configurations, respectively, with respective unit normals $\boldsymbol{n}$ and $\overline{\boldsymbol{n}}$. These quantities satisfy the following relation

$$
\begin{equation*}
\overline{\boldsymbol{n}}^{T} \bar{d} s=\bar{j} \boldsymbol{n}^{T} \boldsymbol{f}^{-1} d s \tag{2.116}
\end{equation*}
$$

Alternatively, by taking the transpose of (2.116), we obtain

$$
\begin{equation*}
\overline{\boldsymbol{n}} \bar{d} s=\bar{j} \boldsymbol{f}^{-T} \boldsymbol{n} d s \tag{2.117}
\end{equation*}
$$

## Incremental stress

## Incremental nominal stress

We will assume that the total nominal stress in $\bar{b}$ is given by

$$
\begin{equation*}
\bar{S}=\boldsymbol{S}+\boldsymbol{s}, \tag{2.118}
\end{equation*}
$$

where $s$ is the nominal stress associated with the perturbation, and, therefore

$$
\begin{equation*}
|s| \ll 1 . \tag{2.119}
\end{equation*}
$$

We justify this assumption by taking the Taylor series of $\overline{\boldsymbol{S}}$ at $\boldsymbol{F}$.
Note that

$$
\begin{equation*}
\overline{\boldsymbol{S}}=\boldsymbol{S}(\overline{\boldsymbol{F}})=\boldsymbol{S}(\boldsymbol{f} \boldsymbol{F})=\boldsymbol{S}(\boldsymbol{F}+\boldsymbol{\gamma} \boldsymbol{F}), \tag{2.120}
\end{equation*}
$$

and let

$$
\begin{equation*}
\boldsymbol{A}=\boldsymbol{F}+\gamma \boldsymbol{F} . \tag{2.121}
\end{equation*}
$$

Now we take the Taylor series of $\overline{\boldsymbol{S}}$ about $\boldsymbol{A}=\boldsymbol{F}$ :

$$
\begin{align*}
\overline{\boldsymbol{S}} & =\left.\boldsymbol{S}(\boldsymbol{A})\right|_{\boldsymbol{A}=\boldsymbol{F}}+\left.\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{A}}\right|_{\boldsymbol{A}=\boldsymbol{F}}:(\boldsymbol{A}-\boldsymbol{F})+\ldots \\
& =\boldsymbol{S}(\boldsymbol{F})+\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{F}}:(\gamma \boldsymbol{F})+\ldots \tag{2.122}
\end{align*}
$$

Hence to first-order, for a compressible material, we have

$$
\begin{equation*}
s=\boldsymbol{L}: \boldsymbol{\Gamma}=\boldsymbol{L}:(\gamma \boldsymbol{F}) \tag{2.123}
\end{equation*}
$$

where $\boldsymbol{L}$ is a fourth-order tensor defined by

$$
\begin{equation*}
\boldsymbol{L}=\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{F}}=\frac{\partial^{2} W}{\partial \boldsymbol{F}^{2}} \tag{2.124}
\end{equation*}
$$

and in Cartesian components, $\boldsymbol{L}$ in index notation is given by

$$
\begin{equation*}
L_{i j k l}=\frac{\partial S_{i j}}{\partial F_{l k}}=\frac{\partial^{2} W}{\partial F_{j i} \partial F_{l k}} \tag{2.125}
\end{equation*}
$$

For an incompressible material,

$$
\begin{equation*}
\boldsymbol{s}=\boldsymbol{L}: \boldsymbol{\Gamma}+q \boldsymbol{F}^{-1}-Q \boldsymbol{F}^{-1} \boldsymbol{\gamma}=\boldsymbol{L}:(\gamma \boldsymbol{F})+q \boldsymbol{F}^{-1}-Q \boldsymbol{F}^{-1} \boldsymbol{\gamma}, \tag{2.126}
\end{equation*}
$$

where $q$ is the perturbation to the Lagrangian multiplier, $Q$, introduced in (2.81).
We also define the push forward of the incremental nominal stress to be

$$
\begin{equation*}
\boldsymbol{\zeta}=J^{-1} \boldsymbol{F} \boldsymbol{s} \tag{2.127}
\end{equation*}
$$

so that

$$
\begin{equation*}
\zeta=M: \gamma \tag{2.128}
\end{equation*}
$$

for a compressible material, where $\boldsymbol{M}$ can be defined in index notation as follows:

$$
\begin{equation*}
M_{i j k l}=J^{-1} L_{m j n l} F_{i m} F_{k n}=J^{-1} \frac{\partial^{2} W}{\partial F_{j m} \partial F_{l n}} F_{i m} F_{k n} . \tag{2.129}
\end{equation*}
$$

For an incompressible material, we have

$$
\begin{equation*}
\boldsymbol{\zeta}=\boldsymbol{M}: \boldsymbol{\gamma}+q \boldsymbol{I}-Q \boldsymbol{\gamma} \tag{2.130}
\end{equation*}
$$

The tensor $\boldsymbol{M}$ can be compared with the elasticity tensor $\boldsymbol{c}$, which will be presented in Section 2.3. These tensors are similar in the sense that they relate a measure of stress to a measure of strain, however, there is a crucial difference between these tensors which makes small-on-large problems more complex than linear elastic problems. The elasticity tensor $\boldsymbol{c}$ possesses the minor symmetries $c_{i j k l}=c_{j i k l}=c_{i j l k}$, however, in general, $\boldsymbol{M}$ does not.

## Incremental Cauchy stress

We will assume that the total Cauchy stress in $\bar{b}$ can be expressed as

$$
\begin{equation*}
\overline{\boldsymbol{T}}=\boldsymbol{T}+\boldsymbol{\tau} \tag{2.131}
\end{equation*}
$$

where $\boldsymbol{\tau}$ is the Cauchy stress associated with the perturbation, and, therefore

$$
\begin{equation*}
|\tau| \ll 1 \tag{2.132}
\end{equation*}
$$

Using (2.88), we see that

$$
\begin{equation*}
\overline{\boldsymbol{T}}=\bar{J}^{-1} \boldsymbol{f} \boldsymbol{F} \overline{\boldsymbol{S}}, \tag{2.133}
\end{equation*}
$$

and, therefore,

$$
\begin{gather*}
\boldsymbol{T}+\boldsymbol{\tau}=\left(J^{-1}+j^{-1}\right)(\boldsymbol{F}+\boldsymbol{\Gamma})(\boldsymbol{S}+\boldsymbol{s}),  \tag{2.134}\\
\Rightarrow \boldsymbol{\tau}=j^{-1} \boldsymbol{F} \boldsymbol{S}+J^{-1} \boldsymbol{\Gamma} \boldsymbol{S}+J^{-1} \boldsymbol{F} \boldsymbol{s}+O\left(|\boldsymbol{u}|^{2}\right), \tag{2.135}
\end{gather*}
$$

since $\boldsymbol{T}=J^{-1} \boldsymbol{F} \boldsymbol{S}$.
So, neglecting $O\left(|\boldsymbol{u}|^{2}\right)$ terms, and using (2.127), we have

$$
\begin{equation*}
\boldsymbol{\tau}=\boldsymbol{\zeta}+j^{-1} \boldsymbol{F} \boldsymbol{S}+J^{-1} \boldsymbol{\Gamma} \boldsymbol{S} \tag{2.136}
\end{equation*}
$$

Using (2.115), we obtain

$$
\begin{align*}
\boldsymbol{\tau} & =\boldsymbol{\zeta}-\operatorname{tr}(\boldsymbol{\gamma}) J^{-1} \boldsymbol{F} \boldsymbol{S}+J^{-1} \boldsymbol{\Gamma} \boldsymbol{S}  \tag{2.137}\\
& \Rightarrow \boldsymbol{\tau}=\boldsymbol{\zeta}-\operatorname{tr}(\gamma) \boldsymbol{T}+J^{-1} \boldsymbol{\Gamma} \boldsymbol{S} \tag{2.138}
\end{align*}
$$

Hence, using (2.103) and (2.88), we obtain

$$
\begin{equation*}
\boldsymbol{\tau}=\boldsymbol{\zeta}-\operatorname{tr}(\gamma) \boldsymbol{T}+\gamma \boldsymbol{T} \tag{2.139}
\end{equation*}
$$

For an incompressible material, $\operatorname{tr}(\gamma)=0$, and so, we have

$$
\begin{equation*}
\boldsymbol{\tau}=\boldsymbol{\zeta}+\gamma \boldsymbol{T} \tag{2.140}
\end{equation*}
$$

## Incremental equations of equilibrium

The equations of equilibrium in the perturbed configuration, in the absence of body forces, are given by

$$
\begin{equation*}
\overline{\operatorname{div}}(\overline{\boldsymbol{T}})=\bar{\rho} \frac{\partial^{2} \overline{\boldsymbol{U}}}{\partial t^{2}} \tag{2.141}
\end{equation*}
$$

where $\overline{\text { div }}$ represents the divergence operator in the perturbed configuration, $\bar{\rho}$ is the density of $\bar{b}, \overline{\boldsymbol{U}}=\boldsymbol{U}+\boldsymbol{u}$ and we remind the reader that $\overline{\boldsymbol{T}}$ is the total stress.

Note that since $\boldsymbol{U}$ is not dependent on time, we have

$$
\begin{equation*}
\frac{\partial^{2} \overline{\boldsymbol{U}}}{\partial t^{2}}=\frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.142}
\end{equation*}
$$

and hence,

$$
\begin{equation*}
\overline{\operatorname{div}}(\overline{\boldsymbol{T}})=\bar{\rho} \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.143}
\end{equation*}
$$

We assume that $\bar{\rho}=\rho+O(|\boldsymbol{u}|)$, and so, neglecting $O\left(|\boldsymbol{u}|^{2}\right)$ terms, we obtain

$$
\begin{equation*}
\overline{\operatorname{div}}(\overline{\boldsymbol{T}})=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}}, \tag{2.144}
\end{equation*}
$$

It can be shown that this can also be written as

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\zeta}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.145}
\end{equation*}
$$

We will prove this in index notation for Cartesian coordinates:
The left side of equation (2.144) can be written in index notation as

$$
\begin{align*}
& \frac{\partial}{\partial \bar{x}_{i}}\left(\bar{T}_{i j}\right)=\frac{\partial x_{k}}{\partial \bar{x}_{i}} \frac{\partial}{\partial x_{k}}\left(\bar{T}_{i j}\right)=f_{i k}^{-T} \frac{\partial}{\partial x_{k}}\left(\bar{T}_{i j}\right)=f_{k i}^{-1} \frac{\partial}{\partial x_{k}}\left(T_{i j}+\tau_{i j}\right)=\left(\delta_{k i}-\gamma_{k i}\right) \frac{\partial}{\partial x_{k}}\left(T_{i j}+\tau_{i j}\right) \\
& \approx \frac{\partial}{\partial x_{i}}\left(T_{i j}\right)+\frac{\partial}{\partial x_{i}}\left(\tau_{i j}\right)-\gamma_{k i} \frac{\partial}{\partial x_{k}}\left(T_{i j}\right)=\frac{\partial}{\partial x_{i}}\left(T_{i j}\right)+\frac{\partial}{\partial x_{i}}\left(\tau_{i j}\right)-\frac{\partial}{\partial x_{k}}\left(\gamma_{k i} T_{i j}\right)+T_{i j} \frac{\partial}{\partial x_{k}}\left(\gamma_{k i}\right) . \tag{2.146}
\end{align*}
$$

Therefore, using the fact that $T_{i j, i}=0$ (from the static equations of equilibrium), we have

$$
\begin{equation*}
\frac{\partial}{\partial \bar{x}_{i}}\left(\bar{T}_{i j}\right)=\frac{\partial}{\partial x_{i}}\left(\tau_{i j}\right)-\frac{\partial}{\partial x_{i}}\left(\gamma_{i k} T_{k j}\right)+T_{i j} \frac{\partial}{\partial x_{k}}\left(\gamma_{k i}\right) . \tag{2.147}
\end{equation*}
$$

By the definition of $\gamma$, we have

$$
T_{i j} \frac{\partial}{\partial x_{k}}\left(\gamma_{k i}\right)=T_{i j} \frac{\partial}{\partial x_{k}}\left(\frac{\partial u_{k}}{\partial x_{i}}\right)=T_{i j} \frac{\partial^{2}}{\partial x_{i} \partial x_{k}}\left(u_{k}\right)=T_{i j} \frac{\partial}{\partial x_{i}}\left(\frac{\partial u_{k}}{\partial x_{k}}\right)=T_{i j} \frac{\partial}{\partial x_{i}}\left(\gamma_{k k}\right)
$$

$$
\begin{equation*}
=\frac{\partial}{\partial x_{i}}\left(T_{i j} \gamma_{k k}\right)-\gamma_{k k} \frac{\partial}{\partial x_{i}}\left(T_{i j}\right)=\frac{\partial}{\partial x_{i}}\left(T_{i j} \gamma_{k k}\right) . \tag{2.148}
\end{equation*}
$$

Hence, (2.147) becomes

$$
\begin{equation*}
\frac{\partial}{\partial \bar{x}_{i}}\left(\bar{T}_{i j}\right)=\frac{\partial}{\partial x_{i}}\left(\tau_{i j}\right)-\frac{\partial}{\partial x_{i}}\left(\gamma_{i k} T_{k j}\right)+\frac{\partial}{\partial x_{i}}\left(T_{i j} \gamma_{k k}\right) . \tag{2.149}
\end{equation*}
$$

This can be written in tensor notation as

$$
\begin{equation*}
\overline{\operatorname{div}}(\overline{\boldsymbol{T}})=\operatorname{div}(\boldsymbol{\tau}-\boldsymbol{\gamma} \boldsymbol{T}+\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{T}) . \tag{2.150}
\end{equation*}
$$

Hence, using (2.139),

$$
\begin{equation*}
\overline{\operatorname{div}}(\overline{\boldsymbol{T}})=\operatorname{div} \boldsymbol{\zeta}, \tag{2.151}
\end{equation*}
$$

and thus,

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\zeta}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.152}
\end{equation*}
$$

### 2.2.3 Incremental boundary conditions

## Hydrostatic pressure boundary conditions

If we have a hydrostatic pressure loading only, on the surface, $s$, of $b$, and this pressure remains unchanged on $\bar{s}$ of $\bar{b}$, then, on $\bar{s}$,

$$
\begin{equation*}
\overline{\boldsymbol{T}} \overline{\boldsymbol{n}} \bar{d} s=-p \overline{\boldsymbol{n}} \bar{d} s, \tag{2.153}
\end{equation*}
$$

where $p$ is the applied pressure, and, on $s$,

$$
\begin{equation*}
\boldsymbol{T} \boldsymbol{n} d s=-p \boldsymbol{n} d s \tag{2.154}
\end{equation*}
$$

So, upon using (2.117) in (2.153), and subtracting (2.154), we obtain

$$
\begin{align*}
& \overline{\boldsymbol{T}} \bar{j} \boldsymbol{f}^{-T} \boldsymbol{n} d s-\boldsymbol{T} \boldsymbol{n} d s=-p \bar{j} \boldsymbol{f}^{-T} \boldsymbol{n} d s+p \boldsymbol{n} d s,  \tag{2.155}\\
\Rightarrow & \left(\bar{j} \boldsymbol{\tau} \boldsymbol{f}^{-T}+\boldsymbol{T}\left(\bar{j} \boldsymbol{f}^{-T}-\boldsymbol{I}\right)\right) \boldsymbol{n}=p\left(\boldsymbol{I}-\bar{j} \boldsymbol{f}^{-T}\right) \boldsymbol{n} . \tag{2.156}
\end{align*}
$$

Now,

$$
\begin{equation*}
\bar{j} \boldsymbol{f}^{-T}-\boldsymbol{I} \approx(1+\operatorname{tr}(\boldsymbol{\gamma}))\left(\boldsymbol{I}-\boldsymbol{\gamma}^{T}\right)-\boldsymbol{I}=\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{I}-\boldsymbol{\gamma}^{T}+O\left(|\boldsymbol{u}|^{2}\right) . \tag{2.157}
\end{equation*}
$$

Hence, (2.156), to first order in $|\boldsymbol{u}|$, becomes

$$
\begin{align*}
& \boldsymbol{\tau} \boldsymbol{n}+\boldsymbol{T}\left(\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{I}-\boldsymbol{\gamma}^{T}\right) \boldsymbol{n}+p\left(\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{I}-\boldsymbol{\gamma}^{T}\right) \boldsymbol{n}=\mathbf{0}  \tag{2.158}\\
& \Rightarrow \boldsymbol{\tau} \boldsymbol{n}+\operatorname{tr}(\boldsymbol{\gamma})(\boldsymbol{T} \boldsymbol{n}+p \boldsymbol{n})-\boldsymbol{T} \boldsymbol{\gamma}^{T} \boldsymbol{n}-p \boldsymbol{\gamma}^{T} \boldsymbol{n}=\mathbf{0} \tag{2.159}
\end{align*}
$$

and, using (2.154), the boundary condition on the incremental Cauchy stress becomes

$$
\begin{equation*}
\boldsymbol{\tau} \boldsymbol{n}=\boldsymbol{T} \boldsymbol{\gamma}^{T} \boldsymbol{n}+p \boldsymbol{\gamma}^{T} \boldsymbol{n} . \tag{2.160}
\end{equation*}
$$

Thus, from (2.139), it can be shown that this is equivalent to

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{T} \boldsymbol{n}+p \boldsymbol{\gamma}^{T} \boldsymbol{n} \tag{2.161}
\end{equation*}
$$

which, upon using (2.154) can be rewritten as

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=p\left(\boldsymbol{\gamma}^{T} \boldsymbol{n}-\operatorname{tr}(\boldsymbol{\gamma}) \boldsymbol{n}\right) \tag{2.162}
\end{equation*}
$$

or for an incompressible material,

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=p \boldsymbol{\gamma}^{T} \boldsymbol{n} . \tag{2.163}
\end{equation*}
$$

Hence, when $p=0$, we have

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=\mathbf{0} \tag{2.164}
\end{equation*}
$$

### 2.2.4 Strain energy functions

Strain energy functions relate the strain energy density of a material to its deformation gradient and, in the case of an isotropic material, can be expressed explicitly as functions of the strain invariants, $I_{1}, I_{2}$ and $I_{3}$, (see Fu and Ogden [34]) or the stretch invariants $i_{1}, i_{2}$ and $i_{3}$. Note that, using (2.52)-(2.54), these strain energy functions can be expressed equivalently in terms of the principal stretches $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$. Objectivity and isotropy of the material under consideration give

$$
\begin{equation*}
W=W\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)=W\left(\lambda_{1}, \lambda_{3}, \lambda_{2}\right)=W\left(\lambda_{2}, \lambda_{1}, \lambda_{3}\right) . \tag{2.165}
\end{equation*}
$$

Much of the literature on strain energy functions is dedicated to modelling the behaviour of rubber, which is an excellent example of a material which exhibits
nonlinear elastic behaviour. Furthermore, the problems in this thesis are concerned with modelling the behaviour of a rubberlike composite material under pre-stress and so below we discuss some proposed forms for the strain energy function of rubberlike materials. For an in depth article on the hyperelasticity of rubber and biological tissues, see Beatty [7].

## Rubber

Here are three simple examples of strain energy functions which are used to model rubberlike materials: the neo-Hookean model,

$$
\begin{equation*}
W=\frac{\mu}{2}\left(I_{1}-3\right), \tag{2.166}
\end{equation*}
$$

the Mooney-Rivlin model,

$$
\begin{equation*}
W=\frac{\mu}{2}\left(S_{1}\left(I_{1}-3\right)+S_{2}\left(I_{2}-3\right)\right), \tag{2.167}
\end{equation*}
$$

and the Varga model,

$$
\begin{equation*}
W=\mu\left(i_{1}-3\right) . \tag{2.168}
\end{equation*}
$$

In these models, $\mu$ represents the initial shear modulus of the material under consideration and $S_{1}$ and $S_{2}$ satisfy $S_{1}+S_{2}=1$ in order to agree with the theory of linear elasticity when the strains under consideration are small (see [79]). These strain energy functions are used to model incompressible materials, for which $i_{3}=I_{3}=1$. Note that rubber is approximately incompressible.

The neo-Hookean model was proposed by Treloar in a paper published in 1943 [100]. It was then shown by Rivlin, in 1948 [86], that the deformation produced on a unit cube of such material by the action of three equally and oppositely directed forces acting normally on its faces is uniquely determined provided that the forces per unit area, measured in the deformed state, are specified. The neo-Hookean model is essentially the extension of Hooke's law (see Section 2.3) into nonlinear elasticity. Similarly, the Mooney-Rivlin model was proposed by Mooney in a paper published in 1940 [61] and the same uniqueness theorem as above was proved by Rivlin in [87] for this model. The Varga model was proposed by Varga in 1966 [106].

In a paper published in 1944 [101], Treloar conducted a series of experiments and showed that the neo-Hookean model exhibits reasonable agreement with the data obtained for rubber under moderate strain, and the neo-Hookean model is now commonly regarded as a valid prototype constitutive model for rubberlike materials. In [106], it was shown that the Varga strain energy function has a similar range of agreement with experimental results as the neo-Hookean model. The Mooney-Rivlin model gives a slight improvement to the range of validity of the neo-Hookean model for rubber, whilst still retaining a certain level of mathematical simplicity, and is widely used in this thesis.

Many generalisations of the above models have been proposed for modelling rubberlike materials, for example the polynomial strain energy function:

$$
\begin{equation*}
W=\sum_{i, j=0}^{n} C_{i j}\left(I_{1}-3\right)^{i}\left(I_{2}-3\right)^{j}, \tag{2.169}
\end{equation*}
$$

where $n$ and $C_{i j}$ are material constants and $C_{00}=0$. This model was proposed by Rivlin and Saunders in a paper published in 1951 [88]. Note that this model reduces to the neo-Hookean model when $n=1$ and $C_{01}=C_{11}=0$ and to the Mooney-Rivlin model when $n=1$ and $C_{11}=0$.

A simpler form of the above is the Yeoh strain energy function:

$$
\begin{equation*}
W=\sum_{i=1}^{3} C_{i}\left(I_{1}-3\right)^{i}, \tag{2.170}
\end{equation*}
$$

where $C_{i}$ are material constants and the quantity $2 C_{1}$ can be interpreted as the initial shear modulus (i.e. $C_{1}=\mu / 2$ ). This model was proposed by Yeoh in a paper published in 1993 [109]. Note that this model reduces to the neo-Hookean model when $C_{2}=C_{3}=0$.

Another proposed model is the Ogden strain energy function:

$$
\begin{equation*}
W=\sum_{p=1}^{N} \frac{\mu_{p}}{\alpha_{p}}\left(\lambda_{1}^{\alpha_{p}}+\lambda_{2}^{\alpha_{p}}+\lambda_{3}^{\alpha_{p}}-3\right), \tag{2.171}
\end{equation*}
$$

where $N, \mu_{p}$ and $\alpha_{p}$ are material constants. This strain energy function was proposed by Ogden in a paper published in 1972 [72], in which he shows excellent agreement of his theory with Treloar's data for simple tension, pure shear and equibiaxial tension.

Note that the Ogden model reduces to the neo-Hookean model when $N=1$ and $\alpha=2$, to the Mooney-Rivlin model when $N=2, \alpha_{1}=2$ and $\alpha_{2}=-2$, and to the Varga model when $N=1$ and $\alpha=1$.

The extra degrees of freedom in the polynomial, Yeoh and Ogden models allow better agreement with experimental data, however, some of the mathematical simplicity of the neo-Hookean, Mooney-Rivlin and Varga models is lost.

The above strain energy functions are all incompressible, and this is a good approximation for rubber when modelling large static deformations. However, rubber is capable of supporting compressional waves, and so if we are interested in modelling these, we must consider a strain energy function which incorporates some compressibility. For compressible materials, we no longer have the condition that $I_{3}=1$, and so potential strain energy functions can also depend on this strain invariant as well as $I_{1}$ and $I_{2}$.

The Blatz-Ko model was proposed for modelling compressible polyurethane:

$$
\begin{equation*}
W=\frac{\mu}{2}\left(\frac{I_{2}}{I_{3}}+2 I_{3}^{\frac{1}{2}}-5\right) \tag{2.172}
\end{equation*}
$$

where $\mu$ denotes the shear modulus of the material under infinitesimal deformations. This model was proposed by Blatz and Ko in a paper published in 1962 [12] and has been adopted in many investigations of finite deformation of compressible materials.

If we are only interested in incorporating a small amount of compressibility (see Section 2.2.5) then it is reasonable to consider strain energy functions which are extensions of either the neo-Hookean or Mooney-Rivlin models, for example. A recently proposed strain energy function of such form is

$$
\begin{equation*}
W=\frac{\mu}{2}\left(S_{1}\left(I_{1}-3 I_{3}^{\frac{1}{3}}\right)+S_{2}\left(I_{2}-3 I_{3}^{\frac{2}{3}}\right)\right)+\frac{\kappa}{2}\left(I_{3}^{\frac{1}{2}}-1\right)^{2}, \tag{2.173}
\end{equation*}
$$

where $\mu$ and $\kappa$ are the ground state shear and bulk moduli, respectively, of the material under consideration, and $S_{1}$ and $S_{2}$ are two material parameters which sum to one. Note that this is a compressible extension of the Mooney-Rivlin strain energy function. This can be seen from that fact the if we take the limit as $I_{3} \rightarrow 1$ in the above we recover equation (2.167). If we choose $S_{1}=1$ and $S_{2}=0$, then we obtain the compressible analogue of the neo-Hookean strain energy function.

Other compressible extensions of incompressible strain energy functions exist, such as this extension of the neo-Hookean model [73]

$$
\begin{equation*}
W=\frac{\mu}{2}\left(I_{1}-3-2 \log J\right)+\frac{\kappa}{2}(J-1)^{2} . \tag{2.174}
\end{equation*}
$$

In Figure 2.5 we plot the stress-strain curves for a compressible neo-Hookean material under uniaxial tension. The solid line represents a material for which $\kappa \gg \mu$ (i.e. a nearly incompressible material - see Section 2.2.5) and the dashed line represents a material for which $\kappa=O(\mu)$.

To obtain these curves, a deformation of the form

$$
\begin{equation*}
X=\frac{x}{\lambda_{1}}, \quad Y=\frac{y}{\lambda_{2}}, \quad Z=\frac{z}{\lambda_{3}}, \tag{2.175}
\end{equation*}
$$

was assumed. We assume that the material is being stretched in the $x$-direction and allowed to contract in the $y$ and $z$-directions. Analytical expressions were obtained for the stresses $T_{11}, T_{22}$ and $T_{33}$ by using (2.174) in (2.77):

$$
\begin{align*}
\frac{T_{11}}{\mu} & =\left(\frac{\lambda_{1}}{\lambda_{2} \lambda_{3}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right),  \tag{2.176}\\
\frac{T_{22}}{\mu} & =\left(\frac{\lambda_{2}}{\lambda_{1} \lambda_{3}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right),  \tag{2.177}\\
\frac{T_{33}}{\mu} & =\left(\frac{\lambda_{3}}{\lambda_{1} \lambda_{2}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right), \tag{2.178}
\end{align*}
$$

and the following conditions were applied:

$$
\begin{equation*}
\lambda_{2}=\lambda_{3}, \quad T_{22}=T_{33}=0 . \tag{2.179}
\end{equation*}
$$

The first condition was applied as we expect the lateral contractions to be the same in the $y$ and $z$-directions. The second condition was applied as we are assuming that the only force applied is in the $x$-direction.

These conditions allowed expressions for $\lambda_{2}$ and $\lambda_{3}$ to be determined as a function of $\lambda_{1}$ :

$$
\begin{equation*}
\lambda_{2}=\lambda_{3}=\frac{1}{\sqrt{2}} \sqrt{\frac{1}{\lambda_{1}}-\frac{\mu}{\kappa \lambda_{1}^{2}}+\frac{\sqrt{4 \kappa \mu \lambda_{1}^{2}+\left(\mu-\kappa \lambda_{1}\right)^{2}}}{\kappa \lambda_{1}^{2}}} \tag{2.180}
\end{equation*}
$$

therefore, they could be eliminated from the expression for $T_{11}$ so that $T_{11}$ could be plotted as a function of $\lambda_{1}$ only.


Figure 2.5: Stress-strain curves for a compressible neo-Hookean material. The solid line represents a material for which $\kappa \gg \mu(\kappa / \mu=10000)$ and the dashed line represents a material for which $\kappa=O(\mu)(\kappa / \mu=1)$.

In the figure, the values of selected for the plots were $\kappa / \mu=10000$ for the solid line and $\kappa / \mu=1$ for the dashed line.

Similarly, we can consider the following compressible extension of the MooneyRivlin strain energy function

$$
\begin{equation*}
W=\frac{\mu}{2}\left(C_{1}\left(I_{1}-3-2 \log J\right)+C_{2}\left(I_{2}-3-4 \log J\right)\right)+\frac{\kappa}{2}(J-1)^{2} . \tag{2.181}
\end{equation*}
$$

In Figure $2.6\left(C_{1}=0.8, C_{2}=0.2\right)$ and Figure $2.7\left(C_{1}=0.1, C_{2}=0.9\right)$ we plot the stress-strain curves for a compressible Mooney-Rivlin material under uniaxial tension. In both figures the solid line represents a material for which $\kappa \gg \mu$ and the dashed line represents a material for which $\kappa=O(\mu)$. The same technique and values of $\mu$ and $\kappa$ were used as for the compressible neo-Hookean strain energy function. The stresses obtained in this case were

$$
\begin{align*}
& \frac{T_{11}}{\mu}=\left(C_{1}\left(\frac{\lambda_{1}}{\lambda_{2} \lambda_{3}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)+C_{2}\left(\frac{\lambda_{1} \lambda_{2}}{\lambda_{3}}+\frac{\lambda_{1} \lambda_{3}}{\lambda_{2}}-\frac{2}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right), \\
& \left.\frac{T_{22}}{\mu}=\left(C_{1}\left(\frac{\lambda_{2}}{\lambda_{1} \lambda_{3}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)\right)+C_{2}\left(\frac{\lambda_{1} \lambda_{2}}{\lambda_{3}}+\frac{\lambda_{2} \lambda_{3}}{\lambda_{1}}-\frac{2}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right), \tag{2.182}
\end{align*}
$$



Figure 2.6: Stress-strain curves for a compressible Mooney-Rivlin material ( $C_{1}=$ $\left.0.8, C_{2}=0.2\right)$. The solid line represents a material for which $\kappa \gg \mu(\kappa / \mu=10000)$ and the dashed line represents a material for which $\kappa=O(\mu)(\kappa / \mu=1)$.

$$
\begin{equation*}
\left.\frac{T_{33}}{\mu}=\left(C_{1}\left(\frac{\lambda_{3}}{\lambda_{1} \lambda_{2}}-\frac{1}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)\right)+C_{2}\left(\frac{\lambda_{1} \lambda_{3}}{\lambda_{2}}+\frac{\lambda_{2} \lambda_{3}}{\lambda_{1}}-\frac{2}{\lambda_{1} \lambda_{2} \lambda_{3}}\right)\right)+\frac{\kappa}{\mu}\left(\lambda_{1} \lambda_{2} \lambda_{3}-1\right) . \tag{2.184}
\end{equation*}
$$

The contractions $\lambda_{2}$ and $\lambda_{3}$ were determined to be

$$
\begin{align*}
& \lambda_{2}=\lambda_{3}= \\
& \frac{1}{\sqrt{2}} \sqrt{\frac{\kappa \lambda_{1}-\mu\left(C_{1}+C_{2} \lambda_{1}^{2}\right)+\sqrt{\left(\kappa \lambda_{1}-\mu\left(C_{1}+C_{2} \lambda_{1}^{2}\right)\right)^{2}+4 \mu\left(C_{1}+2 C_{2}\right)\left(C_{2} \mu+\kappa \lambda_{1}^{2}\right)}}{C_{2} \mu+\kappa \lambda_{1}^{2}}} . \tag{2.185}
\end{align*}
$$

For further information on strain energy functions for rubberlike materials and matching models to experimental data see, for example, Valanis and Landel [103], Blatz et al. [13], James and Treloar [50], Horgan and Saccomandi [45], Ogden et al. [74], Nah et al. [65] or Hoss and Marczak [46].

In this thesis we utilise incompressible neo-Hookean and Mooney-Rivlin strain energy functions. These strain energy functions have been selected due to the fact that they exhibit reasonable agreement with experiments on rubberlike materials, but also retain a degree of mathematical simplicity. We are interested in inhomogeneous


Figure 2.7: Stress-strain curves for a compressible Mooney-Rivlin material ( $C_{1}=$ $\left.0.1, C_{2}=0.9\right)$. The solid line represents a material for which $\kappa \gg \mu(\kappa / \mu=10000)$ and the dashed line represents a material for which $\kappa=O(\mu)(\kappa / \mu=1)$.
deformations and so it was important to use relatively simple strain energy functions in order to give the simplest possible incremental equations. More complex strain energy functions could be used to better match experimental data, however, the neo-Hookean and Mooney-Rivlin models should provide a good first approximation.

### 2.2.5 The incompressible limit

Much work has been done on taking the limit of a compressible material as it tends towards incompressibility. Creating a general theory for a 'nearly incompressible' material is difficult however. In the incompressible limit, we know that certain quantities have certain behaviours. For example, the Poisson's ratio $\nu \rightarrow \frac{1}{2}$ (see Section 2.3), the ratios $\frac{\mu}{\kappa}$ and $\frac{\mu}{\lambda}$ tend to 0 and the third strain invariant $I_{3} \rightarrow 1$. It is this final limit which causes difficulty since the elastic response function $\beta_{0}$ from (2.82) becomes indeterminate in this limit due to the $\frac{\partial W}{\partial I_{3}}$ term. For an incompressible material, we introduce the function $Q$ to account for the indeterminability of $\beta_{0}$. The form of $Q$ must be determined via the incompressibility condition and governing equations.

For some problems involving 'nearly incompressible' nonlinear elastic materials see, for example, the paper by Faulkner, published in 1971 [33], the paper by Chattopadhyay and Rogerson, published in 2001 [16], the paper by Kobayashi and Vanderby, published in 2007 [52] or the paper by Mott et al., published in 2008 [64]. In [33], Faulkner considers finite dynamic deformations of a nearly incompressible elastic spherical shell; in [16], Chattopadhyay and Rogerson discuss wave reflection in nearly incompressible, finitely deformed elastic media; in [52], Kobayashi and Vanderby present an acoustoelastic analysis of reflected waves in nearly incompressible, hyper-elastic materials; and in [64], Mott et al. discuss the behviour of the bulk modulus and Poisson's ratio of an elastic material in the incompressible limit.

### 2.2.6 Static nonlinear elasticity

There is an extremely extensive literature on the static deformation of nonlinear elastic materials. It ranges from the early texts by Green and Zerna [38] (first published in 1954) and Green and Adkins [37] (1960) to recent papers on, for example, the deformation of fiber-reinforced composites [21], [22] or the stress field in a pulled cork [23] (the latter paper involves the use of the 'semi-inverse method' which was reviewed in a paper by De Pascalis et al., published in 2009 [24]).

The problems in this thesis involve an initial finite deformation of either a cylindrical annulus, a cylindrical cavity, or a spherical cavity. Two examples of papers which involve finite deformation in a cylindrical coordinate system are the paper by Tait and Harrow, published in 1985 [98], in which a perturbation method was presented for the analysis of torsion of a compressible hyperelastic cylinder and the paper by Destrade et al., published in 2010 [25], in which the deformation of a sector of a circular cylindrical tube into an intact tube was discussed. Two examples of papers which involve finite deformation of a sphere are the paper by Abeyaratne and Horgan, published in 1984 [1], in which the deformation of a hollow sphere under pressure was considered for a special class of compressible materials called harmonic materials, and the paper by Haughton, published in 1987 [40], in which the inflation and bifurcation
of thick-walled compressible elastic shells was discussed.. Finite deformation of an infinite material containing a spherical cavity is considered in the thesis by Parnell [79].

### 2.3 Linear elasticity

In linear elasticity, it is assumed that the components of the displacement gradient tensor are small, so that

$$
\begin{equation*}
\boldsymbol{D}^{T} \boldsymbol{D} \approx \mathbf{0} \tag{2.186}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\boldsymbol{E} \approx \frac{1}{2}\left(\boldsymbol{D}+\boldsymbol{D}^{T}\right) . \tag{2.187}
\end{equation*}
$$

We, therefore, define the linear strain tensor as

$$
\begin{equation*}
\boldsymbol{e}=\frac{1}{2}\left(\boldsymbol{D}+\boldsymbol{D}^{T}\right), \tag{2.188}
\end{equation*}
$$

which can be expressed in index notation in Cartesian coordinates as

$$
\begin{equation*}
e_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial X_{j}}+\frac{\partial u_{j}}{\partial X_{i}}\right) . \tag{2.189}
\end{equation*}
$$

Note that the displacement $\boldsymbol{u}$ here is defined in the same way as $\boldsymbol{U}$ in Section 2.2, however, we use the lower case $\boldsymbol{u}$ to emphasise that we are in the context of linear elasticity. The displacements in this section are equivalent to the displacements in the small-on-large analysis with no initial large displacement.

We define the linear rotation tensor as

$$
\begin{equation*}
\boldsymbol{\omega}=\frac{1}{2}\left(\boldsymbol{D}-\boldsymbol{D}^{T}\right), \tag{2.190}
\end{equation*}
$$

which can be expressed in index notation in Cartesian coordinates as

$$
\begin{equation*}
\omega_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial X_{j}}-\frac{\partial u_{j}}{\partial X_{i}}\right) . \tag{2.191}
\end{equation*}
$$

The stress-strain relationship for three-dimensional linear elasticity is based on a generalisation of Hooke's law, and can be expressed as

$$
\begin{equation*}
\boldsymbol{\sigma}=\boldsymbol{c}: \boldsymbol{e} \tag{2.192}
\end{equation*}
$$

where $\boldsymbol{\sigma}$ is the linear Cauchy stress and $\boldsymbol{c}$ is a fourth order tensor often referred to as the stiffness tensor or the elasticity tensor. The neo-Hookean and Mooney-Rivlin strain energy functions provide a generalisation of Hooke's law for large deformations.

For small-on-large problems, it is useful to consider the no pre-stress limit. In this case, when the large deformation is zero, the incremental response of the material is just a standard linear elastic response and all of the incremental linear stress tensors are the same; in particular, we have $\boldsymbol{\tau}=\boldsymbol{\sigma}$.

In the case of an isotropic material, equation (2.192) can be reduced to

$$
\begin{equation*}
\boldsymbol{\sigma}=\lambda \operatorname{tr}(\boldsymbol{e}) \boldsymbol{I}+2 \mu \boldsymbol{e}, \tag{2.193}
\end{equation*}
$$

which can be written in index notation as

$$
\begin{equation*}
\sigma_{i j}=\lambda \delta_{i j} e_{k k}+2 \mu e_{i j} \tag{2.194}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta, $\lambda$ is Lamé's first parameter and $\mu$ is Lamé's second parameter, also known as the shear modulus.

The following quantities can be expressed in terms of the Lamé constants: Young's modulus:

$$
\begin{equation*}
E=\frac{\mu(3 \lambda+2 \mu)}{\lambda+\mu} \tag{2.195}
\end{equation*}
$$

Poisson's ratio:

$$
\begin{equation*}
\nu=\frac{\lambda}{2(\lambda+\mu)}, \tag{2.196}
\end{equation*}
$$

bulk modulus:

$$
\begin{equation*}
\kappa=\lambda+\frac{2}{3} \mu . \tag{2.197}
\end{equation*}
$$

The governing equations in linear elasticity are normally expressed in terms of the undeformed coordinates, and so we write

$$
\begin{equation*}
\operatorname{Div} \boldsymbol{\sigma}+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.198}
\end{equation*}
$$

or, in the absence of body forces,

$$
\begin{equation*}
\operatorname{Div} \boldsymbol{\sigma}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.199}
\end{equation*}
$$

where Div represents the divergence operator with respect to the undeformed coordinate system.

It is worth stressing, however, that a linear elastic deformation is equivalent to a small-on-large problem where there is no initial deformation (i.e. the deformed configuration is the same as the undeformed configuration). So we could equivalently write

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\sigma}+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.200}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\sigma}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.201}
\end{equation*}
$$

Upon using (2.193), (2.188) and (2.44), the above equations can be formulated in terms of the displacements, $\boldsymbol{u}$, as follows:

$$
\begin{equation*}
(\lambda+\mu) \nabla(\nabla \cdot \boldsymbol{u})+\mu \nabla^{2} \boldsymbol{u}+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.202}
\end{equation*}
$$

or, in the absence of body forces,

$$
\begin{equation*}
(\lambda+\mu) \nabla(\nabla \cdot \boldsymbol{u})+\mu \nabla^{2} \boldsymbol{u}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{2.203}
\end{equation*}
$$

where $\nabla$ is defined with respect to the undeformed configuration so that, in Cartesian coordinates, for example, we have

$$
\nabla=\left(\begin{array}{c}
\frac{\partial}{\partial X_{1}}  \tag{2.204}\\
\frac{\partial}{\partial X_{2}} \\
\frac{\partial}{\partial X_{3}}
\end{array}\right)=\left(\begin{array}{c}
\frac{\partial}{\partial X} \\
\frac{\partial}{\partial Y} \\
\frac{\partial}{\partial Z}
\end{array}\right) .
$$

The above governing equations can also be expressed in index notation in Cartesian coordinates as

$$
\begin{equation*}
(\lambda+\mu) u_{j, j i}+\mu u_{i, j j}+\rho B_{i}=\rho \ddot{u}_{i}, \tag{2.205}
\end{equation*}
$$

and

$$
\begin{equation*}
(\lambda+\mu) u_{j, j i}+\mu u_{i, j j}=\rho \ddot{u}_{i}, \tag{2.206}
\end{equation*}
$$

respectively.
Upon using the vector identity

$$
\begin{equation*}
\nabla^{2} \boldsymbol{u}=\nabla(\nabla \cdot \boldsymbol{u})-\nabla \times(\nabla \times \boldsymbol{u}) \tag{2.207}
\end{equation*}
$$

(2.198) can be rewritten as

$$
\begin{equation*}
(\lambda+2 \mu) \nabla(\nabla \cdot \boldsymbol{u})-\mu \nabla \times(\nabla \times \boldsymbol{u})+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} . \tag{2.208}
\end{equation*}
$$

Recalling the definition of $\boldsymbol{\omega}$, we can rewrite the above as

$$
\begin{equation*}
(\lambda+2 \mu) \nabla \triangle-2 \mu \nabla \times \boldsymbol{\omega}+\rho \boldsymbol{B}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}}, \tag{2.209}
\end{equation*}
$$

where we have defined the dilatation, $\Delta$, as

$$
\begin{equation*}
\triangle=\nabla \cdot \boldsymbol{u} \tag{2.210}
\end{equation*}
$$

Using the Helmholtz decomposition theorem, $\boldsymbol{u}$ can be rewritten in terms of the scalar and vector potentials $\Phi$ and $\boldsymbol{H}$ as follows:

$$
\begin{equation*}
\boldsymbol{u}=\nabla \Phi+\nabla \times \boldsymbol{H} \tag{2.211}
\end{equation*}
$$

where $\boldsymbol{H}$ satisfies

$$
\begin{equation*}
\nabla \cdot \boldsymbol{H}=0 \tag{2.212}
\end{equation*}
$$

Similarly, $\boldsymbol{B}$ can be rewritten in terms of the scalar and vector potentials $\Psi$ and $\boldsymbol{F}$ as

$$
\begin{equation*}
\boldsymbol{B}=\nabla \Psi+\nabla \times \boldsymbol{F} \tag{2.213}
\end{equation*}
$$

where $\boldsymbol{F}$ satisfies

$$
\begin{equation*}
\nabla \cdot \boldsymbol{F}=0 \tag{2.214}
\end{equation*}
$$

After substituting (2.211) and (2.213) into (2.202), we obtain

$$
\begin{equation*}
(\lambda+\mu) \nabla \nabla \cdot(\nabla \Phi+\nabla \times \boldsymbol{H})+\mu \nabla^{2}(\nabla \Phi+\nabla \times \boldsymbol{H})+\rho(\nabla \Psi+\nabla \times \boldsymbol{F})=\rho(\nabla \ddot{\Phi}+\nabla \times \ddot{\boldsymbol{H}}) . \tag{2.215}
\end{equation*}
$$

This can be rearranged as

$$
\begin{equation*}
\nabla\left((\lambda+2 \mu) \nabla^{2} \Phi+\rho \Psi-\rho \ddot{\Phi}\right)+\nabla \times\left(\mu \nabla^{2} \boldsymbol{H}+\rho \boldsymbol{F}-\rho \ddot{\boldsymbol{H}}\right)=0, \tag{2.216}
\end{equation*}
$$

which will satisfied if

$$
\begin{equation*}
(\lambda+2 \mu) \nabla^{2} \Phi+\rho \Psi=\rho \ddot{\Phi}, \tag{2.217}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu \nabla^{2} \boldsymbol{H}+\rho \boldsymbol{F}=\rho \ddot{\boldsymbol{H}} \tag{2.218}
\end{equation*}
$$

In the absence of body forces, the above equations reduce to

$$
\begin{equation*}
(\lambda+2 \mu) \nabla^{2} \Phi=\rho \ddot{\Phi}, \tag{2.219}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu \nabla^{2} \boldsymbol{H}=\rho \ddot{\boldsymbol{H}} . \tag{2.220}
\end{equation*}
$$

### 2.4 Linear elastic waves in unstressed solids

In this thesis, we are concerned with the propagation of waves through elastic solids in the absence of body forces. In particular, we will consider the case where we have linear, time harmonic waves, so that the displacement $\boldsymbol{u}$ takes the form

$$
\begin{equation*}
\boldsymbol{u}=\Re\left(\hat{\boldsymbol{u}} e^{-i \omega t}\right), \tag{2.221}
\end{equation*}
$$

where $\hat{\boldsymbol{u}}$ is a function of the spatial coordinates only and $\omega$ is the angular frequency of the waves.

## Dilatational waves

If we take the divergence of (2.203) we obtain

$$
\begin{equation*}
(\lambda+\mu) \nabla \cdot(\nabla \nabla \cdot \boldsymbol{u})+\mu \nabla \cdot\left(\nabla^{2} \boldsymbol{u}\right)=\rho \nabla \cdot \boldsymbol{u} \tag{2.222}
\end{equation*}
$$

which can be rewritten in terms of the dilatation as

$$
\begin{equation*}
(\lambda+2 \mu) \nabla^{2} \triangle=\rho \frac{\partial^{2} \triangle}{\partial t^{2}} \tag{2.223}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
\nabla^{2} \triangle=\frac{1}{c_{1}^{2}} \frac{\partial^{2} \triangle}{\partial t^{2}} \tag{2.224}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{1}=\sqrt{\frac{\lambda+2 \mu}{\rho}} \tag{2.225}
\end{equation*}
$$

We recognise (2.224) as the wave equation, where $c_{1}$ is the propagation velocity, hence dilatational waves in a linear elastic material propagate at the velocity $c_{1}$.

## Rotational waves

If we take the curl of (2.203) we similarly obtain

$$
\begin{equation*}
\mu \nabla^{2} \boldsymbol{\omega}=\rho \frac{\partial^{2} \boldsymbol{\omega}}{\partial t^{2}} \tag{2.226}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
\nabla^{2} \boldsymbol{\omega}=\frac{1}{c_{1}^{2}} \frac{\partial^{2} \boldsymbol{\omega}}{\partial t^{2}}, \tag{2.227}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{2}=\sqrt{\frac{\mu}{\rho}} . \tag{2.228}
\end{equation*}
$$

We again recognise (2.227) as a scalar wave equation, where $c_{2}$ is the propagation velocity, hence rotational waves in a linear elastic material propagate at the velocity $c_{2}$.

Specific examples of rotational waves which appear in this thesis are SH waves, SV waves and torsional waves.

### 2.4.1 Propagation and reflection of waves in a half-space

Here we reproduce, with some slight changes and corrections, a problem which appears in [36] in order to illustrate the difference between SH and SV waves.

When elastic waves are incident upon a boundary between two media, some of the energy carried by the waves is reflected from the boundary, and some is transmitted across it. In this thesis, we are generally concerned with waves impinging upon free surfaces and, in this case, all of the energy carried by the wave is reflected. For certain types of boundaries and incident waves, a process called mode conversion occurs, whereby the incident wave is converted into two reflected waves. This process, and the fact that two distinct types of wave can propagate in an elastic material causes elastic wave problems to be more complex than acoustic and many electromagnetic wave problems.

Here we consider time-harmonic plane waves propagating in the half space $y \geq 0$. With no loss of generality, we assume that the wave normal $\boldsymbol{n}$ lies in the $x, y$-plane, which we will call the vertical plane. We call the $x$, $z$-plane, which is the surface of


Figure 2.8: Plane wave, with wave normal $\boldsymbol{n}$ in the $x, y$-(vertical) plane advancing toward a free surface. This figure is a reproduction of the figure given on page 312 of Graff 1991 [36].
the half-space, the horizontal plane. Dilatational wave motion will be in the direction of the wave normal and will, therefore, lie completely in the vertical plane. Shear wave motion may have components both in the vertical plane and parallel to the horizontal plane. The impingement of a general plane wave with normal $\boldsymbol{n}$ upon the free surface $y=0$ is shown in Figure 2.8. The component of the displacement in the normal direction is $u_{n}$ and the transverse components are $u_{v}$ and $u_{z}$, which are in the vertical and horizontal planes, respectively. The motion does not very with $z$ if the wave normal is in the vertical plane since every point of the wave along that axis is undergoing the same motion and has the same phase.

The governing equations for this problem are

$$
\begin{gather*}
u_{x}=\frac{\partial \Phi}{\partial x}+\frac{\partial H_{z}}{\partial y}, \quad u_{y}=\frac{\partial \Phi}{\partial y}-\frac{\partial H_{z}}{\partial x}, \quad u_{z}=-\frac{\partial H_{x}}{\partial y}+\frac{\partial H_{y}}{\partial x}  \tag{2.229}\\
\frac{\partial H_{x}}{\partial x}+\frac{\partial H_{y}}{\partial y}=0  \tag{2.230}\\
\nabla^{2} \Phi=\frac{1}{c_{1}^{2}} \frac{\partial^{2} \Phi}{\partial t^{2}}, \quad \nabla^{2} H_{i}=\frac{1}{c_{2}^{2}} \frac{\partial^{2} H_{i}}{\partial t^{2}}, \quad(i=x, y, z), \tag{2.231}
\end{gather*}
$$

where we have used the $z$ independence of all quantities. Here, $\Phi$ is the scalar potential associated with the dilatational part of the Helmholtz decomposition of the
displacement vector $\boldsymbol{u}$, and $H_{x}, H_{y}$ and $H_{z}$ are the components of the vector potential associated with the rotational part of the Helmholtz decomposition of $\boldsymbol{u}$ (see equation (2.211)). Equation (2.230) results from $\nabla \cdot \boldsymbol{H}=0$.

The Cauchy stresses are given by

$$
\begin{gather*}
\sigma_{x x}=(\lambda+2 \mu)\left(\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}\right)-2 \mu \frac{\partial u_{y}}{\partial y}  \tag{2.232}\\
\sigma_{y y}=(\lambda+2 \mu)\left(\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}\right)-2 \mu \frac{\partial u_{x}}{\partial x}  \tag{2.233}\\
\sigma_{z z}=\frac{\lambda}{2(\lambda+\mu)}\left(\sigma_{x x}+\sigma_{y y}\right),  \tag{2.234}\\
\sigma_{x y}=\mu\left(\frac{\partial u_{x}}{\partial y}+\frac{\partial u_{y}}{\partial x}\right)  \tag{2.235}\\
\sigma_{y z}=\mu \frac{\partial u_{z}}{\partial y}, \quad \sigma_{x z}=\mu \frac{\partial u_{z}}{\partial x} . \tag{2.236}
\end{gather*}
$$

Note that, in [36], $\sigma_{x z}$ was incorrectly given as 0 in the above. From (2.229) - (2.231), we can rewrite the above equations in terms of the potentials $\Phi, H_{x}, H_{y}$ and $H_{z}$ as

$$
\begin{gather*}
\sigma_{x x}=(\lambda+2 \mu)\left(\frac{\partial^{2} \Phi}{\partial x^{2}}+\frac{\partial^{2} \Phi}{\partial y^{2}}\right)-2 \mu\left(\frac{\partial^{2} \Phi}{\partial y^{2}}-\frac{\partial^{2} H_{z}}{\partial x \partial y}\right),  \tag{2.237}\\
\sigma_{y y}=(\lambda+2 \mu)\left(\frac{\partial^{2} \Phi}{\partial x^{2}}+\frac{\partial^{2} \Phi}{\partial y^{2}}\right)-2 \mu\left(\frac{\partial^{2} \Phi}{\partial x^{2}}-\frac{\partial^{2} H_{z}}{\partial x \partial y}\right),  \tag{2.238}\\
\sigma_{z z}=\lambda\left(\frac{\partial^{2} \Phi}{\partial x^{2}}+\frac{\partial^{2} \Phi}{\partial y^{2}}\right),  \tag{2.239}\\
\sigma_{x y}=\mu\left(2 \frac{\partial^{2} \Phi}{\partial x \partial y}+\frac{\partial^{2} H_{z}}{\partial y^{2}}-\frac{\partial^{2} H_{z}}{\partial x^{2}}\right),  \tag{2.240}\\
\sigma_{y z}=\mu\left(-\frac{\partial H_{x}}{\partial y^{2}}+\frac{\partial^{2} H_{y}}{\partial x \partial y}\right), \quad \sigma_{x z}=\mu\left(-\frac{\partial H_{x}}{\partial x \partial y}+\frac{\partial^{2} H_{y}}{\partial x^{2}}\right) . \tag{2.241}
\end{gather*}
$$

Again, in [36], $\sigma_{x z}$ was incorrectly given as 0 .
Finally, we have the boundary conditions given by

$$
\begin{equation*}
\left.\sigma_{y y}\right|_{y=0}=\left.\sigma_{x y}\right|_{y=0}=\left.\sigma_{y z}\right|_{y=0}=0 \tag{2.242}
\end{equation*}
$$

We observe here that the waves we are considering decouple into two distinct types. We note that $u_{x}, u_{y}$ depend only on $\Phi$ and $H_{z}$, which are governed by (2.231), where $i=z$ in the second of the two equations, and the stresses $\sigma_{x x}, \sigma_{y y}$ and $\sigma_{x y}$ only depend on $u_{x}$ and $u_{y}$, and, therefore, only on $\Phi$ and $H_{z}$. We also note that $u_{z}$ depends only
on $H_{x}$ and $H_{y}$ which are governed by the second equation of (2.231), where $i=x, y$, and the stresses $\sigma_{y z}$ and $\sigma_{x z}$ depend only on $u_{z}$, and, hence, only on $H_{x}$ and $H_{y}$. Due to this decoupling, we can resolve the motion into two parts, where one is a plane strain wave motion, with $u_{z}=0, u_{x}=u_{x}(x, y) \neq 0$ and $u_{y}=u_{y}(x, y) \neq 0$, and the other is SH wave motion, where $u_{x}=u_{y}=0$ and $u_{z}=u_{z}(x, y) \neq 0$. This decoupling would become apparent in the boundary condition equations if it was not noted here.

In summary, we have the following:
Plane strain: $u_{z}=0, u_{x}=u_{x}(x, y), u_{y}=u_{y}(x, y)$,

$$
\begin{align*}
& u_{x}=\frac{\partial \Phi}{\partial x}+\frac{\partial H_{z}}{\partial y}, \quad u_{y}=\frac{\partial \Phi}{\partial y}-\frac{\partial H_{z}}{\partial x}  \tag{2.243}\\
& \nabla^{2} \Phi=\frac{1}{c_{1}^{2}} \frac{\partial^{2} \Phi}{\partial t^{2}}, \quad \nabla^{2} H_{z}=\frac{1}{c_{2}^{2}} \frac{\partial^{2} H_{z}}{\partial t^{2}} \tag{2.244}
\end{align*}
$$

with $\sigma_{x x}, \sigma_{y y}, \sigma_{z z}, \sigma_{x y}$ given by equations (2.232) - (2.235) and (2.237) - (2.240). The boundary conditions are

$$
\begin{equation*}
\left.\sigma_{y y}\right|_{y=0}=\left.\sigma_{x y}\right|_{y=0}=0 . \tag{2.245}
\end{equation*}
$$

SH waves: $u_{x}=u_{y}=0, u_{z}=u_{z}(x, y)$,

$$
\begin{gather*}
u_{z}=-\frac{\partial H_{x}}{\partial y}+\frac{\partial H_{y}}{\partial x}, \quad \frac{\partial H_{x}}{\partial x}+\frac{\partial H_{y}}{\partial y}=0  \tag{2.246}\\
\nabla^{2} H_{x}=\frac{1}{c_{2}^{2}} \frac{\partial^{2} H_{x}}{\partial t^{2}}, \quad \nabla^{2} H_{y}=\frac{1}{c_{2}^{2}} \frac{\partial^{2} H_{y}}{\partial t^{2}} \tag{2.247}
\end{gather*}
$$

with $\sigma_{y z}$ given by equations (2.236) and (2.241). The boundary conditions are

$$
\begin{equation*}
\left.\sigma_{y z}\right|_{y=0}=0 . \tag{2.248}
\end{equation*}
$$

Note that we could directly consider the displacement equation of motion in the case of SH waves:

$$
\begin{equation*}
\nabla^{2} u_{z}=\frac{1}{c_{2}^{2}} \frac{\partial^{2} u_{z}}{\partial t^{2}} . \tag{2.249}
\end{equation*}
$$

Now consider the solution in the plane strain case and let

$$
\begin{equation*}
\Phi=f(y) e^{i(\xi x-\omega t)}, \quad H_{z}=h_{z}(y) e^{i(\zeta x-\omega t)} \tag{2.250}
\end{equation*}
$$

Upon substituting the above into the governing equations (2.244), we obtain

$$
\begin{equation*}
\frac{d^{2} f}{d y^{2}}+\alpha^{2} f=0, \quad \frac{d^{2} h_{z}}{d y^{2}}+\beta^{2} h_{z}=0 \tag{2.251}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha^{2}=\frac{\omega^{2}}{c_{1}^{2}}-\xi^{2}, \quad \beta^{2}=\frac{\omega^{2}}{c_{2}^{2}}-\zeta^{2} . \tag{2.252}
\end{equation*}
$$

The plane wave solutions for $\Phi$ and $H_{z}$ are then given by

$$
\begin{align*}
\Phi & =A_{1} e^{i(\xi x-\alpha y-\omega t)}+A_{2} e^{i(\xi x+\alpha y-\omega t)}  \tag{2.253}\\
H_{z} & =B_{1} w^{i(\zeta x-\beta y-\omega t)}+B_{2} e^{i(\zeta x+\beta y-\omega t)} . \tag{2.254}
\end{align*}
$$

We refer to the wave associated with $\Phi$ as a P wave and the wave associated with $H_{z}$ as an SV wave.

If we define $\theta_{1}$ and $\theta_{2}$ as the angles between the $y$-axis and the wave normal of the dilational and shear waves, respectively, then we may write

$$
\begin{array}{ll}
\xi=\gamma_{1} \sin \theta_{1}, & \alpha=\gamma_{1} \cos \theta_{1}, \\
\zeta=\gamma_{2} \sin \theta_{2}, & \beta=\gamma_{2} \cos \theta_{2}, \tag{2.256}
\end{array}
$$

where $\gamma_{1}$ and $\gamma_{2}$ are the wavenumbers of the P and SV waves, respectively. Hence, we may write $\Phi$ and $H_{z}$, respectively, as

$$
\begin{equation*}
\Phi=A_{1} e^{\left(i \gamma_{1}\left(\sin \theta_{1} x-\cos \theta_{1} y-c_{1} t\right)\right.}+A_{2} E^{i \gamma_{1}\left(\sin \theta_{1} x+\cos \theta_{1} y-c_{1} t\right)} \tag{2.257}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{z}=B_{1} e^{i \gamma_{2}\left(\sin \theta_{2} x-\cos \theta_{2} y-c_{2} t\right)}+B_{2} e^{i \gamma_{2}\left(\sin \theta_{2} x+\cos \theta_{2} y-c_{2} t\right)} \tag{2.258}
\end{equation*}
$$

We now apply the solutions (2.257) and (2.258) to the plane-strain boundary conditions (2.245), obtaining

$$
\begin{equation*}
\gamma_{1}^{2}\left(2 \sin ^{2} \theta_{1}-k^{2}\right)\left(A_{1}+A_{2}\right) e^{i \gamma_{1}\left(\sin \theta_{1} x-c_{1} t\right)}-\gamma_{2}^{2} \sin 2 \theta_{2}\left(B_{1}-B_{2}\right) e^{i \gamma_{2}\left(\sin \theta_{2} x-c_{2} t\right)}=0 \tag{2.259}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{1}^{2} \sin 2 \theta_{1}\left(A_{1}-A_{2}\right) e^{i \gamma_{1}\left(\sin \theta_{1} x-c_{1} t\right)}-\gamma_{2}^{2} \cos 2 \theta_{2}\left(B_{1}+B_{2}\right) e^{i \gamma_{2}\left(\sin \theta_{2} x-c_{2} t\right)}=0 \tag{2.260}
\end{equation*}
$$

where

$$
\begin{equation*}
k^{2}=\frac{c_{1}^{2}}{c_{2}^{2}}=\frac{\lambda+2 \mu}{\mu} \tag{2.261}
\end{equation*}
$$

We can immediately factor out $e^{-i \omega t}$ from the above. If these results are to hold for arbitrary $x$, then we must be able to factor $e^{i \gamma_{1} \sin \theta_{1} x}$ and $e^{i \gamma_{2} \sin \theta_{2} x}$ from them. This can only occur if we have

$$
\begin{equation*}
\gamma_{1} \sin \theta_{1}=\gamma_{2} \sin \theta_{2} \tag{2.262}
\end{equation*}
$$

Since $\omega=\gamma_{1} c_{1}=\gamma_{2} c_{2}$, so that $\gamma_{2} / \gamma_{1}=c_{1} / c_{2}=k$, we may write the above as

$$
\begin{equation*}
\frac{\gamma_{2}}{\gamma_{1}}=\frac{\sin \theta_{1}}{\sin \theta_{2}}=k \tag{2.263}
\end{equation*}
$$

This result may be interpreted as the form of Snell's law for elastic waves. With this, the boundary condition equations reduce to

$$
\begin{equation*}
\gamma_{1}^{2}\left(2 \sin ^{2} \theta_{1}-k^{2}\right)\left(A_{1}+A_{2}\right)-\gamma_{2}^{2} \sin 2 \theta_{2}\left(B_{1}-B_{2}\right)=0, \tag{2.264}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{1}^{2} \sin 2 \theta_{1}\left(A_{1}-A_{2}\right)-\gamma_{2}^{2} \cos 2 \theta_{2}\left(B_{1}+B_{2}\right)=0 . \tag{2.265}
\end{equation*}
$$

This governs the reflection of plane waves in a half-space. Note that, in [36], $\gamma_{1}$ was incorrectly given as $\gamma_{2}$ in equation (2.265). We observe from this result that P and SV waves are coupled (i.e. they must both be present to satisfy the boundary conditions). Therefore, if the incident field is either a purely P wave or a purely SV wave, then both P and SV waves will be reflected. As we shall see in Section 2.4.5 and Chapter 6, when a plane wave impinges on a spherical cavity there will also be mode conversion and, therefore, both P waves and shear waves will be scattered.

We now consider the solution in the SH wave case, and let

$$
\begin{equation*}
H_{x}=h_{x}(y) e^{i(\xi x-\omega t)}, \quad H_{y}=h_{y}(y) e^{i(\xi x-\omega t)} . \tag{2.266}
\end{equation*}
$$

Substituting the above into the governing equations (2.247), we obtain

$$
\begin{equation*}
\frac{d^{2} h_{x}}{d y^{2}}+\eta^{2} h_{x}=0, \quad \frac{d^{2} h_{y}}{d y^{2}}+\eta^{2} h_{y}=0 \tag{2.267}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta^{2}=\frac{\omega^{2}}{c_{2}^{2}}-\xi^{2} . \tag{2.268}
\end{equation*}
$$

The solutions are

$$
\begin{equation*}
H_{x}=C_{1} e^{i(\xi x-\eta y-\omega t)}+C_{2} e^{i(\xi x+\eta y-\omega t)}, \tag{2.269}
\end{equation*}
$$

$$
\begin{equation*}
H_{y}=D_{1} e^{i(\xi x-\eta y-\omega t)}+D_{2} e^{i(\xi x+\eta y-\omega t)} . \tag{2.270}
\end{equation*}
$$

These results may also be expressed in terms of an incidence angle $\theta_{3}$, as was done previously for $\Phi$ and $H_{z}$.

Due to the fact that $\nabla \cdot \boldsymbol{H}=0$, not all of the above quantities are independent. This allows us to determine the three components of the displacement from the four components of $\Phi$ and $\boldsymbol{H}$. It was found earlier that $u_{x}$ and $u_{y}$ depend only on $\Phi$ and $H_{z}$, and are, therefore, uniquely determined. This is not the case for $u_{z}$, which depends on $H_{x}$ and $H_{y}$. In the present case of $z$ independence, the divergence condition is given by equation (2.230).

Applying this condition to the results (2.269) and (2.270) gives

$$
\begin{equation*}
i \xi\left(C_{1} e^{-i \eta y}+C_{2} e^{i \eta y}\right)+i \eta\left(-D_{1} e^{-i \eta y}+D_{2} e^{i \eta y}\right)=0 \tag{2.271}
\end{equation*}
$$

where $e^{i(\xi x-\omega t)}$ has been factored out. This re-groups to

$$
\begin{equation*}
\left(\xi C_{1}-\eta D_{1}\right) e^{-i \eta y}+\left(\xi C_{2}+\eta D_{2}\right) e^{i \eta y}=0 . \tag{2.272}
\end{equation*}
$$

In order for the above result to hold for all $y$, we must have $C_{1}, D_{1}, C_{2}$ and $D_{2}$ related by

$$
\begin{equation*}
\xi C_{1}=\eta D_{1}, \quad \xi C_{2}=-\eta D_{2} . \tag{2.273}
\end{equation*}
$$

Therefore, two of the constants of (2.269) and (2.270) may be eliminated. We arbitrarily choose to eliminate $D_{1}$ and $D_{2}$, to obtain

$$
\begin{gather*}
H_{x}=C_{1} e^{i(\xi x-\eta y-\omega t)}+C_{2} e^{i(\xi x+\eta y-\omega t)}  \tag{2.274}\\
H_{y}=\frac{\xi}{\eta} C_{1} e^{i(\xi x-\eta y-\omega t)}-\frac{\xi}{\eta} C_{2} e^{i(\xi x+\eta y-\omega t)} . \tag{2.275}
\end{gather*}
$$

Substituting these results into the boundary condition $\left.\sigma_{y z}\right|_{y=0}=0$ gives

$$
\begin{equation*}
\eta^{2}\left(C_{1}+C_{2}\right)+\xi^{2}\left(C_{1}+C_{2}\right)=0, \tag{2.276}
\end{equation*}
$$

and, therefore,

$$
\begin{equation*}
C_{1}=-C_{2} . \tag{2.277}
\end{equation*}
$$

This result governs the reflection of SH waves in a half-space. The key feature of such waves is that there is no mode conversion on the boundary for an incoming

SH wave field, and so the reflected wave is also an SH wave with a reflection angle equal to the incidence angle. If a shear wave of arbitrary polarisation impinges on a free surface, the SV component of the wave will lose a portion of its energy to the generation of P waves, whereas the SH component will reflect with only a change in phase.

When SH waves are incident upon a cylindrical cavity, with axis aligned with the displacement vector of the wave, the same result is observed, i.e. there is no mode conversion and the scattered wave field still has the same polarisation as the incident field. This problem will be investigated further in Section 2.4.3 and Chapter 4.

### 2.4.2 Torsional wave propagation in an annular circular cylinder

In Chapter 3 we look at the propagation of torsional waves in a pre-stressed nonlinear elastic annular cylinder in the absence of body forces and are interested in the effect of this pre-stress on the dispersion relation for the propagating waves. As a pre-cursor to that chapter, in this section we discuss the propagation of torsional waves in an isotropic linear elastic annular cylinder and the governing equation in this section can be compared with the governing equation in Chapter 3.

For this problem we are interested in time harmonic waves propagating in the longitudinal $(z)$ direction, and so

$$
\begin{equation*}
u_{r}=0, \quad u_{\theta}=\Re\left(v(r) e^{i(k z-\omega t)}\right), \quad u_{z}=0, \tag{2.278}
\end{equation*}
$$

where $k$ is the longitudinal wavenumber and $\omega$ is the frequency of the wave. For conciseness, we shall omit the $\Re$ symbol in the following, however, the fact that we are interested in the real part of the wave should henceforth be considered implicit.

It can be shown by using (2.19) and (2.44) in (2.188) that the components of the linear strain tensor in cylindrical coordinates are as follows:

$$
\begin{gather*}
e_{r r}=\frac{\partial u_{r}}{\partial r}, \quad e_{\theta \theta}=\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}+\frac{u_{r}}{r},  \tag{2.279}\\
e_{z z}=\frac{\partial u_{z}}{\partial z}, \quad e_{r \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u_{r}}{\partial \theta}+\frac{\partial u_{\theta}}{\partial r}-\frac{u_{\theta}}{r}\right), \tag{2.280}
\end{gather*}
$$

$$
\begin{equation*}
e_{r z}=\frac{1}{2}\left(\frac{\partial u_{z}}{\partial r}+\frac{\partial u_{r}}{\partial z}\right), \quad e_{\theta z}=\frac{1}{2}\left(\frac{\partial u_{\theta}}{\partial z}+\frac{1}{r} \frac{\partial u_{z}}{\partial \theta}\right), \tag{2.281}
\end{equation*}
$$

where $u_{r}, u_{\theta}$ and $u_{z}$ are the displacements in the radial, azimuthal and longitudinal directions, respectively.

Upon substituting the above values of $u_{r}, u_{\theta}$ and $u_{z}$ into (2.279) - (2.281), the only non-zero components of the linear strain tensor we obtain are

$$
\begin{equation*}
e_{r \theta}=\frac{1}{2}\left(v^{\prime}(r)-\frac{v(r)}{r}\right) e^{i(k z-\omega t)}, \quad e_{\theta z}=\frac{i k}{2} v(r) e^{i(k z-\omega t)} . \tag{2.282}
\end{equation*}
$$

Hence, upon using (2.193) (since the material we are considering is isotropic), the only non-zero components of the linear Cauchy stress tensor we obtain are

$$
\begin{equation*}
\sigma_{r \theta}=\mu\left(v^{\prime}(r)-\frac{v(r)}{r}\right) e^{-i \omega t}, \quad \sigma_{\theta z}=\mu i k v(r) e^{i(k z-\omega t)} . \tag{2.283}
\end{equation*}
$$

The governing equations for a linear elastic solid can be written in cylindrical coordinates as (see equation (2.25))

$$
\begin{gather*}
\frac{\partial \sigma_{r r}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta}+\frac{\partial \sigma_{r z}}{\partial z}+\frac{\sigma_{r r}-\sigma_{\theta \theta}}{r}+\rho f_{r}=\rho \frac{\partial^{2} u_{r}}{\partial t^{2}}  \tag{2.284}\\
\frac{\partial \sigma_{r \theta}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2}{r} \sigma_{r \theta}+\rho f_{\theta}=\rho \frac{\partial^{2} u_{\theta}}{\partial t^{2}}  \tag{2.285}\\
\frac{\partial \sigma_{r z}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{z z}}{\partial z}+\frac{1}{r} \sigma_{r z}+\rho f_{z}=\rho \frac{\partial^{2} u_{z}}{\partial t^{2}} \tag{2.286}
\end{gather*}
$$

where $f_{r}, f_{\theta}$ and $f_{z}$ are the components of the body force in the radial, azimuthal and longitudinal directions, respectively. In this problem, body forces have negligible effect, and so

$$
\begin{equation*}
f_{r}=f_{\theta}=f_{z}=0 \tag{2.287}
\end{equation*}
$$

Upon substituting the relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ into the above, we find that the radial and longitudinal governing equations are satisfied trivially, and the azimuthal governing equation reduces to

$$
\begin{equation*}
v^{\prime \prime}(r)+\frac{1}{r} v^{\prime}(r)+\left(s^{2}+\frac{1}{r^{2}}\right) v(r)=0, \tag{2.288}
\end{equation*}
$$

where

$$
\begin{equation*}
c=\sqrt{\frac{\mu}{\rho}} \tag{2.289}
\end{equation*}
$$

is the torsional wave velocity, and

$$
\begin{equation*}
s^{2}=\frac{\omega^{2}}{c^{2}}-k^{2} . \tag{2.290}
\end{equation*}
$$

If we make the substitution

$$
\begin{equation*}
x=s r, \tag{2.291}
\end{equation*}
$$

then we can write equation (2.288) as

$$
\begin{equation*}
\frac{d^{2} v}{d x^{2}}+\frac{1}{x} \frac{d v}{d x}+\left(1-\frac{1}{x^{2}}\right) v=0 . \tag{2.292}
\end{equation*}
$$

This is Bessel's equation of order one, and so $v$ must be a linear combination of the order one Bessel functions of the first and second kinds ( $\mathrm{J}_{1}$ and $\mathrm{Y}_{1}$, respectively):

$$
\begin{equation*}
v=C_{1} \mathrm{~J}_{1}(x)+C_{2} \mathrm{Y}_{1}(x)=C_{1} \mathrm{~J}_{1}(s r)+C_{2} \mathrm{Y}_{1}(s r), \tag{2.293}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants.
Let the annular cylinder have inner radius $a$ and outer radius $b$. If we are now interested in applying traction-free boundary conditions on the inner and outer surfaces of the annular cylinder, then the equations to be satisfied on $r=a$ and $r=b$ are

$$
\begin{equation*}
\left.\boldsymbol{\sigma} \cdot \boldsymbol{n}\right|_{r=a, b}=0, \tag{2.294}
\end{equation*}
$$

where $\boldsymbol{n}$ is the outer unit normal to the surface under consideration, so on $r=a$, $\boldsymbol{n}=-\boldsymbol{e}_{\boldsymbol{r}}$ and on $r=b, \boldsymbol{n}=\boldsymbol{e}_{\boldsymbol{r}}$. Therefore, (2.294) reduces to

$$
\begin{equation*}
\left.\sigma_{\theta r}\right|_{r=a}=\left.\sigma_{\theta r}\right|_{r=b}=0, \tag{2.295}
\end{equation*}
$$

which, in turn, reduces to

$$
\begin{equation*}
v^{\prime}(a)-\frac{v(a)}{a}=v^{\prime}(b)-\frac{v(b)}{b}=0 . \tag{2.296}
\end{equation*}
$$

The values of $s$ which satisfy these boundary conditions define the dispersion relations for torsional waves in an annular cylinder via equation (2.290). The resulting dispersion curves will be plotted in Chapter 3 and compared with those for a pre-stressed annular cylinder.


Figure 2.9: SH waves incident on a cylindrical cavity.

### 2.4.3 Scattering of shear waves from a cylindrical cavity

In Chapter 4 we look at the scattering of SH waves from a cylindrical cavity in an infinite, pre-stressed nonlinear elastic material in the absence of body forces. In this section we discuss the scattering of SH waves from a cylindrical cavity in an infinite, isotropic, linear elastic material and the results in this section can be compared with those in Chapter 4. This problem appears in [36], but is reproduced here with slightly different notation.

In Figure 2.9 we illustrate the problem we are considering of SH waves propagating in the positive $x$ direction, and impinging on an infinite, cylindrical cavity of radius $a$.

Time harmonic, SH waves have the following form

$$
\begin{equation*}
u_{r}=0, \quad u_{\theta}=0, \quad u_{z}=\Re\left(w(r, \theta) e^{-i \omega t}\right), \tag{2.297}
\end{equation*}
$$

where $\omega$ is the frequency of the wave. For conciseness, we shall omit the $\Re$ symbol in the following, however, the fact that we are interested in the real part of the wave should henceforth be considered implicit.

Upon substituting the above forms of $u_{r}, u_{\theta}$ and $u_{z}$ into (2.279) - (2.281), the
only non-zero components of the linear strain tensor we obtain are

$$
\begin{equation*}
e_{r z}=\frac{1}{2} \frac{\partial w}{\partial r} e^{-i \omega t}, \quad e_{\theta z}=\frac{1}{2 r} \frac{\partial w}{\partial \theta} e^{-i \omega t} . \tag{2.298}
\end{equation*}
$$

Hence, upon using (2.193) (since the material we are considering is isotropic), the only non-zero components of the linear Cauchy stress tensor we obtain are

$$
\begin{equation*}
\sigma_{r z}=\mu \frac{\partial w}{\partial r} e^{-i \omega t}, \quad \sigma_{\theta z}=\frac{\mu}{r} \frac{\partial w}{\partial \theta} e^{-i \omega t} \tag{2.299}
\end{equation*}
$$

Upon substituting the relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ into (2.284) - (2.286), we find that the radial and azimuthal governing equations are satisfied trivially, and the longitudinal governing equation reduces to

$$
\begin{equation*}
\frac{\partial^{2} w}{\partial r^{2}}+\frac{1}{r} \frac{\partial w}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}+k^{2} w=0 \tag{2.300}
\end{equation*}
$$

where

$$
\begin{equation*}
k=\omega \sqrt{\frac{\rho}{\mu}} . \tag{2.301}
\end{equation*}
$$

If we are now interested in applying a traction free boundary condition on the surface of the cavity, then the equations to be satisfied on $r=a$ are

$$
\begin{equation*}
\left.\boldsymbol{\sigma} \cdot \boldsymbol{n}\right|_{r=a}=0 \tag{2.302}
\end{equation*}
$$

where $\boldsymbol{n}$ is the outer unit normal to the surface under consideration, so on $r=a$, $\boldsymbol{n}=-\boldsymbol{e}_{\boldsymbol{r}}$. Therefore, (2.302) reduces to

$$
\begin{equation*}
\left.\sigma_{\theta r}\right|_{r=a}=0, \tag{2.303}
\end{equation*}
$$

which, in turn, reduces to

$$
\begin{equation*}
\left.\frac{\partial w}{\partial r}\right|_{r=a}=0 . \tag{2.304}
\end{equation*}
$$

An incident SH wave propagating in the positive $x$ direction is given by

$$
\begin{equation*}
w_{i}=W_{0} e^{i(\gamma x-\omega t)} . \tag{2.305}
\end{equation*}
$$

When the incident wave strikes the cavity, reflection will occur, setting up a scattered wavefield,

$$
\begin{equation*}
w_{s}=w_{s}(r, \theta, t) . \tag{2.306}
\end{equation*}
$$

The total displacement will then be given as

$$
\begin{equation*}
w(r, \theta, t)=w_{i}+w_{s} . \tag{2.307}
\end{equation*}
$$

It can be shown that the incident wavefield automatically satisfies the governing equation (2.300) for $\gamma=k$.

We now assume that the scattered field can be written as

$$
\begin{equation*}
w_{s}=R(r) \Theta(\theta) e^{-i \omega t}=W_{s} e^{-i \omega t} \tag{2.308}
\end{equation*}
$$

and substitute this into (2.300), giving

$$
\begin{equation*}
R^{\prime \prime} \Theta+\frac{1}{r} R^{\prime} \Theta+\frac{1}{r^{2}} R \Theta^{\prime \prime}+k^{2} R \Theta=0 . \tag{2.309}
\end{equation*}
$$

This separates to

$$
\begin{equation*}
\Theta^{\prime \prime}+n^{2} \Theta=0, \tag{2.310}
\end{equation*}
$$

and

$$
\begin{equation*}
R^{\prime \prime}+\frac{1}{r} R^{\prime}+\left(k^{2}-\frac{n^{2}}{r^{2}}\right) R=0 . \tag{2.311}
\end{equation*}
$$

The solution of (2.310) is

$$
\begin{equation*}
\Theta=A_{\theta} \cos (n \theta)+B_{\theta} \sin (n \theta) \tag{2.312}
\end{equation*}
$$

Since the incoming wave is propagating in the positive $x$ direction, we require that the scattered field be symmetrical with respect to the plane $y=0$. This implies that we must have $B_{\theta}=0$, while the requirement that $\Theta$ be single valued (that is, $\Theta(\theta)=\Theta(\theta+2 \pi))$ indicates that $n$ must be an integer.

We recognise (2.311) as Bessel's equation of order $n$ and write the solutions in terms of the order $n$ Hankel functions:

$$
\begin{equation*}
R=A_{r} \mathrm{H}_{n}^{(1)}(k r)+B_{r} \mathrm{H}_{n}^{(2)}(k r) . \tag{2.313}
\end{equation*}
$$

This solution form was chosen due to the convenient exponential representation of the asymptotic behaviour. We now impose the requirement that the scattered wavefield must be outward propagating. This condition will enable us to determine which

Hankel function is appropriate. It can be shown (see [36]) that, for large values of the argument $(b \gg 1)$, the Hankel functions have the following approximate expressions,

$$
\begin{align*}
\mathrm{H}_{n}^{(1)}(b) & \sim\left(\frac{2}{\pi b}\right)^{\frac{1}{2}} e^{i\left(b-\frac{\pi}{4}-\frac{n \pi}{2}\right)}(1-\ldots),  \tag{2.314}\\
\mathrm{H}_{n}^{(2)}(b) & \sim\left(\frac{2}{\pi b}\right)^{\frac{1}{2}} e^{-i\left(b-\frac{\pi}{4}-\frac{n \pi}{2}\right)}(1+\ldots) \tag{2.315}
\end{align*}
$$

Recalling that the time dependence is $e^{-i \omega t}$, it is seen that an outward propagating wave must have the form $e^{i(k r-\omega t)}$, so that $\mathrm{H}_{n}^{(1)}$ is appropriate in the present case. Hence let $B_{r}=0$ in (2.313). The scattered field is thus given by

$$
\begin{equation*}
W_{s}(r, \theta)=\sum_{n=0}^{\infty} A_{n} \mathrm{H}_{n}^{(1)}(k r) \cos (n \theta), \tag{2.316}
\end{equation*}
$$

where $A_{n}$ must be determined.
We now have solutions which satisfy the wave equation so that only substitution into the boundary condition remains. The form of $W_{s}(r, \theta)$ is ideally suited for this, but this is not the case for the incoming wave (2.305). What is required is a Besselfunction representation of the plane wave. Writing $w_{i}$ as

$$
\begin{equation*}
w_{i}=W_{i}(r, \theta) e^{-i \omega t}=W_{0} e^{i k r \cos \theta} e^{-i \omega t} \tag{2.317}
\end{equation*}
$$

it can be shown, using the Jacobi-Anger expansion (see [57] or [59]), that

$$
\begin{equation*}
W_{i}(r, \theta)=W_{0} \sum_{n=0}^{\infty} \epsilon_{n}(-1)^{n} \mathrm{~J}_{n}(k r) \cos (n \theta) \tag{2.318}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{0}=1, \tag{2.319}
\end{equation*}
$$

and

$$
\begin{equation*}
\epsilon_{n}=2 i^{n}, \quad n>0 . \tag{2.320}
\end{equation*}
$$

In (2.316) and (2.317), we have the incident and scattered wavefields similarly represented. Substitution into the boundary conditions, where the time dependence has been omitted, gives

$$
\begin{equation*}
\left.\frac{\partial W_{i}}{\partial r}\right|_{r=a}+\left.\frac{\partial W_{s}}{\partial r}\right|_{r=a}=0 \tag{2.321}
\end{equation*}
$$

Using (2.316) and (2.318), we obtain

$$
\begin{equation*}
\left.\sum_{n=0}^{\infty}\left(W_{0} \epsilon_{n}(-1)^{n} \frac{d \mathrm{~J}_{n}(k r)}{d r}+A_{n} \frac{\mathrm{H}_{n}^{(1)}(k r)}{d r}\right) \cos (n \theta)\right|_{r=a}=0 \tag{2.322}
\end{equation*}
$$

Solving the above for $A_{n}$, we obtain

$$
\begin{equation*}
A_{n}=-\frac{\epsilon_{n}(-1)^{n} W_{0} \mathrm{~J}_{n}^{\prime}(k a)}{\mathrm{H}_{n}^{(1) \prime}(k a)} \tag{2.323}
\end{equation*}
$$

This analytical expression for the scattering coefficients $A_{n}$ can be plotted as a function of $k a$, and in Chapter 4 we compare the scattering coefficients for pre-stressed cavities with the above results for an unstressed cavity.

### 2.4.4 Scattering of dilatational waves from a spherical cavity

In Chapter 6, we consider the scattering of waves from a spherical cavity in a prestressed nonlinear elastic material in the absence of body forces. Here we discuss the unstressed state. First, in this section, we discuss the scattering of P waves from a spherical cavity in an infinite, isotropic, linear elastic material; then, in the following section, we will discuss the scattering of shear waves. The problem of scattering from an inclusion with different material parameters is presented in [36].

In Figure 2.10 we illustrate the problem we are considering of P waves propagating in the positive $z$ direction, and impinging on a spherical cavity of radius $a$.

Due to the axisymmetry of the present problem, we have

$$
\begin{equation*}
\boldsymbol{u}=u_{r}(r, \theta, t) \boldsymbol{e}_{r}+u_{\theta}(r, \theta, t) \boldsymbol{e}_{\theta}, \tag{2.324}
\end{equation*}
$$

and since we are interested in time-harmonic waves we have

$$
\begin{equation*}
u_{r}=\Re\left(u(r, \theta) e^{-i \omega t}\right), \quad u_{\theta}=\Re\left(v(r, \theta) e^{-i \omega t}\right), \tag{2.325}
\end{equation*}
$$

where $\omega$ is the angular frequency of the wave. For conciseness, we shall omit the $\Re$ symbol in the following, however, the fact that we are interested in the real part of the wave should henceforth be considered implicit.

It can be shown (see [36]) that the components of the linear strain tensor in spherical coordinates are as follows:

$$
\begin{equation*}
e_{r r}=\frac{\partial u_{r}}{\partial r}, \quad e_{\theta \theta}=\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}+\frac{u_{r}}{r}, \tag{2.326}
\end{equation*}
$$



Figure 2.10: P waves incident on a spherical cavity.

$$
\begin{gather*}
e_{\phi \phi}=\frac{1}{r \sin \theta} \frac{\partial u_{\phi}}{\partial \phi}+\frac{u_{r}}{r}+u_{\theta} \frac{\cot \theta}{r}, \quad e_{r \phi}=\frac{1}{2}\left(\frac{1}{r \sin \theta} \frac{\partial u_{r}}{\partial \phi}-\frac{u_{\phi}}{r}+\frac{\partial u_{\phi}}{\partial r}\right),  \tag{2.327}\\
e_{r \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u_{r}}{\partial \theta}-\frac{u_{\theta}}{r}+\frac{\partial u_{\theta}}{\partial r}\right), \quad e_{\phi \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u_{\phi}}{\partial \theta}-\frac{u_{\phi} \cot \theta}{r}+\frac{1}{r \sin \theta} \frac{\partial u_{\theta}}{\partial \phi}\right), \tag{2.328}
\end{gather*}
$$

where $u_{r}, u_{\theta}$ and $u_{\phi}$ are the displacements in the radial, polar and azimuthal directions, respectively.

Upon substituting the above forms of $u_{r}, u_{\theta}$ and $u_{\phi}=0$ into (2.326) - (2.328), the only non-zero components of the linear strain tensor we obtain are

$$
\begin{gather*}
e_{r r}=\frac{\partial u}{\partial r} e^{-i \omega t}, \quad e_{\theta \theta}=\left(\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{u}{r}\right) e^{-i \omega t},  \tag{2.329}\\
e_{\phi \phi}=\left(\frac{u}{r}+\frac{v \cot \theta}{r}\right) e^{-i \omega t}, \quad e_{r \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u}{\partial \theta}-\frac{v}{r}+\frac{\partial v}{\partial r}\right) e^{-i \omega t} . \tag{2.330}
\end{gather*}
$$

Hence, upon using (2.193) (since the material we are considering is isotropic), the only non-zero components of the linear Cauchy stress tensor we obtain are

$$
\begin{gather*}
\sigma_{r r}=\lambda \Delta+2 \mu \frac{\partial u}{\partial r} e^{-i \omega t}  \tag{2.331}\\
\sigma_{\theta \theta}=\lambda \triangle+2 \mu\left(\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{u}{r}\right) e^{-i \omega t},  \tag{2.332}\\
\sigma_{\phi \phi}=\lambda \triangle+2 \mu\left(\frac{u}{r}+\frac{\cot \theta}{r} v\right) e^{-i \omega t},  \tag{2.333}\\
\sigma_{r \theta}=\mu\left(\frac{1}{r} \frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right) e^{-i \omega t}, \tag{2.334}
\end{gather*}
$$

where

$$
\begin{equation*}
\triangle=\nabla \cdot \boldsymbol{u}=\left(\frac{\partial u}{\partial r}+\frac{2}{r} u+\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r} v\right) e^{-i \omega t} \tag{2.335}
\end{equation*}
$$

The governing equations for a linear elastic solid can be written in spherical coordinates as

$$
\begin{gather*}
\frac{\partial \sigma_{r r}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{r \phi}}{\partial \phi}+\frac{2 \sigma_{r r}-\sigma_{\theta \theta}-\sigma_{\phi \phi}+\sigma_{r \theta} \cot \theta}{r}+\rho f_{r}=\rho \frac{\partial^{2} u_{r}}{\partial t^{2}}  \tag{2.336}\\
\frac{\partial \sigma_{r \theta}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta \phi}}{\partial \phi}+\frac{3 \sigma_{r \theta}+\left(\sigma_{\theta \theta}-\sigma_{\phi \phi}\right) \cot \theta}{r}+\rho f_{\theta}=\rho \frac{\partial^{2} u_{\theta}}{\partial t^{2}}  \tag{2.337}\\
\frac{\partial \sigma_{r \phi}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \phi}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi \phi}}{\partial \phi}+\frac{3 \sigma_{r \phi}+2 \sigma_{\theta \phi} \cot \theta}{r}+\rho f_{\phi}=\rho \frac{\partial^{2} u_{\phi}}{\partial t^{2}} \tag{2.338}
\end{gather*}
$$

where $f_{r}, f_{\theta}$ and $f_{\phi}$ are the components of the body force in the radial, polar and azimuthal directions, respectively. In this problem, there are no body forces, and so

$$
\begin{equation*}
f_{r}=f_{\theta}=f_{\phi}=0 . \tag{2.339}
\end{equation*}
$$

Upon substituting the relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ into the above, we find that the azimuthal governing equation is satisfied trivially, and the radial and polar governing equations reduce to

$$
\begin{align*}
& (\lambda+2 \mu)\left(\frac{\partial^{2} u}{\partial r^{2}}+\frac{2}{r} \frac{\partial u}{\partial r}-\frac{2}{r^{2}} u+\frac{1}{r} \frac{\partial^{2} v}{\partial r \partial \theta}-\frac{1}{r^{2}} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r} \frac{\partial v}{\partial r}-\frac{\cot \theta}{r^{2}} v\right)+ \\
& \mu\left(\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}-\frac{1}{r} \frac{\partial^{2} v}{\partial r \partial \theta}-\frac{1}{r^{2}} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r^{2}} \frac{\partial u}{\partial \theta}-\frac{\cot \theta}{r} \frac{\partial v}{\partial r}-\frac{\cot \theta}{r^{2}} v\right)=-\rho \omega^{2} u \tag{2.340}
\end{align*}
$$

and

$$
\begin{align*}
& (\lambda+2 \mu)\left(\frac{1}{r} \frac{\partial^{2} u}{\partial r \partial \theta}+\frac{2}{r^{2}} \frac{\partial u}{\partial \theta}+\frac{1}{r^{2}} \frac{\partial^{2} v}{\partial \theta^{2}}+\frac{\cot \theta}{r^{2}} \frac{\partial v}{\partial \theta}-\frac{\csc ^{2} \theta}{r^{2}} v\right)+ \\
& \mu\left(\frac{\partial^{2} v}{\partial r^{2}}-\frac{1}{r} \frac{\partial^{2} u}{\partial r \partial \theta}+\frac{2}{r} \frac{\partial v}{\partial r}\right)=-\rho \omega^{2} v, \tag{2.341}
\end{align*}
$$

respectively.
Using separation of variables, it can be shown that a solution to the above equations can be written in the form

$$
\begin{equation*}
u(r, \theta)=\sum_{n=0}^{\infty} f_{n}(r) P_{n}(\cos \theta), \quad v(r, \theta)=\sum_{n=0}^{\infty} g_{n}(r) \frac{d}{d \theta}\left(P_{n}(\cos \theta)\right), \tag{2.342}
\end{equation*}
$$

where $P_{n}(\cos \theta)$ is a Legendre polynomial of order $n$.

Using the above form, equations (2.340) and (2.341) reduce to

$$
\begin{align*}
& (\lambda+2 \mu)\left(f^{\prime \prime}(r)+\frac{2}{r} f^{\prime}(r)-\frac{2}{r^{2}} f(r)-\frac{m}{r} g^{\prime}(r)+\frac{m}{r^{2}} g(r)\right)+ \\
& \mu\left(\frac{m}{r} g^{\prime}(r)+\frac{m}{r^{2}} g(r)-\frac{m}{r^{2}} f(r)\right)=-\rho \omega^{2} f(r), \tag{2.343}
\end{align*}
$$

and

$$
\begin{align*}
& (\lambda+2 \mu)\left(\frac{1}{r} f^{\prime}(r)+\frac{2}{r^{2}} f(r)-\frac{m}{r^{2}} g(r)\right)+ \\
& \mu\left(g^{\prime \prime}(r)+\frac{2}{r} g^{\prime}(r)-\frac{1}{r} f^{\prime}(r)\right)=-\rho \omega^{2} g(r), \tag{2.344}
\end{align*}
$$

where we have dropped the subscript $n$ on $f$ and $g$ and $m=n(n+1)$.
It follows from the derivation in [36] of a solution in terms of potentials that the incoming plane P wave can be expressed as (2.342) with

$$
\begin{equation*}
f_{n}^{(i)}(r)=(2 n+1) \frac{d}{d r}\left(j_{n}(K r)\right), \quad g_{n}^{(i)}(r)=\frac{(2 n+1) j_{n}(K r)}{r}, \tag{2.345}
\end{equation*}
$$

where $j_{n}$ is an order $n$ spherical Bessel function of the first kind and

$$
\begin{equation*}
K=\sqrt{\frac{\rho \omega^{2}}{\lambda+2 \mu}} . \tag{2.346}
\end{equation*}
$$

Similarly, the outgoing, scattered solution can be expressed as

$$
\begin{equation*}
f_{n}^{(s)}(r)=A_{n} \frac{d}{d r}\left(h_{n}(K r)\right)-\frac{m B_{n} h_{n}(k r)}{r}, \tag{2.347}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{n}^{(s)}(r)=\frac{A_{n} h_{n}(K r)}{r}-\frac{B_{n}}{r} \frac{d}{d r}\left(r h_{n}(k r)\right), \tag{2.348}
\end{equation*}
$$

where $A_{n}$ and $B_{n}$ are scattering coefficients, $h_{n}$ is an order $n$ spherical Hankel function of the first kind and

$$
\begin{equation*}
k=\sqrt{\frac{\rho \omega^{2}}{\mu}} . \tag{2.349}
\end{equation*}
$$

The scattering coeffifients can then be determined by applying traction-free boundary conditions:

$$
\begin{equation*}
\boldsymbol{\sigma} \cdot \boldsymbol{n}=0, \tag{2.350}
\end{equation*}
$$

on $r=a$.

In this case $\boldsymbol{n}=-\boldsymbol{e}_{r}$, and therefore the boundary conditions can be written as

$$
\begin{equation*}
\left.\sigma_{r r}\right|_{r=a}=\left.\sigma_{r \theta}\right|_{r=a}=0 . \tag{2.351}
\end{equation*}
$$

The scattering coefficients can then be determined by subsituting the derived expressions for $u_{r}$ and $u_{\theta}$ into the above.

### 2.4.5 Scattering of shear waves from a spherical cavity

As mentioned above, in Chapter 6 we look at the scattering of shear waves from a spherical cavity in an infinite, pre-stressed nonlinear elastic material in the absence of body forces. In this section we discuss the scattering of shear waves from a spherical cavity in an infinite, isotropic, linear elastic material and the results in this section can be compared with the results in Chapter 6. One of the difficulties in Chapter 6 is that we assume that the host medium is incompressible and so P waves in that medium have an infinite wavelength. It is, therefore, of interest to consider the infinite wavelength (i.e. small wavenumber) limit of the results in this section.

Now we are considering shear waves, we no longer have the axisymmetry of the previous section, and so we consider a more general form for the displacement:

$$
\begin{equation*}
\boldsymbol{u}=u_{r}(r, \theta, \phi) \boldsymbol{e}_{r}+u_{\theta}(r, \theta, \phi) \boldsymbol{e}_{\theta}+u_{\phi}(r, \theta, \phi) \boldsymbol{e}_{\phi}, \tag{2.352}
\end{equation*}
$$

and since we are interested in time-harmonic waves we have

$$
\begin{equation*}
u_{r}=\Re\left(u(r, \theta, \phi) e^{-i \omega t}\right), \quad u_{\theta}=\Re\left(v(r, \theta, \phi) e^{-i \omega t}, \quad u_{\phi}=\Re\left(w(r, \theta, \phi) e^{-i \omega t}\right),\right. \tag{2.353}
\end{equation*}
$$

where $\omega$ is the angular frequency of the wave. For conciseness, we shall omit the $\Re$ symbol in the following, however, the fact that we are interested in the real part of the wave should henceforth be considered implicit.

Upon substituting the above forms of $u_{r}, u_{\theta}$ and $u_{\phi}$ into (2.326) - (2.328), we obtain the following expressions for the components of the linear strain tensor:

$$
\begin{gather*}
e_{r r}=\frac{\partial u}{\partial r} e^{-i \omega t}, \quad e_{\theta \theta}=\left(\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{u}{r}\right) e^{-i \omega t},  \tag{2.354}\\
e_{\phi \phi}=\left(\frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi}+\frac{u}{r}+\frac{v \cot \theta}{r}\right) e^{-i \omega t}, \tag{2.355}
\end{gather*}
$$

$$
\begin{gather*}
e_{r \phi}=\frac{1}{2}\left(\frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi}-\frac{w}{r}+\frac{\partial w}{\partial r}\right) e^{-i \omega t},  \tag{2.356}\\
e_{r \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u}{\partial \theta}-\frac{v}{r}+\frac{\partial v}{\partial r}\right) e^{-i \omega t},  \tag{2.357}\\
e_{\phi \theta}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial w}{\partial \theta}-\frac{w \cot \theta}{r}+\frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi}\right) e^{-i \omega t}, \tag{2.358}
\end{gather*}
$$

Hence, upon using (2.193) (since the material we are considering is isotropic), the only non-zero components of the linear Cauchy stress tensor we obtain are

$$
\begin{gather*}
\sigma_{r r}=\lambda \Delta+2 \mu \frac{\partial u}{\partial r} e^{-i \omega t},  \tag{2.359}\\
\sigma_{\theta \theta}=\lambda \Delta+2 \mu\left(\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{u}{r}\right) e^{-i \omega t},  \tag{2.360}\\
\sigma_{\phi \phi}=\lambda \triangle+2 \mu\left(\frac{u}{r}+\frac{\cot \theta}{r} v+\frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi}\right) e^{-i \omega t},  \tag{2.361}\\
\sigma_{r \phi}=\mu\left(\frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi}-\frac{w}{r}+\frac{\partial w}{\partial r}\right) e^{-i \omega t},  \tag{2.362}\\
\sigma_{r \theta}=\mu\left(\frac{1}{r} \frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right) e^{-i \omega t},  \tag{2.363}\\
\sigma_{\phi \theta}=\mu\left(\frac{1}{r} \frac{\partial w}{\partial \theta}-\frac{w \cot \theta}{r}+\frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi}\right) e^{-i \omega t}, \tag{2.364}
\end{gather*}
$$

where

$$
\begin{equation*}
\triangle=\nabla \cdot \boldsymbol{u}=\left(\frac{\partial u}{\partial r}+\frac{2}{r} u+\frac{1}{r} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r} v+\frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi}\right) e^{-i \omega t} . \tag{2.365}
\end{equation*}
$$

The governing equations for a linear elastic solid can be written in spherical coordinates as (see equation (2.26))

$$
\begin{gather*}
\frac{\partial \sigma_{r r}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{r \phi}}{\partial \phi}+\frac{2 \sigma_{r r}-\sigma_{\theta \theta}-\sigma_{\phi \phi}+\sigma_{r \theta} \cot \theta}{r}+\rho f_{r}=\rho \frac{\partial^{2} u_{r}}{\partial t^{2}}  \tag{2.366}\\
\frac{\partial \sigma_{r \theta}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta \phi}}{\partial \phi}+\frac{3 \sigma_{r \theta}+\left(\sigma_{\theta \theta}-\sigma_{\phi \phi}\right) \cot \theta}{r}+\rho f_{\theta}=\rho \frac{\partial^{2} u_{\theta}}{\partial t^{2}}  \tag{2.367}\\
\frac{\partial \sigma_{r \phi}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \phi}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi \phi}}{\partial \phi}+\frac{3 \sigma_{r \phi}+2 \sigma_{\theta \phi} \cot \theta}{r}+\rho f_{\phi}=\rho \frac{\partial^{2} u_{\phi}}{\partial t^{2}} \tag{2.368}
\end{gather*}
$$

where $f_{r}, f_{\theta}$ and $f_{\phi}$ are the components of the body force in the radial, polar and azimuthal directions, respectively. In this problem, there are no body forces, and so

$$
\begin{equation*}
f_{r}=f_{\theta}=f_{\phi}=0 . \tag{2.369}
\end{equation*}
$$

Upon substituting the relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ into the above, we obtain the following three equations

$$
\begin{gather*}
(\lambda+2 \mu)\left(\frac{\partial^{2} u}{\partial r^{2}}+\frac{2}{r} \frac{\partial u}{\partial r}-\frac{2}{r^{2}} u+\frac{1}{r} \frac{\partial^{2} v}{\partial r \partial \theta}-\frac{1}{r^{2}} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r} \frac{\partial v}{\partial r}-\frac{\cot \theta}{r^{2}} v-\frac{1}{r^{2} \sin \theta} \frac{\partial w}{\partial \phi}+\right. \\
\left.\frac{1}{r \sin \theta} \frac{\partial^{2} w}{\partial r \partial \phi}\right)+\mu\left(\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}-\frac{1}{r} \frac{\partial^{2} v}{\partial r \partial \theta}-\frac{1}{r^{2}} \frac{\partial v}{\partial \theta}+\frac{\cot \theta}{r^{2}} \frac{\partial u}{\partial \theta}-\frac{\cot \theta}{r} \frac{\partial v}{\partial r}-\frac{\cot \theta}{r^{2}} v-\right. \\
\left.\frac{1}{r \sin \theta} \frac{\partial^{2} w}{\partial r \partial \phi}+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} u}{\partial \phi^{2}}-\frac{1}{r^{2} \sin \theta} \frac{\partial w}{\partial \phi}\right)=-\rho \omega^{2} u, \quad(2.370)  \tag{2.370}\\
(\lambda+2 \mu)\left(\frac{1}{r} \frac{\partial^{2} u}{\partial r \partial \theta}+\frac{2}{r^{2}} \frac{\partial u}{\partial \theta}+\frac{1}{r^{2}} \frac{\partial^{2} v}{\partial \theta^{2}}+\frac{\cot \theta}{r^{2}} \frac{\partial v}{\partial \theta}-\frac{\csc ^{2} \theta}{r^{2}} v+\frac{1}{r^{2} \sin \theta} \frac{\partial^{2} w}{\partial \theta \partial \phi}-\right. \\
\left.\frac{\cot \theta}{r^{2} \sin \theta} \frac{\partial w}{\partial \phi}\right)+\mu\left(\frac{\partial^{2} v}{\partial r^{2}}-\frac{1}{r} \frac{\partial^{2} u}{\partial r \partial \theta}+\frac{2}{r} \frac{\partial v}{r^{2} \sin 2^{2} \theta}-\frac{1}{r^{2} \sin \theta} \frac{\partial^{2} w}{\partial \theta \partial \phi}-\frac{\cot \theta}{r^{2} \sin \theta} \frac{\partial w}{\partial \phi}\right)=-\rho \omega^{2} v, \quad(2.371) \\
\frac{\lambda+2 \mu}{r \sin \theta}\left(\frac{\partial^{2} u}{\partial r \partial \phi}+\frac{2}{r} \frac{\partial u}{\partial \phi}+\frac{1}{r} \frac{\partial^{2} v}{\partial \theta \partial \phi}+\frac{\cot \theta}{r} \frac{\partial v}{\partial \phi}+\frac{1}{r \sin \theta} \frac{\partial^{2} w}{\partial \phi^{2}}\right)+  \tag{2.371}\\
\mu\left(\frac{\partial^{2} w}{\partial r^{2}}+\frac{2}{r} \frac{\partial w}{\partial r}-\frac{w}{r^{2}}+\frac{1}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}+\frac{\cot \theta}{r^{2}} \frac{\partial w}{\partial \theta}-\frac{1}{r \sin \theta} \frac{\partial^{2} u}{\partial r \partial \phi}-\frac{1}{r^{2} \sin \theta} \frac{\partial^{2} v}{\partial \theta \partial \phi}+\right. \\
\left.\left.\frac{\cot \theta}{r^{2} \sin \theta} \frac{\partial v}{\partial \phi}-\frac{w \cot ^{2} \theta}{r^{2}}\right)=-\rho{\omega^{2} w, \quad(2.372)}^{2}\right)
\end{gather*}
$$

We now expand $\boldsymbol{u}$ in terms of the vector spherical harmonics, $\boldsymbol{Y}_{l m}, \boldsymbol{\Psi}_{l m}$ and $\boldsymbol{\Phi}_{l m}$, which are defined as follows:

$$
\begin{equation*}
\boldsymbol{Y}_{l m}=Y_{l}^{m} \boldsymbol{e}_{r}, \quad \boldsymbol{\Psi}_{l m}=r \nabla Y_{l}^{m}, \quad \boldsymbol{\Phi}_{l m}=\boldsymbol{r} \times \nabla Y_{l}^{m} \tag{2.373}
\end{equation*}
$$

where $\boldsymbol{e}_{r}$ is a unit vector in the radial direction, $\boldsymbol{r}$ is a position vector, and $Y_{l}^{m}$ is a scalar spherical harmonic, defined as follows:

$$
\begin{equation*}
Y_{l}^{m}=Y_{l}^{m}(\theta, \phi)=P_{l}^{m}(\cos \theta) e^{i m \phi} \tag{2.374}
\end{equation*}
$$

where $P_{l}^{m}$ is an associated Legendre polynomial of degree $l$ and order $m$.
The expansion is as follows:

$$
\begin{equation*}
\boldsymbol{u}=\Re\left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(f_{l}(r) \boldsymbol{Y}_{l m}+g_{l}(r) \boldsymbol{\Psi}_{l m}+h_{l}(r) \boldsymbol{\Phi}_{l m}\right) e^{-i \omega t}\right) \tag{2.375}
\end{equation*}
$$

so that $u, v$ and $w$ can be expressed as

$$
\begin{gather*}
u(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l}(r) P_{l}^{m}(\cos \theta) e^{i m \phi},  \tag{2.376}\\
v(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(g_{l}(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+h_{l}(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}\right),  \tag{2.377}\\
w(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(g_{l}(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}-h_{l}(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}\right) . \tag{2.378}
\end{gather*}
$$

Using the above expansion, the governing equations reduce to

$$
\begin{gather*}
\left((\lambda+2 \mu)\left(f^{\prime \prime}(r)+\frac{2}{r} f^{\prime}(r)-\frac{2}{r^{2}} f(r)-\frac{l(l+1)}{r} g^{\prime}(r)+\frac{l(l+1)}{r^{2}} g(r)\right)+\right. \\
\left.\mu l(l+1)\left(\frac{1}{r} g^{\prime}(r)+\frac{1}{r^{2}} g(r)-\frac{1}{r^{2}} f(r)\right)\right) P_{l}^{m}(\cos \theta) e^{i m \phi}=-\rho \omega^{2} f(r) P_{l}^{m}(\cos \theta) e^{i m \phi} \tag{2.379}
\end{gather*}
$$

$$
\begin{align*}
\left(( \lambda + 2 \mu ) \left(\frac{1}{r}\right.\right. & \left.f^{\prime}(r)+\frac{2}{r^{2}} f(r)-\frac{l(l+1)}{r^{2}} g(r)\right)+ \\
& \left.\mu\left(g^{\prime \prime}(r)+\frac{2}{r} g^{\prime}(r)-\frac{1}{r} f^{\prime}(r)\right)\right) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+ \\
& \left(h^{\prime \prime}(r)+\frac{2}{r} h^{\prime}(r)-\frac{l(l+1)}{r^{2}} h(r)\right) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}= \\
& -\rho \omega^{2}\left(g(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+h(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}\right), \tag{2.380}
\end{align*}
$$

and

$$
\begin{align*}
& \left((\lambda+2 \mu)\left(\frac{1}{r} f^{\prime}(r)+\frac{2}{r^{2}} f(r)-\frac{l(l+1)}{r^{2}} g(r)\right)+\right. \\
& \left.\mu\left(g^{\prime \prime}(r)+\frac{2}{r} g^{\prime}(r)-\frac{1}{r} f^{\prime}(r)\right)\right) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}- \\
& \left(h^{\prime \prime}(r)+\frac{2}{r} h^{\prime}(r)-\frac{l(l+1)}{r^{2}} h(r)\right) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}= \\
&  \tag{2.381}\\
& \quad-\rho \omega^{2}\left(g(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}-h(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}\right),
\end{align*}
$$

where we have dropped the subscripts on $f$ and $g$.

Due to the orthogonality of $\frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}$ and $\frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}$, the above equations can be separated as follows:

$$
\begin{align*}
&(\lambda+2 \mu)\left(f^{\prime \prime}(r)+\frac{2}{r} f^{\prime}(r)-\frac{2}{r^{2}} f(r)-\frac{l(l+1)}{r} g^{\prime}(r)+\frac{l(l+1)}{r^{2}} g(r)\right)+ \\
& \mu l(l+1)\left(\frac{1}{r} g^{\prime}(r)+\frac{1}{r^{2}} g(r)-\frac{1}{r^{2}} f(r)\right)=-\rho \omega^{2} f(r),  \tag{2.382}\\
&(\lambda+2 \mu)\left(\frac{1}{r} f^{\prime}(r)+\frac{2}{r^{2}} f(r)-\frac{l(l+1)}{r^{2}} g(r)\right)+ \\
& \mu\left(g^{\prime \prime}(r)+\frac{2}{r} g^{\prime}(r)-\frac{1}{r} f^{\prime}(r)\right)=-\rho \omega^{2} g(r), \tag{2.383}
\end{align*}
$$

and

$$
\begin{equation*}
h^{\prime \prime}(r)+\frac{2}{r} h^{\prime}(r)-\frac{l(l+1)}{r^{2}} h(r)=-\rho \omega^{2} h(r) . \tag{2.384}
\end{equation*}
$$

These equations can be compared with those derived in Chapter 6 .
In [29] it is stated that an incoming shear wave propagating in the $z$ direction and polarised such that the displacements are in the $x$ direction can be expressed as

$$
\begin{equation*}
\boldsymbol{u}^{(i)}=\sum_{n=1}^{\infty} \frac{(2 n+1) i^{n}}{n(n+1)}\left(\boldsymbol{M}_{o 1 n}^{1}(\boldsymbol{r})-i \boldsymbol{N}_{e 1 n}^{1}(\boldsymbol{r})\right), \tag{2.385}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{M}_{\sigma m n}^{1}=(n(n+1))^{\frac{1}{2}} \boldsymbol{C}_{m n}^{\sigma} j_{n}(k r), \tag{2.386}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{N}_{\sigma m n}^{1}=n(n+1) \boldsymbol{P}_{m n}^{\sigma} \frac{1}{k r} j_{n}(k r)+(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{1}{k r} \frac{d}{d r}\left(r j_{n}(k r)\right) . \tag{2.387}
\end{equation*}
$$

Here the label $\sigma$ is either $e$ (even) or $o$ (odd) and designates whether the even (real) or odd (imaginary) part of the azimuthal function is to be employed, $k$ is defined via

$$
\begin{equation*}
k^{2}=\frac{\rho \omega^{2}}{\mu}, \tag{2.388}
\end{equation*}
$$

and $\boldsymbol{P}_{m n}, \boldsymbol{B}_{m n}$ and $\boldsymbol{C}_{m n}$ are defined by

$$
\begin{equation*}
\boldsymbol{P}_{m n}=\boldsymbol{e}_{r} Y_{n}^{m}(\theta, \phi) \tag{2.389}
\end{equation*}
$$

$$
\begin{align*}
\boldsymbol{B}_{m n} & =\frac{(n(n+1))^{\frac{1}{2}}}{(2 n+1) \sin \theta}\left(\boldsymbol{e}_{\theta}\left(\frac{n-m+1}{n+1} Y_{n+1}^{m}-\frac{n+m}{n} Y_{n-1}^{m}\right)+\boldsymbol{e}_{\phi} \frac{m(2 n+1)}{n(n+1)} i Y_{n}^{m}\right) \\
& =(n(n+1))^{-\frac{1}{2}} \frac{d}{d \theta}\left(P_{n}^{m}(\cos \theta)\right) e^{i m \phi} \boldsymbol{e}_{\theta}+(n(n+1))^{-\frac{1}{2}} \frac{P_{n}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi} \boldsymbol{e}_{\phi} \\
& =\boldsymbol{e}_{r} \times \boldsymbol{C}_{m n},  \tag{2.390}\\
\boldsymbol{C}_{m n} & =\frac{(n(n+1))^{\frac{1}{2}}}{(2 n+1) \sin \theta}\left(\boldsymbol{e}_{\theta} \frac{m(2 n+1)}{n(n+1)} i Y_{n}^{m}-\boldsymbol{e}_{\phi}\left(\frac{n-m+1}{n+1} Y_{n+1}^{m}-\frac{n+m}{n} Y_{n-1}^{m}\right)\right) \\
& =(n(n+1))^{-\frac{1}{2}} \frac{P_{n}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi} \boldsymbol{e}_{\theta}-(n(n+1))^{-\frac{1}{2}} \frac{d}{d \theta}\left(P_{n}^{m}(\cos \theta)\right) e^{i m \phi} \boldsymbol{e}_{\phi} . \tag{2.391}
\end{align*}
$$

By comparing equations (2.385) with equations (2.376) - (2.378), we observe that

$$
\begin{gather*}
f_{l}^{(i)}(r)=\left\{\begin{array}{cl}
(2 l+1) i^{l-1} \frac{j_{l}(k r)}{k r} & l \geq 1, \\
0 & l \leq 0,
\end{array}\right.  \tag{2.392}\\
g_{l}^{(i)}(r)=\left\{\begin{array}{cc}
\frac{(2 l+1) i^{l-1}}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r j_{l}(k r)\right) & l \geq 1, \\
0 & l \leq 0,
\end{array}\right. \tag{2.393}
\end{gather*}
$$

and

$$
h_{l}^{(i)}(r)=\left\{\begin{array}{cc}
\frac{(2 l+1) i^{l}}{l(l+1)} j_{l}(k r) & l \geq 1,  \tag{2.394}\\
0 & l \leq 0
\end{array}\right.
$$

It is also stated in [29] that the scattered wave displacements can be expressed as

$$
\begin{equation*}
\boldsymbol{u}^{(s)}=\sum_{n} \frac{(2 n+1) i^{n}}{n(n+1)}\left(\hat{a_{n}} \boldsymbol{M}_{o 1 n}^{3}(\boldsymbol{r})-i \hat{b_{n}} \boldsymbol{N}_{e 1 n}^{3}(\boldsymbol{r})+\hat{d_{n}} \boldsymbol{L}_{e 1 n}^{3}(\boldsymbol{r})\right), \tag{2.395}
\end{equation*}
$$

where $\hat{a_{n}}, \hat{b_{n}}$ and $\hat{d_{n}}$ are expansion coefficients to be determined from the boundary conditions,

$$
\begin{gather*}
\boldsymbol{M}_{\sigma m n}^{3}=(n(n+1))^{\frac{1}{2}} \boldsymbol{C}_{m n}^{\sigma} h_{n}(k r),  \tag{2.396}\\
\boldsymbol{N}_{\sigma m n}^{3}=n(n+1) \boldsymbol{P}_{m n}^{\sigma} \frac{1}{k r} h_{n}(k r)+(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{1}{k r} \frac{d}{d r}\left(r h_{n}(k r)\right), \tag{2.397}
\end{gather*}
$$

and

$$
\begin{equation*}
\boldsymbol{L}_{\sigma m n}^{3}=\boldsymbol{P}_{m n}^{\sigma} \frac{1}{k} \frac{d}{d r}\left(h_{n}(k r)\right)+(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{h_{n}(k r)}{k r} . \tag{2.398}
\end{equation*}
$$

This corresponds to

$$
f_{l}^{(s)}(r)=\left\{\begin{array}{cc}
b_{l} \frac{h_{l}(k r)}{r}+d_{l} \frac{d}{d r}\left(h_{l}(K r)\right) & l \geq 1,  \tag{2.399}\\
0 & l \leq 0,
\end{array}\right.
$$

$$
g_{l}^{(s)}(r)=\left\{\begin{array}{cc}
\frac{b_{l}}{r} \frac{d}{d r}\left(r h_{l}(k r)\right)+d_{l} \frac{h_{l}(K r)}{r} & l \geq 1,  \tag{2.400}\\
0 & l \leq 0,
\end{array}\right.
$$

and

$$
h_{l}^{(s)}(r)=\left\{\begin{array}{cc}
a_{l} h_{l}(k r) & l \geq 1,  \tag{2.401}\\
0 & l \leq 0,
\end{array}\right.
$$

where

$$
\begin{equation*}
K^{2}=\frac{\rho \omega^{2}}{\lambda+2 \mu} . \tag{2.402}
\end{equation*}
$$

The scattering coefficients can then be determined by applying traction-free boundary conditions:

$$
\begin{equation*}
\boldsymbol{\sigma} \cdot \boldsymbol{n}=0 \tag{2.403}
\end{equation*}
$$

on $r=a$. In this case $\boldsymbol{n}=-\boldsymbol{e}_{r}$, and therefore the boundary conditions can be written as

$$
\begin{equation*}
\left.\sigma_{r r}\right|_{r=a}=\left.\sigma_{r \theta}\right|_{r=a}=\left.\sigma_{r \phi}\right|_{r=a}=0 . \tag{2.404}
\end{equation*}
$$

The scattering coefficients can then be determined by substituting the derived expressions for $u_{r}, u_{\theta}$ and $u_{\phi}$ into the above. Upon doing so, we obtain (see [29]):

$$
\begin{equation*}
a_{l}=-\frac{\frac{d}{d a}\left(\frac{j_{l}(k a)}{a}\right)}{\frac{d}{d a}\left(\frac{h_{l}(k a)}{a}\right)}, \tag{2.405}
\end{equation*}
$$

and $b_{l}$ and $d_{l}$ can be determined solving the following matrix equation:

$$
\left(\begin{array}{ll}
E_{11} & E_{12}  \tag{2.406}\\
E_{21} & E_{22}
\end{array}\right)\binom{b_{l}}{d_{l}}=\binom{\delta_{1}}{\delta_{2}},
$$

where

$$
\begin{align*}
E_{11}=(\lambda+2 \mu) \frac{2 l(l+1)}{2 l+1} \frac{d}{d a}\left(A_{1}(a)\right)+\lambda \frac{2 l(l+1)}{2 l+1} \frac{A_{1}(a)}{a} & +\lambda\left(1-2 l^{2}\right) \frac{B_{2}(a)}{a} \\
& +\lambda l(l+1) \frac{D_{2}(a)}{a},  \tag{2.407}\\
E_{12}=(\lambda+2 \mu) \frac{2 l(l+1)}{2 l+1} \frac{d}{d a}\left(\left(A_{2}(a)\right)+\lambda \frac{2 l(l+1)}{2 l+1} \frac{A_{2}(a)}{a}\right. & +\lambda\left(1-2 l^{2}\right) \frac{B_{3}(a)}{a} \\
& +\lambda l(l+1) \frac{D_{3}(a)}{a} \tag{2.408}
\end{align*}
$$

$$
\begin{gather*}
E_{21}=a^{2} \frac{d}{d a}\left(\frac{D_{2}(a)}{a}\right)-A_{1}(a),  \tag{2.409}\\
E_{22}=a^{2} \frac{d}{d a}\left(\frac{D_{3}(a)}{a}\right)-A_{2}(a),  \tag{2.410}\\
\delta_{1}=-\left((\lambda+2 \mu) \frac{2 l(l+1)}{2 l+1} \frac{d}{d a} E_{1}(a)+\lambda \frac{2 l(l+1)}{2 l+1} \frac{E_{1}(a)}{a}+\lambda\left(1-2 l^{2}\right) \frac{F_{2}(a)}{a}\right. \\
\left.+\lambda l(l+1) \frac{G_{1}(a)}{a}\right),  \tag{2.411}\\
\delta_{2}=E_{1}(a)-a^{2} \frac{d}{d a} \frac{G_{1}(a)}{a}, \tag{2.412}
\end{gather*}
$$

and

$$
\begin{gather*}
A_{1}(r)=-i^{l+1}(2 l+1) \frac{h_{l}(k r)}{k r},  \tag{2.413}\\
A_{2}(r)=\frac{i^{n}(2 l+1)}{l(l+1)} \frac{1}{K} \frac{d}{d r}\left(h_{l}(K r)\right),  \tag{2.414}\\
B_{2}(r)=-i^{l+1} \frac{1}{k r} \frac{d}{d r}\left(r h_{l}(k r)\right),  \tag{2.415}\\
B_{3}(r)=i^{n} \frac{h_{l}(K r)}{K r},  \tag{2.416}\\
D_{2}(r)=\frac{i^{l+1}(2 l+1)}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r h_{l}(k r)\right),  \tag{2.417}\\
D_{3}(r)=-\frac{i^{l}(2 l+1)}{l(l+1)} \frac{h_{l}(K r)}{K r},  \tag{2.418}\\
E_{1}(r)=-i^{l+1}(2 l+1) \frac{j_{l}(k r)}{k r},  \tag{2.419}\\
F_{2}(r)=-i^{l+1} \frac{1}{k r} \frac{d}{d r}\left(r j_{l}(k r)\right),  \tag{2.420}\\
G_{1}(r)=\frac{i^{l+1}(2 l+1)}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r j_{l}(k r)\right) . \tag{2.421}
\end{gather*}
$$

The explicit analytical expressions for $b_{l}$ and $d_{l}$ are too long to reproduce here, but they can be easily calculated using the above expressions. In Chapter 6 we plot the scattering coefficients and cross-sections for a pre-stressed, incompressible material and compare them with the case of no pre-stress.

For some papers featuring wave scattering from spheres, see Ying and Truell [110], Einspruch et al. [29]. Iwashimizu [47], [48], Norris [66] or Ávila-Carrera and SánchezSesma [5]. In [110], Ying and Truell discuss the scattering of a plane longitudinal wave by a spherical obstacle in an isotropic elastic solid; in [29], Einspruch et al. do
the same, but for a transverse incoming wave; in [47] and [48], Iwashimizu considers the scattering of longitudinal and shear waves by a movable rigid sphere and scattering of shear waves by an elastic sphere, respectively; in [66], Norris discusses the scattering of elastic waves by spherical inclusions with applications to low frequency wave propagation in composites; and in [5], Ávila-Carrera and Sánchez-Sesma review the classical solution for the scattering and diffraction of P and S waves by a spherical obstacle.

### 2.4.6 Linear elastic waves in unstressed anisotropic elastic solids

All of the problems above refer to unstressed isotropic materials. In the case of anisotropy, waves propagate with different speeds in different directions. For example, in transversely isotropic materials, shear waves polarised along and perpendicular to the axis of symmetry of the material propagate with different wave speeds, as we show below.

The stress strain relationship for a transversely isotropic material can be written in contracted notation (see [107], [53], [49] or [18]) as

$$
\left(\begin{array}{c}
\sigma_{11}  \tag{2.422}\\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{13} \\
\sigma_{23} \\
\sigma_{12}
\end{array}\right)=\left(\begin{array}{cccccc}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(c_{11}-c_{12}\right)
\end{array}\right)\left(\begin{array}{c}
e_{11} \\
e_{22} \\
e_{33} \\
2 e_{13} \\
2 e_{23} \\
2 e_{12}
\end{array}\right) .
$$

Now let us consider two time-harmonic, plane shear waves in Cartesian coordinates, both polarised in the $y$ direction, but propagating in different directions (the $x$ and $z$ directions respectively). We can express these waves respectively as

$$
\begin{align*}
& \boldsymbol{u}_{1}=u_{1}(x, t) \boldsymbol{e}_{y}=e^{i\left(k_{1} x-\omega t\right)} \boldsymbol{e}_{y},  \tag{2.423}\\
& \boldsymbol{u}_{2}=u_{2}(z, t) \boldsymbol{e}_{y}=e^{i\left(k_{2} z-\omega t\right)} \boldsymbol{e}_{y} . \tag{2.424}
\end{align*}
$$

Using (2.423) and (2.424) in (2.201), the equations governing these waves can be reduced to

$$
\begin{align*}
\frac{\partial \sigma_{y x}}{\partial x} & =\rho \frac{\partial^{2} u_{1}}{\partial t^{2}}  \tag{2.425}\\
\frac{\partial \sigma_{y z}}{\partial z} & =\rho \frac{\partial^{2} u_{2}}{\partial t^{2}} \tag{2.426}
\end{align*}
$$

Using (2.422) in the above, and by inserting the definition of the linear strains, we obtain

$$
\begin{gather*}
\frac{1}{2}\left(c_{11}-c_{12}\right) \frac{\partial^{2} u_{1}}{\partial x^{2}}=\rho \frac{\partial^{2} u_{1}}{\partial t^{2}},  \tag{2.427}\\
c_{44} \frac{\partial^{2} u_{2}}{\partial z^{2}}=\rho \frac{\partial^{2} u_{2}}{\partial t^{2}} . \tag{2.428}
\end{gather*}
$$

Substituting (2.423) and (2.424) into the above, we obtain

$$
\begin{gather*}
-\frac{k_{1}^{2}}{2}\left(c_{11}-c_{12}\right)=-\rho \omega^{2},  \tag{2.429}\\
-c_{44} k_{2}^{2}=-\rho \omega^{2} . \tag{2.430}
\end{gather*}
$$

Therefore,

$$
\begin{gather*}
k_{1}^{2}=\frac{\omega^{2}}{c_{1}^{2}}=\frac{2 \rho \omega^{2}}{c_{11}-c_{12}}  \tag{2.431}\\
k_{2}^{2}=\frac{\omega^{2}}{c_{2}^{2}}=\frac{\rho \omega^{2}}{c_{44}} . \tag{2.432}
\end{gather*}
$$

Hence,

$$
\begin{gather*}
c_{1}=\sqrt{\frac{c_{11}-c_{12}}{2 \rho}},  \tag{2.433}\\
c_{2}=\sqrt{\frac{c_{44}}{\rho}} \tag{2.434}
\end{gather*}
$$

As expected, we have observed that waves propagating in the $x$ direction travel at a different speed to those propagating in the $z$ direction.

Wave propagation in anisotropic materials is of relevance to this thesis since the application of homogeneous pre-stress in a nonlinear elastic material results in an induced anisotropy. For some more examples of wave and pulse propagation in anisotropic materials, see the papers by Morse [63], Norris [67] and Chadwick [15]. In [63], Morse discusses compressional waves along an anisotropic circular cylinder with hexagonal symmetry, in [67], Norris presents a general theory for pulse propagation
in anisotropic solids, and in [15], Chadwick discusses the propagation of plane waves in transversely isotropic media.

Of particular interest is wave propagation and scattering in composite materials. Composites often induce anisotropy due to their distributed microstructure, although, uniform distributions of, for example, spherical inclusions can ensure macroscopic isotropy. For some examples of wave propagation and scattering in composite materials, see Norris [66], Shindo et al. [93], Sato and Shindo [89], [90] or Parnell and Abrahams [83]. In [66], Norris discusses the scattering of elastic waves by spherical inclusions, with an emphasis on low frequency wave propagation in composites. In [93], Shindo et al. discuss multiple scattering of antiplane shear waves in a fiber-reinforced composite medium with graded interfacial layers, and in [89] and [90], this work was extended to the case of general plane waves in fiber-reinforced and particle-reinforced media. Finally, in [83], Parnell and Abrahams discuss mutliple point scattering to determine the effective wavenumber and effective material properties of an inhomogeneous slab.

The effects of anisotropy on wave propagation can be exploited in certain situations, for example, in elastodynamic cloaking theory. Interest in cloaking theory and its practical realisation has grown significantly since the early theoretical work of Leonhart [54] and Pendry et al. [85] in optics and electromagnetism, respectively. Methods have been largely based on the idea of coordinate transformations, [39], which motivate the design of cloaking metamaterials that are able to guide waves around a specific region of space. Since the early work, research has focused on the possibility of cloaking in the context of acoustics, [20], [17], [68], surface waves in fluids, [31], heat transfer, [56], fluid flow, [102], and linear elastodynamics, [60], [14], [4], [70], and it is the latter application which is the concern of Chapter 5. In [60], in was shown that elastodynamic cloaking is made difficult due to the lack of invariance of Navier's equations under general coordinate transformations which retain the symmetries of the elastic modulus tensor. A special case is that of flexural waves in thin plates [32]. Invariance of the governing equations can be achieved for a more specific class of transformations if assumptions are relaxed on the minor symmetries of the
elastic modulus tensor as was described for the in-plane problem in [14]. Cosserat materials were exploited in [70].

As noted in [14], another special case for elastodynamics is the antiplane elastic wave problem, where cloaking can readily be achieved from a cylindrical region (using a cylindrical cloak) in two dimensions by virtue of the duality between antiplane waves and acoustics in this dimension. Consider an unbounded homogeneous elastic material with shear modulus $\mu_{0}$ and density $\rho_{0}$ and introduce a Cartesian coordinate system $(X, Y, Z)$ and cylindrical polar coordinate system $(R, \Theta, Z)$ with some common origin $\boldsymbol{O}$. Planar variables are related in the usual manner, $X=R \cos \Theta$ and $Y=R \sin \Theta$. Suppose that there is a time-harmonic line source, polarised in the $Z$ direction and located at ( $R_{0}, \Theta_{0}$ ), with angular frequency $\omega$ and amplitude $C$ (which is a force per unit length in the $Z$ direction). This generates antiplane elastic waves with the only non-zero displacement component in the $Z$ direction of the form $U=\Re\left(W(X, Y) e^{-i \omega t}\right)$. The displacement $W$ is governed by

$$
\begin{equation*}
\nabla_{\boldsymbol{X}} \cdot\left(\mu_{0} \nabla_{\boldsymbol{X}} W\right)+\rho_{0} \omega^{2} W=C \delta\left(\boldsymbol{X}-\boldsymbol{X}_{0}\right), \tag{2.435}
\end{equation*}
$$

where $\nabla_{\boldsymbol{X}}$ is the gradient operator in the "untransformed" frame, $\boldsymbol{X}=(X, Y)$ and $\boldsymbol{X}_{0}=\left(X_{0}, Y_{0}\right)$. Note that, when $\left(X_{0}, Y_{0}\right)=(-\infty, 0)$, we obtain the plane wave problem considered in Section 2.4.3.

The assumed mapping for a cloak for antiplane waves (cf. acoustics) expressed in plane cylindrical polar coordinates, takes the form

$$
\begin{equation*}
r=g(R), \quad \theta=\Theta, \quad z=Z, \quad \text { for } \quad 0 \leq R \leq R_{2} \tag{2.436}
\end{equation*}
$$

and the identity mapping for all $R>R_{2}$ for some chosen monotonically increasing function $g(R)$ with $g(0) \equiv r_{1} \in\left[0, R_{2}\right], g\left(R_{2}\right)=R_{2} \in \mathbb{R}$ such that $R_{2}<R_{0}$, i.e., the line source remains outside the cloaking region. The cloaking region is thus defined by $r \in\left[r_{1}, r_{2}\right]$, where $r_{2}=R_{2}$. We use upper and lower case variables for the untransformed and transformed problems, respectively. Under this mapping, the form of the governing equation (2.435) remains unchanged for $R=r>R_{2}$, whereas for $0 \leq R \leq R_{2}$, corresponding to the transformed domain $r_{1} \leq r \leq R_{2}$, the transformed
equation takes the form (in transformed cylindrical polar coordinates $(r, \theta=\Theta)$ )

$$
\begin{equation*}
\frac{1}{r} \frac{\partial}{\partial r}\left(r \mu_{r}(r) \frac{\partial w}{\partial r}\right)+\frac{\mu_{\theta}(r)}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}+d(r) \omega^{2} w=0 \tag{2.437}
\end{equation*}
$$

where (see equations (26) and (27) in [70])

$$
\begin{equation*}
\mu_{r}(r)=\frac{\mu_{0}^{2}}{\mu_{\theta}(r)}=\mu_{0} \frac{R}{r} \frac{d g}{d R}, \quad d(r)=\rho_{0} \frac{R}{r}\left(\frac{d g}{d R}\right)^{-1} . \tag{2.438}
\end{equation*}
$$

Hence, both the shear modulus and the density must be inhomogeneous and the shear modulus must be anisotropic. Material properties of this form cannot be constructed exactly since the shear modulus $\mu_{\theta}$ becomes unbounded as $r \rightarrow r_{1}$ (the inner boundary of the cloak). In this limit, the density behaves as $d=\left(\text { pcr }_{1}\right)^{-1} \rho_{0} R^{2-p}+\ldots$, where $p, c>0$ define the mapping in the vicinity of the inner boundary according to $r=$ $r_{1}+c R^{p}+\ldots$ as $R \rightarrow 0$. In practice, of course, approximations are required, as described in, e.g., [31], [92], and [111]. Note that, as expected [70], the total mass is conserved since, regardless of the mapping, the integral of the density $d(r)$ over $r \in\left[r_{1}, r_{2}\right]$ is $\pi R_{2}^{2} \rho_{0}$.

In general, both anisotropy and inhomogeneity are required for cloaking, which can be acheived by creating a material with a complex microstructure. Varying the microstructure of a material enables waves to be tuned by creating stop bands and pass bands in their dispersion relations. Pre-stress can also be used as a tuning mechanism by inducing anisotropy and inhomogeneity as we shall see in the following section and in Chapters 3 and 5 .

### 2.5 Linear elastic waves in pre-stressed solids

Of particular relevance to this thesis is the effect of pre-stress on the propagation and scattering of time-harmonic waves in nonlinear elastic solids. Early work in this area concentrated mainly on homogeneous pre-stress. Homogeneous pre-stress induces anisotropy and, thus, modifies the wave speed in different directions, as we show below.

Consider an infinite incompressible, neo-Hookean solid, which is stretched in the $z$ direction with stretch factor $L$. This deformation can be described in Cartesian
coordinates by the follwoing equations:

$$
\begin{equation*}
x=L^{-\frac{1}{2}} X, \quad y=L^{-\frac{1}{2}} Y, \quad z=L Z \tag{2.439}
\end{equation*}
$$

The Cauchy stress for such a deformation is given by

$$
\boldsymbol{T}=\left(\begin{array}{ccc}
\mu L^{-1}+Q & 0 & 0  \tag{2.440}\\
0 & \mu L^{-1}+Q & 0 \\
0 & 0 & \mu L^{2}+Q
\end{array}\right)
$$

If we assume that the applied stress is only in the $z$ direction, then we must have $Q=-\mu L^{-1}$, and, therefore,

$$
\boldsymbol{T}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{2.441}\\
0 & 0 & 0 \\
0 & 0 & \mu\left(L^{2}-L^{-1}\right)
\end{array}\right)
$$

We now use the theory of small-on-large to consider the propagation of two timeharmonic, plane shear waves in Cartesian coordinates, both polarised in the $y$ direction, but propagating in different directions (the $x$ and $z$ directions respectively). We can express these waves respectively as

$$
\begin{align*}
& \boldsymbol{u}_{1}=u_{1}(x, t) \boldsymbol{e}_{y}=e^{i\left(k_{1} x-\omega t\right)} \boldsymbol{e}_{y},  \tag{2.442}\\
& \boldsymbol{u}_{2}=u_{2}(z, t) \boldsymbol{e}_{y}=e^{i\left(k_{2} z-\omega t\right)} \boldsymbol{e}_{y} . \tag{2.443}
\end{align*}
$$

Using (2.442) and (2.443) in (2.152), the equations governing these waves can be reduced to

$$
\begin{align*}
& \frac{\partial \zeta_{y x}}{\partial x}=\rho \frac{\partial^{2} u_{1}}{\partial t^{2}}  \tag{2.444}\\
& \frac{\partial \zeta_{y z}}{\partial z}=\rho \frac{\partial^{2} u_{2}}{\partial t^{2}} . \tag{2.445}
\end{align*}
$$

By using (2.130) in the above, and inserting the correct values for $\gamma$, we obtain

$$
\begin{align*}
& M_{1212} \frac{\partial^{2} u_{1}}{\partial x^{2}}=\rho \frac{\partial^{2} u_{1}}{\partial t^{2}},  \tag{2.446}\\
& M_{3232} \frac{\partial^{2} u_{2}}{\partial z^{2}}=\rho \frac{\partial^{2} u_{2}}{\partial t^{2}} . \tag{2.447}
\end{align*}
$$

Note that we have assumed that $q$ is a constant, since the static deformation is not spatially dependent.

Substituting (2.442) and (2.443) into the above, we obtain

$$
\begin{align*}
& -M_{1212} k_{1}^{2}=-\rho \omega^{2},  \tag{2.448}\\
& -M_{3232} k_{2}^{2}=-\rho \omega^{2} . \tag{2.449}
\end{align*}
$$

Using (2.129), it can be shown that

$$
\begin{gather*}
M_{1212}=\mu L^{-1},  \tag{2.450}\\
M_{3232}=\mu L^{2} . \tag{2.451}
\end{gather*}
$$

Therefore,

$$
\begin{align*}
k_{1}^{2} & =\frac{\omega^{2}}{c_{1}^{2}}=\frac{L \rho \omega^{2}}{\mu}  \tag{2.452}\\
k_{2}^{2} & =\frac{\omega^{2}}{c_{2}^{2}}=\frac{\rho \omega^{2}}{L^{2} \mu} . \tag{2.453}
\end{align*}
$$

Hence,

$$
\begin{align*}
c_{1} & =\sqrt{\frac{\mu}{L \rho}}  \tag{2.454}\\
c_{2} & =\sqrt{\frac{L^{2} \mu}{\rho}} \tag{2.455}
\end{align*}
$$

Just as in the above example for waves propagating in a transversely isotropic material, we have observed that waves propagating in the $x$ direction travel at a different speed to those propagating in the $z$ direction. This is due the fact that the pre-stress has induced anisotropy in the host material.

Recently there has been an increased focus on inhomogeneous pre-stress. Of considerable interest is the question of whether pre-stress can be used to tune materials or in other words to effect their properties in a such way that the material develops some desirable characteristics. In Parnell [81] it is demonstrated how pre-stress can be used to cloak a cylindrical void in an infinite neo-Hookean material from incoming time-harmonic SH waves and in Parnell et al. [84] the same result is shown for a finite cloak around a cylindrical void.

For further examples of papers on waves in nonlinear elastic solids see, for example, Ogden [71] or Abhyankar and Hanagud [2]. In [71], Ogden considers waves in isotropic Hadamard, Green and harmonic materials and in [2], Abhyankar and Hanagud consider coupled waves in a bar of Mooney-Rivlin material.

### 2.6 Structure of the thesis

In this chapter the literature relevant to the problems discussed in this thesis has been reviewed and the appropriate background material has been provided. At each stage we have also motivated the problems that we will study, referencing the important aspect of a more simple problem that we will build on. In the following chapters we will discuss some specific problems in detail which have not previously appeared in the literature. The focus of these problems is wave propagation in pre-stressed nonlinear elastic materials where inhomogeneous pre-stress has a significant influence.

In Chapter 3 we discuss torsional wave propagation in a pre-stressed, annular, circular cylinder. First we discuss the static deformation of an annular circular cylinder under a homogeneous longitudinal stretch and a pressure applied to its inner and outer surfaces. This deformation is radially inhomogenous and leads to an inhomogeneous stress field. Next, the theory of small-on-large is implemented to determine how this pre-stress affects the torsional wave equation and this pre-stressed equation is analysed via (a) numerical methods, and (b) a Liouville-Green approximation. Finally, we use these numerical and asymptotic solutions to determine the effect of the pre-stress on the dispersion curves for torsional waves.

In Chapter 4 we discuss the scattering of SH waves from an infinite cylindrical cavity in a pre-stressed, infinite, nonlinear elastic material. First we discuss the deformation of the infinite material under a homogeneous longitudinal stretch and a pressure applied at infinity. Again, the deformation is radially inhomogeneous and leads to an inhomogeneous stress field. Next, the theory of small-on-large is implemented to determine how this pre-stress affects the SH wave equation. An analytical
solution can be found if the material under consideration is neo-Hookean, but, unfortunately, this is not the case if the material is Mooney-Rivlin. For a Mooney-Rivlin material, we analyse the governing equation via (a) a hybrid analytical-numerical method and (b) by discretising the material into multiple layers and treating each one as homogeneous; as the number of layers increases, the solution obtained becomes more accurate. Finally, we determine the effect of the pre-stress on the scattering coefficients.

In Chapter 5 we build on the work of Chapter 4 and discuss how to employ prestress to generate finite cloaks for antiplane elastic waves in a material characterised by a neo-Hookean strain energy function. This work is summarised in the paper by Parnell et al. [84]. We then extend this work to show how imperfect cloaks are generated when the Mooney-Rivlin strain energy function is used.

In Chapter 6 we discuss the scattering of shear waves from a spherical cavity in a pre-stressed, infinite, nonlinear elastic material. First we discuss the deformation of the infinite material under a pressure applied at infinity. Again, the deformation is radially inhomogeneous and leads to an inhomogeneous stress field. Next, the theory of small-on-large is implemented to determine how this pre-stress affects the governing equations for waves in an incompressible material in spherical coordinates. We reduce the governing equations down to ordinary differential equations and analyse them numerically, and finally, we determine the effect of the pre-stress on the scattering coefficients.

We shall conclude in Chapter 7 by discussing the common features of the problems discussed, and will indicate possible extensions of the work undertaken in this thesis. We will also discuss how the problems considered can assist with the modelling of composite materials, such as the one mentioned in Chapter 1. In particular, we note that a compressed inclusion does not behave like a small inclusion.

## Chapter 3

## Torsional wave propagation in a pre-stressed annular circular

## cylinder

### 3.1 Overview

In Chapter 2, we saw how one can obtain the dispersion curves for torsional waves in an unstressed annular cylinder. Here we shall consider how these can be modified by the application of pre-stress. In particular we shall see that pre-stress acts as a tuning mechanism, enabling us to turn waves "on and off" at a given frequency. Hydrostatic pressure is applied to the inner and outer surfaces of an incompressible nonlinear elastic annular cylinder, of circular cross-section, whose constitutive behaviour is governed by a Mooney-Rivlin strain energy function. The pressure difference creates an inhomogeneous deformation field and modifies the inner and outer radii of the annular cylinder. We deduce the effect that this pre-stress, and a given axial stretch, has on the propagation of small-amplitude torsional waves through the medium. We use the theory of small-on-large to determine the linear wave equation that governs incremental torsional waves and then determine the dispersion relation for the pre-stressed annulus by using an approximate scheme (the Liouville-Green transformation). We show that this scheme compares well to numerical solutions except in
regions very close to turning points. In particular we stress that the inhomogeneous deformation makes the coefficients of the governing ODE spatially dependent and affects the location of the roots of the dispersion relation. We observe that, if the pressure on the outer surface of the annular cylinder is greater (smaller) than that on the inner, then the cut-on frequencies are spaced further apart (closer) than they would be in the stress-free case. This result could potentially be used to tune the propagation characteristics of the cylinder over a range of frequencies.

### 3.2 Introduction

Over the past few decades, much interest has been centred on the effect of pre-stress on the propagation of incremental linear waves in elastic media using the theory of small-on-large [38], [73], where a small perturbation is applied to a body which has undergone a finite deformation. Since the perturbation is considered to be small in relation to the initial deformation, a linearisation is applied in order to determine the characteristics of wave propagation in the pre-stressed material. Attention has been focused mainly on the effect of homogeneous deformation on wave propagation, which induces anisotropy (examples of this are given in [27] and [51]). However, pre-stress in an inhomogeneous material almost always leads to inhomogeneous deformations, except in special cases (see [80], where the deformation of a one-dimensional composite bar is assumed to be piecewise homogeneous). Dey [28] analysed the case of initial tension in a solid rod of circular cross-section in the context of linear elasticity. The interest was to determine how the presence of the tension affects the subsequent propagation of torsional waves through the material. This work utilised the incremental deformation theory derived by Biot [11]. We emphasise that the pre-stress was homogeneous in this case, which meant that the incremental equations were straightforward to solve. Of great interest, however, is how an initial inhomogenous pre-stress affects subsequent torsional waves which propagate through inhomogeneous media. In this chapter we consider the nonlinear deformation of a cylindrical annulus and show that such a deformation leads to a more complicated azimuthal governing

ODE whose coefficients are spatially dependent. As mentioned previously, nonlinear pre-stress can be useful in practice, allowing us to tune materials in order to permit or restrict waves of specific frequency ranges. Parnell described this property in [80] and it is discussed further in subsequent articles in different contexts [9], [35].

Torsional wave propagation in the linear elastic (unstressed) regime is relatively well understood [10]. However, if the host material is nonlinear-elastic (e.g. rubber) we can expect pressures applied to the surfaces of an annular cylinder to lead to a nonlinear deformation if such pressures are of the same order of magnitude as the shear modulus of the material and hence we can postulate the following question: How does the difference in pressure on the surfaces of a nonlinear-elastic, annular cylinder affect the propagation of subsequent linear elastic torsional waves? In this chapter we shall consider a torsional wave propagating in an annular cylinder which is capable of finite deformation and is incompressible and Mooney-Rivlin in its constitutive behaviour. A related initial value problem in the case of no pre-stress is discussed in [30], and for a solid cylinder on pp. 148-155 of [10]. A related pre-stress problem is that which was studied in [78]. However, in [78], the pre-stress was assumed to be uniform, i.e. stress distributions in the host materials were homogeneous. This is, therefore, a simpler problem than that to be discussed in this article, since the only effect of this pre-stress would have been to induce anisotropy in the media under consideration.

In Section 3.3 we use the condition of incompressibility to determine the form of the applied radial deformation. The static equations of equilibrium are determined and they are used to determine the effect of the applied pressures and longitudinal stretch on the radii. In Section 3.4 we consider the propagation of small-amplitude, time-harmonic waves through the finitely-deformed medium and give the relevant governing equation and discuss the case of no pre-stress. In Section 3.4.3 we discuss the governing ODE, which is hard to solve due to the spatial dependence of its coefficients, as mentioned above, and in Section 3.4.4 we derive an approximate solution for the ODE using the Liouville-Green method. In Section 3.5 we show the effect of the pre-stress on the dispersion curves using both a numerical method and the

Liouville-Green approximation.

### 3.3 Initial finite (static) deformation

Consider an incompressible, annular cylinder with circular cross-section and initial inner and outer radii $A$ and $B$ respectively. We will assume that the annular cylinder is isotropic and that its constitutive behaviour may be described by a strain energy function, $W=W\left(I_{1}, I_{2}, I_{3}\right)$, where $I_{j}$ are the principal strain invariants of the deformation [38], [73]. Since it is incompressible, $I_{3}=1$, and thus $W=W\left(I_{1}, I_{2}\right)$. We suppose that radial pressures are applied on the inner and outer radii of the cylinder (which would occur, for example, if the cylinder were immersed in an inviscid fluid) so that under such loading the inner and outer radii are deformed to $a$ and $b$, respectively. The above deformation can be described by

$$
\begin{equation*}
R=R(r), \quad \Theta=\theta, \quad Z=\frac{z}{L} \tag{3.1}
\end{equation*}
$$

where $(R, \Theta, Z)$ and $(r, \theta, z)$ are cylindrical polar coordinates in the undeformed and deformed configurations respectively and $R(r)$ is a function to be determined from the radial equation of equilibrium. Note the convention introduced in (3.1) above, i.e. that upper case variables correspond to the undeformed configuration whilst lower case corresponds to the deformed configuration. Note, also, that it will be convenient for us to derive equations in terms of coordinates in the deformed configuration, hence the form assumed in (3.1). Position vectors in the undeformed and deformed configurations are

$$
\boldsymbol{X}=\left(\begin{array}{c}
R \cos \Theta  \tag{3.2}\\
R \sin \Theta \\
Z
\end{array}\right)=\left(\begin{array}{c}
R(r) \cos \theta \\
R(r) \sin \theta \\
z / L
\end{array}\right), \quad \boldsymbol{x}=\left(\begin{array}{c}
r \cos \theta \\
r \sin \theta \\
z
\end{array}\right)
$$

Using (3.1), it can be shown that the principal stretches for this deformation in the radial, azimuthal and longitudinal directions are respectively

$$
\begin{equation*}
\lambda_{r}=\frac{d r}{d R}=\frac{1}{R^{\prime}(r)}, \quad \quad \lambda_{\theta}=\frac{r}{R}=\frac{r}{R(r)}, \quad \lambda_{z}=L \tag{3.3}
\end{equation*}
$$

where prime denotes differentiation with respect to $r$. We define the deformation gradient tensor by $\boldsymbol{F}=\operatorname{Grad} \boldsymbol{x}$, where Grad represents the gradient operator with respect to the undeformed configuration. In our case, we have

$$
\boldsymbol{F}=\left(\begin{array}{ccc}
\lambda_{r} & 0 & 0  \tag{3.4}\\
0 & \lambda_{\theta} & 0 \\
0 & 0 & \lambda_{z}
\end{array}\right)=\left(\begin{array}{ccc}
1 / R^{\prime}(r) & 0 & 0 \\
0 & r / R(r) & 0 \\
0 & 0 & L
\end{array}\right)
$$

For an incompressible material, we must have $J=\operatorname{det} \boldsymbol{F}=1$, and so

$$
\begin{equation*}
\lambda_{r} \lambda_{\theta} \lambda_{z}=\frac{L r}{R(r) R^{\prime}(r)}=1, \tag{3.5}
\end{equation*}
$$

which is an ordinary differential equation that we can solve straightforwardly to obtain

$$
\begin{equation*}
R(r)=\sqrt{L\left(r^{2}+\alpha\right)}, \tag{3.6}
\end{equation*}
$$

where $\alpha$ is a constant defined by

$$
\begin{equation*}
\alpha=\frac{A^{2}}{L}-a^{2}=\frac{B^{2}}{L}-b^{2} . \tag{3.7}
\end{equation*}
$$

Note that (3.6) gives the same form for $R$ as in [82] ( $\alpha$ in this chapter is equivalent to $M$ in [82]).

From [26], the Cauchy stress tensor for an incompressible material is given by

$$
\begin{equation*}
\boldsymbol{T}=\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}+Q \boldsymbol{I} \tag{3.8}
\end{equation*}
$$

where $W$ is the strain energy function of the material under consideration, $\boldsymbol{I}$ is the identity tensor and $Q$ is a Lagrange multiplier associated with the incompressibility constraint and referred to as an arbitrary hydrostatic pressure. As discussed in the introduction, 3.2, we will use the Mooney-Rivlin strain energy function:
$W=\frac{\mu}{2}\left(S_{1}\left(I_{1}-3\right)+S_{2}\left(I_{2}-3\right)\right)=\frac{\mu}{2}\left(S_{1}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right)+S_{2}\left(\lambda_{r}^{2} \lambda_{\theta}^{2}+\lambda_{r}^{2} \lambda_{z}^{2}+\lambda_{\theta}^{2} \lambda_{z}^{2}-3\right)\right)$.

Componentwise, (3.8) is equivalent to

$$
\begin{equation*}
T_{i j}=F_{i \alpha} \frac{\partial W}{\partial F_{j \alpha}}+Q \delta_{i j}, \tag{3.10}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta. We note that since $\boldsymbol{F}$ is diagonal, so is $\boldsymbol{T}$ and therefore for the strain energy function in (3.9) we find that

$$
\begin{align*}
& T_{r r}=\left[\chi S_{1}+\left(\frac{1}{L}+L^{2} \chi\right) S_{2}\right] \frac{\mu}{L}+Q \\
& T_{\theta \theta}=\left[\frac{S_{1}}{\chi}+\left(\frac{1}{L}+\frac{L^{2}}{\chi}\right) S_{2}\right] \frac{\mu}{L}+Q  \tag{3.11}\\
& T_{z z}=\left[L S_{1}+\left(2+\frac{\alpha^{2}}{r^{4} \chi}\right) S_{2}\right] L \mu+Q
\end{align*}
$$

where for convenience we have defined $\chi=\chi(r)=\left(r^{2}+\alpha\right) / r^{2}$. In the case considered here, the static equations of equilibrium, $\operatorname{div} \boldsymbol{T}=\mathbf{0}$ (where div signifies the divergence operator with respect to the deformed configuration), reduce to

$$
\begin{equation*}
\frac{\partial T_{r r}}{\partial r}+\frac{1}{r}\left(T_{r r}-T_{\theta \theta}\right)=0, \quad \frac{\partial T_{\theta \theta}}{\partial \theta}=0, \quad \frac{\partial T_{z z}}{\partial z}=0 \tag{3.12}
\end{equation*}
$$

which then become

$$
\begin{equation*}
\frac{\partial Q}{\partial r}=\frac{\mu \alpha^{2} S}{r^{3}\left(\alpha+r^{2}\right)}, \quad \frac{\partial Q}{\partial \theta}=\frac{\partial Q}{\partial z}=0 \tag{3.13}
\end{equation*}
$$

where we have defined $S=S_{1} / L+L S_{2}$. Therefore, $Q$ depends only on $r$ and can be expressed as

$$
\begin{equation*}
Q=-\frac{\mu S}{2}\left(\log \left(\frac{r^{2}}{r^{2}+\alpha}\right)+\frac{\alpha}{r^{2}}\right)+Q_{0} \tag{3.14}
\end{equation*}
$$

where $Q_{0}$ is a constant that can be deduced from the boundary conditions. This result can be used in (3.12) to determine explicit expressions for the stresses $T_{r r}, T_{\theta \theta}$ and $T_{z z}$.

If we label the hydrostatic pressures applied to the inner and outer surfaces of the annulus, respectively, as $p_{i n}$ and $p_{o u t}$, then, upon applying the boundary conditions $\left.T_{r r}\right|_{r=a}=-p_{\text {in }}$ and $\left.T_{r r}\right|_{r=b}=-p_{o u t}$, and rewriting the result in terms of the undeformed radii $A$ and $B$ and the parameter $\alpha$, we find that

$$
\begin{equation*}
p_{\text {out }}-p_{\text {in }}=\mu S\left(\frac{\left(B^{2}-A^{2}\right) \alpha}{2\left(A^{2}-L \alpha\right)\left(B^{2}-L \alpha\right)}+\log \left(\frac{A}{B} \sqrt{\frac{B^{2}-L \alpha}{A^{2}-L \alpha}}\right)\right) \tag{3.15}
\end{equation*}
$$

Therefore, if we know the undeformed radii $A$ and $B$, the applied pressure difference $p_{o u t}-p_{\text {in }}$, the material constants $\mu, S_{1}$ and $S_{2}$ and the applied stretch $L$, (3.15) can be used to determine the value of $\alpha$, which can in turn be used to determine the deformed radii $a$ and $b$ from (3.7).


Figure 3.1: Torsional waves in a pre-stressed annular cylinder.

### 3.4 Incremental deformations

We now consider the propagation of small-amplitude time-harmonic waves through the finitely-deformed medium. We use the theory of small-on-large, i.e. linearisation about a non-linear deformation state [73]. The total displacement field may be represented by

$$
\begin{equation*}
\bar{U}=\boldsymbol{U}+\boldsymbol{u} \tag{3.16}
\end{equation*}
$$

where $\boldsymbol{U}$ is the displacement field derived from the finite deformation (3.1) and $\boldsymbol{u}$ is the incremental displacement. Let us assume that the incremental displacement is of the form

$$
\begin{equation*}
\boldsymbol{u}=\Re\left((0, v(r), 0) e^{i(k z-\omega t)}\right), \tag{3.17}
\end{equation*}
$$

so that it is a torsional wave (see Figure 3.1). We will also assume that $|\boldsymbol{u}| \ll|\boldsymbol{U}|$. The gradient of $\boldsymbol{u}$ with respect to the deformed configuration is given by

$$
\gamma=\operatorname{grad} \boldsymbol{u}=\left(\begin{array}{ccc}
0 & -\frac{v}{r} & 0  \tag{3.18}\\
\frac{d v}{d r} & 0 & i k v \\
0 & 0 & 0
\end{array}\right) e^{i(k z-\omega t)}
$$

The push forward of the nominal stress for an incompressible material is given by

$$
\begin{equation*}
\boldsymbol{\zeta}=\boldsymbol{M}: \boldsymbol{\gamma}+q \boldsymbol{I}-Q \boldsymbol{\gamma} \tag{3.19}
\end{equation*}
$$

where $\boldsymbol{M}$ is the push forward of the elasticity tensor defined by

$$
\begin{equation*}
M_{i j k l}=J^{-1} \frac{\partial^{2} W}{\partial F_{j m} \partial F_{l n}} F_{i m} F_{k n} \tag{3.20}
\end{equation*}
$$

often denoted by $\mathcal{A}_{\mathbf{0}}$ and $q$ is the increment of $Q$. We will assume that $q=q(r, z)$ (i.e. independent of $\theta$ since there is no $\theta$ dependence in either the pre-stress or the perturbation).

The incremental equations of motion are then given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\zeta}=\rho \frac{\partial^{2} u}{\partial t^{2}} \tag{3.21}
\end{equation*}
$$

where $\rho$ is the density of the body, which remains constant throughout the deformation, since we are considering an incompressible material.

Using (3.21), we observe that the radial and axial equations give respectively

$$
\begin{equation*}
\frac{\partial \zeta_{11}}{\partial r}=0, \quad \frac{\partial \zeta_{33}}{\partial z}=0 \tag{3.22}
\end{equation*}
$$

and the azimuthal equation is given by

$$
\begin{equation*}
\frac{\partial \zeta_{12}}{\partial r}+\frac{\partial \zeta_{32}}{\partial z}+\frac{\zeta_{12}+\zeta_{21}}{r}=-\rho \omega^{2} v(r) e^{i(k z-\omega t)} \tag{3.23}
\end{equation*}
$$

Now, from [26], we have

$$
\begin{equation*}
J M_{i i j j}=\lambda_{i} \lambda_{j} W_{i j}, \tag{3.24}
\end{equation*}
$$

and, when $i \neq j, \lambda_{i} \neq \lambda_{j}$,

$$
\begin{equation*}
J M_{i j i j}=\frac{\lambda_{i} W_{i}-\lambda_{j} W_{j}}{\lambda_{i}^{2}-\lambda_{j}^{2}} \lambda_{i}^{2}, \quad J M_{i j j i}=\frac{\lambda_{j} W_{i}-\lambda_{i} W_{j}}{\lambda_{i}^{2}-\lambda_{j}^{2}} \lambda_{i} \lambda_{j}, \tag{3.25}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{i}=\frac{\partial W}{\partial \lambda_{i}}, \quad W_{i j}=\frac{\partial^{2} W}{\partial \lambda_{i} \partial \lambda_{j}}, \tag{3.26}
\end{equation*}
$$

and we note that, in the above, there is no implied summation over repeated indices and, for all other combinations of $i, j, k$ and $l$, we have $M_{i j k l}=0$.

Therefore, using (3.19), we have

$$
\begin{equation*}
\zeta_{11}=q, \quad \zeta_{22}=q, \quad \zeta_{33}=q, \tag{3.27}
\end{equation*}
$$

$$
\begin{align*}
& \zeta_{12}=\left(\frac{S_{2} \mu+L^{2} Q}{L^{2} r} v(r)+\frac{1}{L}\left(1+\frac{\alpha}{r^{2}}\right)\left(S_{1}+L^{2} S_{2}\right) \mu v^{\prime}(r)\right) e^{i(k z-\omega t)},  \tag{3.28}\\
& \zeta_{21}=-\left(\frac{\left(S_{1}+L^{2} S_{2}\right) \mu r}{L\left(\alpha+r^{2}\right)} v(r)+\frac{S_{2} \mu+L^{2} Q}{L^{2}} v^{\prime}(r)\right) e^{i(k z-\omega t)},  \tag{3.29}\\
& \zeta_{23}=-i k\left(\frac{L r^{2} S_{2} \mu}{\alpha+r^{2}}+Q\right) v(r) e^{i(k z-\omega t)},  \tag{3.30}\\
& \zeta_{32}=i k L\left(\frac{\alpha S_{2}}{r^{2}}+L S_{1}+S_{2}\right) \mu v(r) e^{i(k z-\omega t)}, \tag{3.31}
\end{align*}
$$

which referring to (3.23) leads to the following governing equation for $v(r)$ :

$$
\begin{equation*}
\left(1+\frac{\alpha}{r^{2}}\right) v^{\prime \prime}(r)+\frac{1}{r}\left(1-\frac{\alpha}{r^{2}}\right) v^{\prime}(r)+\left(\beta^{2}-\frac{1+\alpha \delta}{r^{2}}+\frac{\alpha}{r^{4}}\right) v(r)=0 \tag{3.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta^{2}=\frac{k_{0}^{2}-k^{2} L^{2} S_{1}}{S}-\delta, \quad \delta=\frac{k^{2} L S_{2}}{S} \tag{3.33}
\end{equation*}
$$

where $k_{0}^{2}=\rho \omega^{2} / \mu$ is the zeroth order torsional wavenumber of an unstressed cylinder, $\rho$ represents the mass density, $\alpha$ is defined in (3.7) and we remind the reader that $S=S_{1} / L+L S_{2}$.

Finally (3.22) give

$$
\begin{equation*}
\frac{\partial q}{\partial r}=\frac{\partial q}{\partial z}=0 \tag{3.34}
\end{equation*}
$$

so that $q=q(r, z)$ must be a constant. The boundary conditions, to be discussed next, dictate this constant.

### 3.4.1 Boundary conditions

We apply the condition that the perturbation $v(r)$ does not affect the pressure on the surfaces $r=a, b$ of the deformed annulus. For this to hold, it can be shown that the following condition must be satisfied on the boundaries:

$$
\begin{equation*}
\boldsymbol{\tau} \boldsymbol{n}=\boldsymbol{T} \boldsymbol{\gamma}^{T} \boldsymbol{n}+p \boldsymbol{\gamma}^{T} \boldsymbol{n}, \tag{3.35}
\end{equation*}
$$

where $p$ represents either $p_{\text {in }}$ or $p_{\text {out }}$, depending on which boundary is under consideration, and $\boldsymbol{n}$ is the outer unit normal to the boundary in the deformed configuration. $\boldsymbol{\tau}$ is the perturbed Cauchy stress, given by

$$
\begin{equation*}
\boldsymbol{\tau}=\boldsymbol{\zeta}+\boldsymbol{\gamma} \boldsymbol{T} . \tag{3.36}
\end{equation*}
$$

Equation (3.35) simplifies to

$$
\begin{equation*}
v^{\prime}(a)-\frac{v(a)}{a}=v^{\prime}(b)-\frac{v(b)}{b}=0, \tag{3.37}
\end{equation*}
$$

and $q=0$.

### 3.4.2 The case of no pre-stress

To study the effects of pre-stress it is first useful to examine torsional waves in the absence of such static loads. Zero pre-stress corresponds to the situation when $\alpha=0$ and $L=1$ and in this case (3.32) reduces to

$$
\begin{equation*}
v^{\prime \prime}(r)+\frac{1}{r} v^{\prime}(r)+\left(\frac{\omega^{2}}{c^{2}}-k^{2}-\frac{1}{r^{2}}\right) v(r)=0 \tag{3.38}
\end{equation*}
$$

where $c^{2}=\mu / \rho$. This is Bessel's equation and is the standard linear elastic torsional wave equation [10]. The solution of (3.38) is thus

$$
\begin{equation*}
v(r)=C_{1} \mathrm{~J}_{1}(s r)+C_{2} \mathrm{Y}_{1}(s r), \tag{3.39}
\end{equation*}
$$

where $J_{1}$ and $Y_{1}$ are the first order Bessel functions of the first and second kind respectively and $s^{2}=\omega^{2} / c^{2}-k^{2}$. Imposing the conditions (??), we obtain the classical dispersion relation

$$
\begin{equation*}
\frac{a s \mathrm{~J}_{0}(a s)-2 \mathrm{~J}_{1}(a s)}{b s \mathrm{~J}_{0}(b s)-2 \mathrm{~J}_{1}(b s)}=\frac{a s \mathrm{Y}_{0}(a s)-2 \mathrm{Y}_{1}(a s)}{b s \mathrm{Y}_{0}(b s)-2 \mathrm{Y}_{1}(b s)} \tag{3.40}
\end{equation*}
$$

This dispersion relation leads to the well-known result that a thicker cylindrical annulus (i.e. smaller value of $a / b$, with $a=0$ in the solid cylinder limit) will allow more modes of propagation for a given fixed frequency. Our main interest is thus how this dispersion relation is modified by the nonlinear elastic pre-stress and its dependence on the Mooney-Rivlin strain energy function parameter $S_{1}=1-S_{2}$. In Figure 3.2, we plot the distance from origin to the first, second and third roots of the dispersion relation as a function of $a / b$ with $b=1$.

### 3.4.3 Non-zero pre-stress: Singularity structure

Note that in the case of non-zero pre-stress the governing equation (3.32) has regular singular points at $r=0, r=\sqrt{\alpha} i$ and $r=-\sqrt{\alpha} i$ and an irregular singular point at


Figure 3.2: The first (solid), second (dashed) and third (dotted) cut-on frequencies of equation (3.40) as a function of $\frac{a}{b}$.
$r=\infty$. All other points are ordinary points. To the authors' knowledge (3.32) is not a well-known and classified ODE, so we will investigate the solution. Note that it is difficult to solve due to the spatial dependence of its coefficients and the singular nature of the zero pre-stress limit.

Frobenius' method can be used to determine an approximate solution in the case when $r<\alpha$ up to two arbitrary constants in terms of an infinite series. Unfortunately, however, this condition is extremely resrtictive. In order to correctly predict the dispersion relations using this method, the entire annulus must be within the range of validity of the expansion and, therefore, we must have $b<\alpha$. This condition will, in general, only hold for an extremely large pre-stress, so, instead, in the following section, we present a Liouville-Green approximate solution for the ODE.

### 3.4.4 Liouville-Green solution

Here we consider an approach to determining solutions to the ODE, (3.32), based on the Liouville-Green approximation. The Liouville-Green approximation is closely related to WKB method (see Appendix A).

Let us restate the original ODE for incremental waves :

$$
\begin{equation*}
\left(1+\frac{\alpha}{r^{2}}\right) v^{\prime \prime}(r)+\frac{1}{r}\left(1-\frac{\alpha}{r^{2}}\right) v^{\prime}(r)+\left(\beta^{2}-\frac{1+\alpha \delta}{r^{2}}+\frac{\alpha}{r^{4}}\right) v(r)=0 \tag{3.41}
\end{equation*}
$$

First divide (3.41) by the term multiplying the highest derivative so that it can be written in the form

$$
\begin{equation*}
v^{\prime \prime}(r)+P(r) v^{\prime}(r)+Q(r) v(r)=0 \tag{3.42}
\end{equation*}
$$

with $P(r)$ and $Q(r)$ defined as

$$
\begin{equation*}
P(r)=\frac{1}{r} \frac{r^{2}-\alpha}{r^{2}+\alpha}, \quad Q(r)=\frac{1}{r^{2}+\alpha}\left(r^{2} \beta^{2}-\left((1-\alpha \delta)-\frac{\alpha}{r^{2}}\right)\right) . \tag{3.43}
\end{equation*}
$$

We can now eliminate the first derivative in (3.42) using the following transformation of the dependent variable $v$,

$$
\begin{equation*}
v(r)=w(r) e^{-\frac{1}{2} \int P(r) d r} \tag{3.44}
\end{equation*}
$$

so that (3.42) may be transformed into the following ODE

$$
\begin{equation*}
w^{\prime \prime}(r)+q(r) w(r)=0 . \tag{3.45}
\end{equation*}
$$

Here the polynomial quotient function $q(r)$ is defined as

$$
\begin{equation*}
q(r)=\frac{\hat{q}(\hat{r})}{b^{2}} \tag{3.46}
\end{equation*}
$$

where

$$
\begin{gather*}
\hat{q}(\hat{r})=\frac{4 \hat{\beta}^{2} \hat{r}^{6}+\left(4 \hat{\alpha}\left(\hat{\beta}^{2}+\hat{\delta}\right)-3\right) \hat{r}^{4}+2 \hat{\alpha}(2 \hat{\alpha} \hat{\delta}-3) \hat{r}^{2}+\hat{\alpha}^{2}}{4 \hat{r}^{2}\left(\hat{r}^{2}+\hat{\alpha}\right)^{2}},  \tag{3.47}\\
\hat{r}=\frac{r}{b},  \tag{3.48}\\
\hat{\alpha}=\frac{\alpha}{b^{2}}=\frac{1}{L}\left(\frac{B}{b}\right)^{2}-1,  \tag{3.49}\\
\hat{\beta}^{2}=b^{2} \beta^{2}=\frac{\hat{k}_{0}^{2}-\hat{k}^{2} L^{2} S_{1}}{S}-\hat{\delta},  \tag{3.50}\\
\hat{\delta}=b^{2} \delta=\frac{\hat{k}^{2} L S_{2}}{S}  \tag{3.51}\\
\hat{k}_{0}^{2}=b^{2} k_{0}^{2} \tag{3.52}
\end{gather*}
$$

and

$$
\begin{equation*}
\hat{k}^{2}=b^{2} k^{2} \tag{3.53}
\end{equation*}
$$

Note that the above variables are all non-dimensional. We have introduced the hat notation $(\hat{\bullet})$ to signify this.

Assuming for now that $q(r)$ is a positive and twice continuously differentiable function, we apply the Liouville-Green transformation [76] by defining

$$
\begin{equation*}
\xi(r)=\int \sqrt{q(\varrho)} d \varrho, \quad \mathcal{W}(\xi)=\left\{\xi^{\prime}(r)\right\}^{1 / 2} w(r(\xi)) \tag{3.54}
\end{equation*}
$$

Upon doing this we obtain the governing equation

$$
\begin{equation*}
\frac{d^{2} \mathcal{W}}{d \xi^{2}}+(1-\varphi(\xi)) \mathcal{W}(\xi)=0 \tag{3.55}
\end{equation*}
$$

where
$\varphi(\xi(r))=\psi(r)=\frac{4 q(r) q^{\prime \prime}(r)-5\left\{q^{\prime}(r)\right\}^{2}}{16\{q(r)\}^{3}}=-\frac{1}{\{q(r)\}^{3 / 4}} \frac{d^{2}}{d r^{2}}\left(\frac{1}{\{q(r)\}^{1 / 4}}\right)$.
If the function $\varphi$ is assumed small and hence neglected, then independent solutions of (3.55) are $\mathrm{e}^{ \pm i \xi}$. Restoring the original dependent $(w)$ and independent $(r)$ variables we obtain

$$
\begin{equation*}
w(r)=C_{1}\left(\frac{1}{q(r)}\right)^{1 / 4} \mathrm{e}^{i \xi(r)}+C_{2}\left(\frac{1}{q(r)}\right)^{1 / 4} \mathrm{e}^{-i \xi(r)} \tag{3.57}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are constants of integration. Notice that the restriction $q(r)>0$ ensures that the solutions (3.57) are wave-like. When $q(r)$ changes its sign in the interval of $r$ under consideration, we have then what are conventionally called turning points or transition points of the ODE (i.e. zeros of $q$ ). In that case the appropriate approximation functions are the Airy (one turning point) or Weber parabolic cylinder functions (two turning points) [76]. The error in this approximation is obviously dependent on the behaviour of the neglected function $\varphi(\xi(r))=\psi(r)$. If $\{q(r)\}^{-1 / 4}$ is small or slowly varying we have then a good approximation for the ODE. Finally, when $q(r)$ is positive and the interval under consideration is far from the turning point(s) it can be shown that the functions $\mathrm{e}^{ \pm i \xi}$ are suitable ones for the approximate solution of the ODE.

Our final step is to transform back to the original dependent variable $v$ using the following transformation

$$
\begin{equation*}
v(r)=w(r) \mathrm{e}^{-\frac{1}{2} \int P(\hat{r}) d r}=w(r) \mathrm{e}^{-\frac{1}{2} \log \left(\left(\alpha+r^{2}\right) / r\right)} . \tag{3.58}
\end{equation*}
$$

Upon doing this we obtain the final form of the approximate solution of the ODE

$$
\begin{equation*}
v(r)=C_{1}\left(\frac{1}{q(r)}\right)^{1 / 4} \mathrm{e}^{i \xi(r)-\frac{1}{2} \log \left(\left(\alpha+r^{2}\right) / r\right)}+C_{2}\left(\frac{1}{q(r)}\right)^{1 / 4} \mathrm{e}^{-i \xi(r)-\frac{1}{2} \log \left(\left(\alpha+r^{2}\right) / r\right)} \tag{3.59}
\end{equation*}
$$

Notice that the Liouville-Green approximation implies the transformation of the independent variable $r$ into $\xi(r)$, obtained via the integration of the square root of the function $q(r)$. For many problems, an explicit form for this integral is not generally available. However in Appendix B we show that for the present problem this integral can be explicitly evaluated in terms of the three incomplete elliptic integrals and elementary functions in the form

$$
\begin{align*}
& \xi(r)=\frac{|\beta|}{2}\left\{\frac{2}{d} \frac{\sqrt{1-m \sin ^{2} \phi} \sqrt{1-\sin ^{2} \phi}}{\sin \phi}+\frac{2}{d} E(\phi \mid m)+\mathcal{D}_{1} F(\phi \mid m)+\right. \\
&\left.\mathcal{D}_{2} \Pi\left(n_{1} ; \phi \mid m\right)+\mathcal{D}_{3} \Pi\left(n_{2} ; \phi \mid m\right)\right\}, \tag{3.60}
\end{align*}
$$

where $\phi=\phi(r)$ is a known function, given in equation (B.9). The other terms present in (3.60) are also defined in Appendix B.

In Figures 3.3 and 3.4 we give some plots of the regions where $\hat{q}(\hat{r})<0$ (and hence $q(r)<0$ ) for specific values of $\hat{\alpha}$ and $\hat{\delta}$ (the regions below the curves are the regions where $\hat{q}(\hat{r})$ is negative). In the plots $\hat{r}$ ranges over the horizontal axis and $\hat{\beta}$ ranges over the vertical axis. We observe that increasing values of $\hat{\alpha}$ and $\hat{\delta}$ improve the range over which the method is expected to work. Increasing $|\hat{\alpha}|$ corresponds to increasing pre-stress, $\hat{\alpha}>0$ corresponds to $p_{\text {out }}>p_{\text {in }}$ and $\hat{\alpha}<0$ corresponds to $p_{\text {out }}<p_{\text {in }}$. Increasing $\hat{\delta}$ corresponds to increasing dependence of the particular strain energy function on the parameter $S_{2}$.

We know that, for a stress-free annular cylinder with $a=0$ (i.e. the solid cylinder limit), the smallest non-zero value $\hat{\beta}$ can take is $\approx 5.14$. Increasing the inner radius from 0 increases the smallest non-zero value of $\hat{\beta}$ (see Figure 3.2 and note that in this figure, increasing $\omega / c$ corresponds to increasing $\hat{\beta}$ ). We have also shown that increasing the value of $\hat{\alpha}$ improves the range of validity of the Liouville-Green method. Therefore, in Figure 3.5, we show that for $\hat{\alpha}$ greater than $\approx 0.087$ the Liouville-Green method should work for all annular cylinders. We also show that for a greater value of


Figure 3.3: Curves of $\hat{q}(\hat{r})=0$. Solid line: $\hat{\alpha}=0, \hat{\delta}=0$; dashed line: $\hat{\alpha}=0.01$, $\hat{\delta}=0$.


Figure 3.4: Curves of $\hat{q}(\hat{r})=0$. Solid line: $\hat{\alpha}=0.1, \hat{\delta}=0$; dashed line: $\hat{\alpha}=0.1$.


Figure 3.5: Curves of $\hat{q}(\hat{r})=0$. Dashed line: $\hat{\alpha}=0.087, \hat{\delta}=0$; dotted line: $\hat{\alpha}=$ 0.0409 .
$\hat{\delta}$ this critical value of $\hat{\alpha}$ is decreased. Note that Figure 3.7 reveals that, for negative pressures, the distance to the first non-zero root of the dispersion curves decreases slowly, and so we expect the region where the Liouville-Green method is inaccurate increases slowly too.

### 3.5 Dispersion curves: Predictions via numerical and Liouville-Green solutions

### 3.5.1 Neo-Hookean case

In the neo-Hookean case, we have $S_{2}=0$ and, therefore, $\delta=0$, so the dispersion relations can be determined simply by determining the values of $\beta$ which satisfy the boundary conditions. Using a numerical solver one can obtain an interpolating polynomial solution for $v(r)$. Alternatively, the Liouville-Green solution described in the previous section can be used. Both methods can be employed to find values of $\beta$ which satisfy the boundary conditions, and thus define the dispersion curves for the problem.

The numerical solver used in this case was NDSolve in Mathematica 7. The numerical method used by NDSolve is selected automatically unless specified by the user, and was not specified in this case. For ordinary differential equations, NDSolve by default uses an LSODA (Livermore solver for ordinary differential equations) approach, switching between a non-stiff Adams method and a stiff Gear backward differentiation formula method. The AccuracyGoal, which specifies the absolute local error allowed at each step in finding a solution, and the PrecisionGoal, which specifies the relative local error allowed were left at their default settings, which are both equal to MachinePrecision/2. The value of MachinePrecision on the computer which was used in this case was $\left(53 \log _{10} 2\right) \approx 16$. With AccuracyGoal set to $a$ and PrecisionGoal set to $p$, Mathematica attempts to make the numerical error in a result of size $x$ be less than $10^{-a}+|x| 10^{-p}$. NDSolve by default uses an infinity norm to measure the error for this method.

For both methods, we apply the boundary condition derived in Appendix 3.4.1 on $r=a$ and also arbitrarily choose $v(a)=1$. We then plot $v^{\prime}(b)-v(b) / b$ as a function of $\beta$ in order to determine the values of $\beta$ which correspond to the cut-on frequencies for the problem (i.e. the roots of these plots). In Figure 3.6 we plot $v^{\prime}(b)-v(b) / b$ as a function of $\beta$ for $a=0.5, b=1, S_{2}=0, \alpha=1$. Note that, as expected, the agreement is better for larger values of $\beta$. We note that the numerical solver always predicts $\beta=0$ as a root, which is the fundamental mode. In this case we have

$$
\begin{equation*}
k_{0}^{2}-k^{2} L^{2}=0, \tag{3.61}
\end{equation*}
$$

and hence

$$
\begin{equation*}
k^{2}=\frac{k_{0}^{2}}{L^{2}} . \tag{3.62}
\end{equation*}
$$

We observe from (3.62) that the only modification to the fundamental mode is a change in gradient, which is dependent on the longitudinal stretch factor, $L$.

By plotting the dispersion curves for various values of $a, b, L$ and $\alpha$, it can be observed that a positive value of $\alpha$ (which corresponds to ( $p_{\text {out }}>p_{\text {in }}$ ) causes the roots of the dispersion curves to be spaced further apart, whilst a negative value of $\alpha$ (which corresponds to $\left(p_{\text {out }}<p_{\text {in }}\right)$ causes them to be spaced more closely. It


Figure 3.6: Prediction of cut-on frequencies using Liouville-Green method (black) and a numerical solver (dashed) for $a=0.5, b=1, S_{2}=0, \alpha=1$
can also be observed that for $L>1$ (corresponding to a longitudinal stretch), the cut-on frequencies are closer together and the dispersion curves are less steep, whilst for $L<1$ (corresponding to a longitudinal compression) the cut-on frequencies are further apart and the dispersion curves are steeper.

In Figure 3.7, we plot the first, second and third cut-on frequencies as a function of $\frac{p_{\text {out }}-p_{\text {in }}}{S \mu}$ for a neo-Hookean $\left(S_{1}=1, S_{2}=0\right)$ annular cylinder with $a=0.5$ and $b=1$ using the numerical solution. In Figure 3.8, we do the same with $A=0.5$ and $B=1$.

In Figure 3.9 we give the dispersion curves for a neo-Hookean $\left(S_{1}=1, S_{2}=0\right)$ annular cylinder with $a=0.5$ and $b=1$, with $L=1$ and $\frac{p_{o u t}-p_{\text {in }}}{S \mu}=0,1.96$, and 3.55. In Figure 3.10 we give the same, but for $A=0.5$ and $B=1$. Figures 3.9 and 3.10 were produced using NDSolve rather than the Liouville-Green method.

### 3.5.2 Mooney-Rivlin case

In the Mooney-Rivlin case, $\delta \neq 0$, and so the dispersion relations cannot be determined by the value of $\beta$ alone. In this case we evaluate the boundary condition $v^{\prime}(b)-v(b) / b=0$ as a two-dimensional function of $k$ and $k_{0}=\omega / c$ and plot the regions where this function equals zero. These plots are our dispersion curves. In


Figure 3.7: First (solid), second (dashed) and third (dotted) cut-on frequencies for a pre-stressed cylinder as a function of $\frac{p_{\text {out }}-p_{\text {in }}}{S \mu}$ with $a=0.5$ and $b=1$.


Figure 3.8: First (solid), second (dashed) and third (dotted) cut-on freqeuncies for a pre-stressed cylinder as a function of $\frac{p_{\text {out }}-p_{\text {in }}}{S \mu}$ with $A=0.5$ and $B=1$.


Figure 3.9: Dispersion curves for a neo-Hookean ( $S_{1}=1, S_{2}=0$ ) annular cylinder of deformed inner radius 0.5 and outer radius 1 , with $L=1$ and $\frac{p_{\text {out }}-p_{i n}}{S \mu}=0$ (black), 1.96 (dashed), and 3.55 (dotted).


Figure 3.10: Dispersion curves for a neo-Hookean ( $S_{1}=1, S_{2}=0$ ) annular cylinder of initial inner radius 0.5 and outer radius 1 , with $L=1$ and $\frac{p_{\text {out }}-p_{\text {in }}}{S \mu}=0$ (black), 1.96 (dashed), and 3.55 (dotted).


Figure 3.11: Dispersion curves for a Mooney-Rivlin ( $S_{1}=0.8, S_{2}=0.2$ ) annular cylinder of deformed inner radius 0.5 and outer radius 1 , with $L=1$ and $\frac{p_{\text {out }}-p_{i n}}{S \mu}=0$ (black), 1.96 (dashed), and 3.55 (dotted).

Figure 3.11 we plot the dispersion curves for a Mooney-Rivlin ( $S_{1}=0.8, S_{2}=0.2$ ) annular clyinder with $a=0.5$ and $b=1$, with $L=1$ and $\frac{p_{\text {out }}-p_{\text {in }}}{S \mu}=0,1.96$, and 3.55. Figure 3.11 was produced using NDSolve.

The trends observed in the Mooney-Rivlin case are the same as those in the neoHookean case except that in the neo-Hookean case (i.e. when $S_{1}=1, S_{2}=0$ ), $\alpha$ does not affect the gradients of the dispersion curves, whereas in the Mooney-Rivlin case, a positive value of $\alpha$ decreases their gradients and a negative value increases them.

### 3.6 Conclusions

In this chapter we have studied the problem of torsional wave propagation in a prestressed, Mooney-Rivlin, annular cylinder. The pre-stress consists of a uniform longitudinal stretch and hydrostatic pressures imposed on the inner and outer surfaces of the cylinder, thus altering the radii. Importantly, the latter generates an inhomogeneous deformation in the host domain.

The theory of small-on-large was used to derive the incremental equation in the
pre-stressed configuration. It was then discussed that this equation was difficult to solve due to the spatial dependence of its coefficients and the singular limit of the equation in the case of zero pre-stress. In Section 3.4.4 we presented a Liouville-Green approximation to the solution of the ODE and discussed when we expect this to be a good approximation. It was shown that for $\hat{\alpha}>0.087$, we expect the Liouville-Green approximation to be good for an annular cylinder of any size.

We noted that a positive value of $\alpha$ (which corresponds to $p_{\text {out }}>p_{\text {in }}$ ) causes the roots of the dispersion curves to be spaced further apart, whilst a negative value of $\alpha$ (which corresponds to $p_{\text {out }}<p_{\text {in }}$ ) causes them to be spaced more closely. In the neo-Hookean case (i.e. when $S_{1}=1, S_{2}=0$ ), $\alpha$ does not affect the gradients of the dispersion curves, whereas in the Mooney-Rivlin case, a positive value of $\alpha$ decreases their gradients and a negative value increases them. We also noted that for $L>1$ (corresponding to a longitudinal stretch) the cut-on frequencies move closer together and the dispersion curves are less steep, whilst for $L<1$ (corresponding to a longitudinal compression) the cut-on frequencies move further apart and the dispersion curves are steeper.

The dependence of the cut-on frequencies on the pre-stress could potentially be used to tune which modes are able to propagate over a given range of frequencies. For example, if it was required to reduce the number of torsional modes which propagate along an annular cylinder at a given frequency, we have demonstrated that applying a large enough pressure on the outer surface would achieve this. Potential areas of further work would be a study into the effect of the elastic parameter $S_{2}$ on the stated results, and an investigation of whether the behaviour of the cylinder would be similar for other choices of strain energy function.

## Chapter 4

## Scattering of shear waves from a cylindrical cavity in a pre-stressed <br> host medium

### 4.1 Overview

In Chapter 2, we discussed the scattering of out-of-plane shear waves from a cylindrical cavity in a stress-free linear elastic material. In this chapter we investigate the scattering of these waves in a pre-stressed medium. The circular cylindrical cavity, whose length to cross section ratio is so large that end effects can be neglected and so can be considered as having infinite length, is located in an unbounded, nonlinear elastic material. Pressure is applied in the "far-field" and on the surface of the cavity, and a longitudinal stretch is applied, resulting in a nonlinear pre-stress throughout the material. The main aim of this chapter is to study the effect of this pre-stress on the scattering of incoming incremental horizontally polarised shear waves by the cavity. The incremental displacements satisfy an ordinary differential equation whose coefficients are spatially dependent. If a neo-Hookean strain energy function is used to model the material under consideration an analytical solution may be found (see [82]); however, if a Mooney-Rivlin strain energy function is employed the governing ordinary differential equation is more complicated and it appears that explicit solutions
cannot be determined. Due to the difficulty in solving such an ordinary differential equation we analyse it numerically using two methods - firstly, using a numerical solver and, secondly, by a semi-analytical method. In order to use a numerical solver we separate the material into an inhomogeneous region in the vicinity of the cavity and a homogeneous region far from the cavity, and then apply a continuity condition on the boundary between the two regions. In order to use the semi-analytical method we discretise the material local to the cavity into $N$ layers and treat each layer as homogeneous. Upon applying continuity of traction and displacement on the boundary of each layer it is possible to obtain an approximate solution to the spatially dependent ordinary differential equation.

### 4.2 Introduction

In this chapter, we will consider the canonical problem of the effect of pre-stress on the scattering of antiplane waves from a single cylindrical cavity. The pre-stress we are considering is hydrostatic pressure applied at infinity and on the surface of the cavity along with a longitudinal stretch of factor $L$. We assume the host material can be modelled by a Mooney-Rivlin strain energy function and show the effect of this pre-stress on the scattering coefficients for an incident plane wave propagating in the positive $x$-direction.

Scattering of horizontally polarised shear waves by a cylindrical cavity in a linear elastic material is well understood (see Chapter 2), as is wave scattering in linear elastic materials in general. There are papers dealing with scattering of both longitudinal and shear waves in many different geometries in linear elastic materials, [110], [29], [19].

Recently, however, attention has focused on wave propagation and scattering in pre-stressed nonlinear elastic materials. Most papers in the literature focus on the effect of homogeneous pre-stress on the subsequent propagation of incremental waves ([51], [27], for example). It is of interest, however, to study the effect of inhomogeneous pre-stress on wave propagation and scattering. In [82], Parnell and Abrahams
discussed the effect of an inhomogeneous pre-stress on the scattering of horizontally polarised shear waves in a neo-Hookean material. It was shown that a pressure applied at infinity and a longitudinal stretch has no effect on the scattering coefficients for these waves; perhaps a surprising and non-intuitive result. For unstressed materials, there is a strong correlation between the cavitiy size and magnitude of the scattering coefficients for SH waves. The applied pressure will significantly modify the cavity radius, and so, for the scattering coefficients to be completely unchanged by this is a quite unexpected result.

In this chapter we extend the work in [82] to the case of a Mooney-Rivlin material. Unfortunately, for a Mooney-Rivlin material, an analytic expression for the scattering cofficients could not be found. Instead, the effect of the pre-stress on the scattering coefficients was determined using two methods: firstly, numerically, and secondly, by discretising the material into several layers and treating each as homogeneous.

In Section 4.3 we will analyse the initial deformation resulting from the pre-stress before discussing the incremental deformation in Section 4.4. In Section 4.5.2, we briefly present the result for a neo-Hookean material given in [82], before discussing the methods we used for a Mooney-Rivlin material in Sections 4.6 and 4.7. Finally, in Section 4.8, we plot some results, and compare the pre-stressed scattering coefficients with the scattering coefficients of an unstressed material with the same void radius.

### 4.3 Initial finite (static) deformation

Consider an infinite, incompressible material containing a cylindrical cavity of initial radius $A$ and infinite length. We will assume that the host material is isotropic and that its constitutive behaviour may be described by a strain energy function, $W=W\left(I_{1}, I_{2}, I_{3}\right)$, where $I_{j}$ are the principal strain invariants of the deformation [38], [73]. Since it is incompressible, $I_{3}=1$, and thus $W=W\left(I_{1}, I_{2}\right)$. A pressure, $p_{\infty}$, is applied at infinity, a pressure, $p_{a}$, is applied on the surface of the cavity and a longitudinal stretch $L$ is also applied. The radius of the void after deformation will be denoted $a$.

The above deformation is described by

$$
\begin{equation*}
R=R(r), \quad \Theta=\theta, \quad Z=\frac{z}{L}, \tag{4.1}
\end{equation*}
$$

where $(R, \Theta, Z)$ and $(r, \theta, z)$ are cylindrical polar coordinates in the undeformed and deformed configurations respectively and $R(r)$ is a function to be determined from the incompressibility condition. Note the convention introduced in (4.1) above, i.e. that upper case variables correspond to the undeformed configuration whilst lower case corresponds to the deformed configuration. It will be convenient for us to derive equations in terms of coordinates in the deformed configuration as this will become our reference state when we consider the incremental SH-waves. Position vectors in the undeformed (upper case) and deformed (lower case) configurations are

$$
\boldsymbol{X}=\left(\begin{array}{c}
R \cos \Theta  \tag{4.2}\\
R \sin \Theta \\
Z
\end{array}\right)=\left(\begin{array}{c}
R(r) \cos \theta \\
R(r) \sin \theta \\
\frac{z}{L}
\end{array}\right), \quad \boldsymbol{x}=\left(\begin{array}{c}
r \cos \theta \\
r \sin \theta \\
z
\end{array}\right)
$$

Using (4.1), it can be shown that the principal stretches for this deformation in the radial, azimuthal and longitudinal directions, respectively, are

$$
\begin{equation*}
\lambda_{r}=\frac{d r}{d R}=\frac{1}{R^{\prime}(r)}, \quad \quad \lambda_{\theta}=\frac{r}{R}=\frac{r}{R(r)}, \quad \quad \lambda_{z}=L \tag{4.3}
\end{equation*}
$$

The deformation gradient tensor, $\boldsymbol{F}$, is given by

$$
\begin{equation*}
\boldsymbol{F}=\operatorname{Grad} \boldsymbol{x}, \tag{4.4}
\end{equation*}
$$

where Grad represents the gradient operator in the undeformed configuration. In our case, we have

$$
\boldsymbol{F}=\left(\begin{array}{ccc}
\lambda_{r} & 0 & 0  \tag{4.5}\\
0 & \lambda_{\theta} & 0 \\
0 & 0 & \lambda_{z}
\end{array}\right)=\left(\begin{array}{ccc}
\frac{1}{R^{\prime}(r)} & 0 & 0 \\
0 & \frac{r}{R(r)} & 0 \\
0 & 0 & L
\end{array}\right)
$$

For an incompressible material, we must have $J=\operatorname{det} \boldsymbol{F}=1$, and so

$$
\begin{equation*}
\lambda_{r} \lambda_{\theta} \lambda_{z}=\frac{L r}{R(r) R^{\prime}(r)}=1 . \tag{4.6}
\end{equation*}
$$

Solving the above, we obtain

$$
\begin{equation*}
R(r)=\sqrt{L\left(r^{2}+\alpha\right)}, \tag{4.7}
\end{equation*}
$$

where $\alpha$ is a constant defined by

$$
\begin{equation*}
\alpha=\frac{A^{2}}{L}-a^{2} . \tag{4.8}
\end{equation*}
$$

Note that (4.7) gives the same form for $R$ as in [82] ( $\alpha$ here is equivalent to $M$ in [82]).

From [26], the Cauchy stress tensor for an incompressible material is given by

$$
\begin{equation*}
\boldsymbol{T}=\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}+Q \boldsymbol{I} \tag{4.9}
\end{equation*}
$$

where $W$ is the strain energy function of the material under consideration, $\boldsymbol{I}$ is the identity tensor and $Q$ is a Lagrange multiplier associated with the incompressibility constraint and referred to as an arbitrary hydrostatic pressure.

Equation (4.9) can be written in index notation as

$$
\begin{equation*}
T_{i j}=F_{i \alpha} \frac{\partial W}{\partial F_{j \alpha}}+Q \delta_{i j} \tag{4.10}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta. We note that since $\boldsymbol{F}$ is diagonal, so is $\boldsymbol{T}$. In our case, we have

$$
\boldsymbol{T}=\left(\begin{array}{ccc}
\lambda_{r} \frac{\partial W}{\partial \lambda_{r}}+Q & 0 & 0  \tag{4.11}\\
0 & \lambda_{\theta} \frac{\partial W}{\partial \lambda_{\theta}}+Q & 0 \\
0 & 0 & \lambda_{z} \frac{\partial W}{\partial \lambda_{z}}+Q
\end{array}\right)
$$

The static equations of equilibrium are then given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{T}=\mathbf{0} \tag{4.12}
\end{equation*}
$$

where div signifies the divergence operator with respect to the deformed configuration. In our case, (4.12) reduces to

$$
\begin{equation*}
\frac{\partial T_{r r}}{\partial r}+\frac{1}{r}\left(T_{r r}-T_{\theta \theta}\right)=0, \quad \frac{\partial T_{\theta \theta}}{\partial \theta}=0, \quad \frac{\partial T_{z z}}{\partial z}=0 . \tag{4.13}
\end{equation*}
$$

Since the principal stretches depend on $r$ only, the latter two equations in the above tell us that $Q$ is a function of $r$ only. A specific strain energy function must be used in order to evaluate $Q$ explicitly, along with the conditions $\left.T_{r r}\right|_{r=a}=-p_{a}$ and $T_{r r} \rightarrow-p_{\infty}$ as $r \rightarrow \infty$.

### 4.3.1 Specific strain energy functions

Here we give specific results for different types of strain energy function.

## Neo-Hookean strain energy function

The neo-Hookean strain energy function is given by:

$$
\begin{equation*}
W=\frac{\mu}{2}\left(I_{1}-3\right)=\frac{\mu}{2}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right), \tag{4.14}
\end{equation*}
$$

where $\mu$ is the ground state shear modulus of the material under consideration. For this strain energy function we have

$$
\begin{gather*}
T_{r r}=\frac{1}{L}\left(1+\frac{\alpha}{r^{2}}\right) \mu+Q,  \tag{4.15}\\
T_{\theta \theta}=\frac{r^{2}}{L\left(\alpha+r^{2}\right)} \mu+Q,  \tag{4.16}\\
T_{z z}=L^{2} \mu+Q \tag{4.17}
\end{gather*}
$$

and the radial equation of equilibrium becomes

$$
\begin{equation*}
\frac{\partial Q}{\partial r}=\frac{\alpha^{2} \mu}{L r^{3}\left(\alpha+r^{2}\right)} . \tag{4.18}
\end{equation*}
$$

Hence $Q$ depends only on $r$ and is determined by the equation

$$
\begin{equation*}
Q=\int \frac{\alpha^{2} \mu}{L r^{3}\left(\alpha+r^{2}\right)} d r=-\frac{\mu}{2 L}\left(\frac{\alpha}{r^{2}}+\log \left(\frac{r^{2}}{r^{2}+\alpha}\right)\right)+Q_{0} \tag{4.19}
\end{equation*}
$$

Upon applying $\left.T_{r r}\right|_{r=a}=-p_{a}$, we determine $Q_{0}$ to be such that

$$
\begin{equation*}
T_{r r}=\frac{\mu}{2 L}\left(\alpha\left(\frac{1}{r^{2}}-\frac{1}{a^{2}}\right)-\log \left(\frac{r^{2}}{a^{2}}\right)+\log \left(\frac{r^{2}+\alpha}{a^{2}+\alpha}\right)\right)-p_{a} . \tag{4.20}
\end{equation*}
$$

Hence, enforcing the condition that $T_{r r} \rightarrow-p_{\infty}$ as $r \rightarrow \infty$, we obtain

$$
\begin{equation*}
\frac{p_{\infty}-p_{a}}{\mu}=\frac{1}{2 L}\left(\frac{A^{2}}{L a^{2}}-1+\log \left(\frac{A^{2}}{L a^{2}}\right)\right) . \tag{4.21}
\end{equation*}
$$

In Figure 4.1 we plot $a / A$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$. Note that the results in this section agree with those obtained in [82].


Figure 4.1: Plot of $a / A$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for the three prescribed values of $L=0.7,1,1.3$.

## Mooney-Rivlin strain energy function

The Mooney-Rivlin strain energy function is given by:

$$
\begin{align*}
W & =\frac{\mu}{2}\left(S_{1}\left(I_{1}-3\right)+S_{2}\left(I_{2}-3\right)\right) \\
& =\frac{\mu}{2}\left(S_{1}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right)+S_{2}\left(\lambda_{r}^{2} \lambda_{\theta}^{2}+\lambda_{r}^{2} \lambda_{z}^{2}+\lambda_{\theta}^{2} \lambda_{z}^{2}-3\right)\right), \tag{4.22}
\end{align*}
$$

where $\mu$ is the shear modulus of the material under consideration, and $S_{1}$ and $S_{2}$ are two constants which sum to one. For this strain energy function we have

$$
\begin{gather*}
T_{r r}=\frac{1}{L}\left(1+\frac{\alpha}{r^{2}}\right) S_{1} \mu+\left(\frac{1}{L^{2}}+L\left(1+\frac{\alpha}{r^{2}}\right)\right) S_{2} \mu+Q  \tag{4.23}\\
T_{\theta \theta}=\frac{r^{2}}{L\left(\alpha+r^{2}\right)} S_{1} \mu+\left(\frac{1}{L^{2}}+\frac{L r^{2}}{\alpha+r^{2}}\right) S_{2} \mu+Q  \tag{4.24}\\
T_{z z}=L^{2} S_{1} \mu+L\left(2+\frac{\alpha^{2}}{r^{2}\left(\alpha+r^{2}\right)}\right) S_{2} \mu+Q \tag{4.25}
\end{gather*}
$$

and we also obtain the following from the radial equilibrium equation:

$$
\begin{equation*}
\frac{\partial Q}{\partial r}=\frac{\alpha^{2}}{L r^{3}\left(\alpha+r^{2}\right)}\left(S_{1}+L^{2} S_{2}\right) \mu=\frac{\mu S \alpha^{2}}{r^{3}\left(\alpha+r^{2}\right)}, \tag{4.26}
\end{equation*}
$$

where

$$
\begin{equation*}
S=\frac{S_{1}}{L}+L S_{2} \tag{4.27}
\end{equation*}
$$

Using (4.26), we can determine $Q$ :

$$
\begin{equation*}
Q=\int \frac{\alpha^{2} S \mu}{r^{3}\left(\alpha+r^{2}\right)} d r=-\frac{S \mu}{2}\left(\log \left(\frac{r^{2}}{r^{2}+\alpha}\right)+\frac{\alpha}{r^{2}}\right)+Q_{0} \tag{4.28}
\end{equation*}
$$

Then, upon applying $\left.T_{r r}\right|_{r=a}=-p_{a}$, we can determine $Q_{0}$ and, therefore, give an explicit expression for $T_{r r}$ :

$$
\begin{equation*}
T_{r r}=\frac{S \mu}{2}\left(\alpha\left(\frac{1}{r^{2}}-\frac{1}{a^{2}}\right)-\log \left(\frac{r^{2}}{a^{2}}\right)+\log \left(\frac{r^{2}+\alpha}{a^{2}+\alpha}\right)\right)-p_{a} \tag{4.29}
\end{equation*}
$$

Finally, upon applying the condition that $T_{r r} \rightarrow-p_{\infty}$ as $r \rightarrow \infty$, we obtain

$$
\begin{equation*}
\frac{p_{\infty}-p_{a}}{S \mu}=\frac{1}{2}\left(\frac{A^{2}}{L a^{2}}-1+\log \left(\frac{A^{2}}{L a^{2}}\right)\right) . \tag{4.30}
\end{equation*}
$$

Note that when $L=1, S=1$ and (4.21) and (4.30) are equivalent.
If we were to plot $a / A$ for this strain energy function we would obtain the same graph as in Figure 4.1 but with $\left(p_{\infty}-p_{a}\right) / \mu$ replaced with $\left(p_{\infty}-p_{a}\right) / L S \mu$. Note that by setting $S_{1}=1$ and $S_{2}=0$ we obtain the results for a neo-Hookean material.

### 4.4 Incremental deformation

We now consider the propagation of small-amplitude, time-harmonic waves through the finitely-deformed medium. We use the theory of small-on-large, i.e. linearisation about a non-linear deformation state [73]. The total displacement field may be represented by

$$
\begin{equation*}
\bar{U}=\boldsymbol{U}+\boldsymbol{u} . \tag{4.31}
\end{equation*}
$$

where $\boldsymbol{U}$ is the displacement field derived from the finite deformation (4.1) and $\boldsymbol{u}$ is the incremental displacement. Let us assume that the incremental displacement is of the form

$$
\begin{equation*}
\boldsymbol{u}=\Re\left\{(0,0, w(r, \theta)) e^{-i \omega t}\right\} \tag{4.32}
\end{equation*}
$$

so that it is an SH-wave (see Figure 4.2). We will also assume that $|\boldsymbol{u}| \ll|\boldsymbol{U}|$.
The gradient of $\boldsymbol{u}$, with respect to the deformed configuration, is given by

$$
\boldsymbol{\gamma}=\operatorname{grad} \boldsymbol{u}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{4.33}\\
0 & 0 & 0 \\
\frac{\partial w}{\partial r} & \frac{1}{r} \frac{\partial w}{\partial \theta} & 0
\end{array}\right) e^{-i \omega t}
$$



Figure 4.2: SH waves incident on a cylindrical cavity.

The push forward of the incremental nominal stress (see Chapter 2) for an incompressible material is given by

$$
\begin{equation*}
\boldsymbol{\zeta}=\boldsymbol{M}: \boldsymbol{\gamma}+q \boldsymbol{I}-Q \boldsymbol{\gamma} \tag{4.34}
\end{equation*}
$$

where $\boldsymbol{M}$ is the push forward of the elasticity tensor defined by

$$
\begin{equation*}
M_{i j k l}=J^{-1} \frac{\partial^{2} W}{\partial F_{j m} \partial F_{l n}} F_{i m} F_{k n} \tag{4.35}
\end{equation*}
$$

and $q$ is the incremental form of $Q$.
Now, from [26], we have

$$
\begin{equation*}
J M_{i i j j}=\lambda_{i} \lambda_{j} W_{i j}, \tag{4.36}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{i j}=\frac{\partial^{2} W}{\partial \lambda_{i} \partial \lambda_{j}}, \tag{4.37}
\end{equation*}
$$

and, when $i \neq j, \lambda_{i} \neq \lambda_{j}$,

$$
\begin{gather*}
J M_{i j i j}=\frac{\lambda_{i} W_{i}-\lambda_{j} W_{j}}{\lambda_{i}^{2}-\lambda_{j}^{2}} \lambda_{i}^{2}  \tag{4.38}\\
J M_{i j j i}=\frac{\lambda_{j} W_{i}-\lambda_{i} W_{j}}{\lambda_{i}^{2}-\lambda_{j}^{2}} \lambda_{i} \lambda_{j}, \tag{4.39}
\end{gather*}
$$

with

$$
\begin{equation*}
W_{i}=\frac{\partial W}{\partial \lambda_{i}} . \tag{4.40}
\end{equation*}
$$

In our case, the components of $\boldsymbol{M}$ are functions of $r$ only, since they are dependent on the principal stretches, which here are functions of $r$ only. We will also assume that $q=q(r, \theta)$ (i.e. independent of $z$ since there is no $z$ dependence in either the pre-stress or the perturbation).

From the above, in our case, we have

$$
\boldsymbol{\zeta}=\left(\begin{array}{ccc}
q & 0 & M_{1313} \gamma_{31}  \tag{4.41}\\
0 & q & M_{2323} \gamma_{32} \\
\left(M_{3113}-Q\right) \gamma_{31} & \left(M_{3223}-Q\right) \gamma_{32} & 0
\end{array}\right) .
$$

The incremental equations of motion are then given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\zeta}=\rho \boldsymbol{u}_{, t t}, \tag{4.42}
\end{equation*}
$$

where $\rho$ is the density of the body, which remains constant throughout the deformation, since we are considering an incompressible material.

Using (4.42), we observe that the radial and azimuthal equations give

$$
\begin{equation*}
\frac{\partial \zeta_{11}}{\partial r}=0 \tag{4.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial \zeta_{22}}{\partial \theta}=0 \tag{4.44}
\end{equation*}
$$

which reduce to

$$
\begin{equation*}
\frac{\partial q}{\partial r}=\frac{\partial q}{\partial \theta}=0 \tag{4.45}
\end{equation*}
$$

Hence $q$ is a constant.
The longitudinal equation is given by

$$
\begin{equation*}
\frac{\partial \zeta_{13}}{\partial r}+\frac{1}{r} \frac{\partial \zeta_{23}}{\partial \theta}+\frac{\zeta_{13}}{r}=-\rho \omega^{2} w(r, \theta) e^{-i \omega t}, \tag{4.46}
\end{equation*}
$$

which reduces to

$$
\begin{equation*}
M_{1313} \frac{\partial^{2} w}{\partial r^{2}}+\left(M_{1313}^{\prime}+\frac{M_{1313}}{r}\right) \frac{\partial w}{\partial r}+\frac{M_{2323}}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}=-\rho w^{2} w, \tag{4.47}
\end{equation*}
$$

where the prime notation denotes differentiation with respect to $r$.
This can be rewritten as

$$
\begin{equation*}
\frac{1}{r} \frac{\partial}{\partial r}\left(r \mu_{r}(r) \frac{\partial w}{\partial r}\right)+\frac{\mu_{\theta}(r)}{r^{2}} \frac{\partial^{2} w}{\partial \theta^{2}}+\rho \omega^{2} w=0 \tag{4.48}
\end{equation*}
$$

where we define $\mu_{r}(r)=M_{1313}$ and $\mu_{\theta}(r)=M_{2323}$ to be the anisotropic shear moduli of the pre-stressed body.

Therefore, upon selecting a specific strain energy function, the above can be used to determine the governing equation for $w(r, \theta)$.

### 4.4.1 Boundary conditions

Equation (4.48) is the governing equation for $w(r, \theta)$ and we will apply the condition that the perturbation does not affect the pressure on the surface $r=a$. For this to hold, it is shown in chapter 2 that, for an incompressible material, the following condition must be satisfied on the boundary:

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=p \boldsymbol{\gamma}^{T} \boldsymbol{n}, \tag{4.49}
\end{equation*}
$$

where $p$ is the applied pressure on the boundary of the void, which in our case is $p_{a}, \boldsymbol{\zeta}$ is the push forward of the incremental nominal stress, and $\boldsymbol{n}$ is the outer unit normal to the boundary in the deformed configuration, which in our case is $\boldsymbol{e}_{r}$. These equations simplify to

$$
\begin{equation*}
\left.\frac{\partial w}{\partial r}\right|_{r=a}=0 \tag{4.50}
\end{equation*}
$$

and

$$
\begin{equation*}
q=0 \tag{4.51}
\end{equation*}
$$

### 4.4.2 Specific strain energy functions

Here we give specific results for different types of strain energy function.

## Neo-Hookean strain energy function

Upon substitution of the neo-Hookean strain energy function (4.14) into the above equations, we obtain the following anisotropic shear moduli:

$$
\begin{equation*}
\mu_{r}(r)=\frac{\mu}{L}\left(1+\frac{\alpha}{r^{2}}\right), \tag{4.52}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{\theta}(r)=\frac{\mu}{L}\left(1-\frac{\alpha}{r^{2}+\alpha}\right) . \tag{4.53}
\end{equation*}
$$

In this case, the governing equation for $w(r, \theta)$ can be written as

$$
\begin{equation*}
\left(1+\frac{\alpha}{r^{2}}\right) \frac{\partial^{2} w}{\partial r^{2}}+\frac{1}{r}\left(1-\frac{\alpha}{r^{2}}\right) \frac{\partial w}{\partial r}+\frac{1}{r^{2}+\alpha} \frac{\partial^{2} w}{\partial \theta^{2}}+k^{2} w=0 \tag{4.54}
\end{equation*}
$$

where

$$
\begin{equation*}
k^{2}=L K^{2}, \quad K=\omega \sqrt{\frac{\rho}{\mu}} . \tag{4.55}
\end{equation*}
$$

## Mooney-Rivlin strain energy function

Substituting the Mooney-Rivlin strain energy function (4.22) into the displacement equation (4.48), we obtain the following anisotropic shear moduli:

$$
\begin{equation*}
\mu_{r}(r)=\frac{T \mu}{L^{2}}\left(1+\frac{m}{r^{2}}\right), \tag{4.56}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{\theta}(r)=\frac{T \mu}{L^{2}}\left(1-\frac{m}{r^{2}+\alpha}\right), \tag{4.57}
\end{equation*}
$$

where

$$
\begin{equation*}
m=\frac{\alpha L S_{1}}{T} \tag{4.58}
\end{equation*}
$$

and

$$
\begin{equation*}
T=1+(L-1) S_{1} . \tag{4.59}
\end{equation*}
$$

In this case, the equation governing $w(r, \theta)$ can be written as

$$
\begin{equation*}
\left(1+\frac{m}{r^{2}}\right) \frac{\partial^{2} w}{\partial r^{2}}+\frac{1}{r}\left(1-\frac{m}{r^{2}}\right) \frac{\partial w}{\partial r}+\frac{r^{2}+\alpha-m}{r^{2}\left(r^{2}+\alpha\right)} \frac{\partial^{2} w}{\partial \theta^{2}}+k^{2} w=0 \tag{4.60}
\end{equation*}
$$

where

$$
\begin{equation*}
k^{2}=\frac{L^{2} K^{2}}{T} \tag{4.61}
\end{equation*}
$$

and $K^{2}$ is defined in (4.55).
Note that by setting $S_{1}=1$ we obtain the results for a neo-Hookean material.

### 4.5 Analytical solutions

### 4.5.1 Neo-Hookean case

It was observed in [82] that, in the neo-Hookean case, an analytical solution for $w(r, \theta)$ could be obtained. In fact, the scattering coefficients in the deformed configuration $a_{n}$ for a wave produced by a line source at $\left(r_{0}, \theta_{0}\right)$ in the deformed configuration were shown to be unchanged from the scattering coefficients in the undeformed configuration $A_{n}$ :

$$
\begin{equation*}
a_{n}=A_{n}=\frac{\pi C}{2 \mu i^{n-1}} \frac{\mathrm{~J}_{n}^{\prime}(K A)}{\mathrm{H}_{n}^{(1) \prime}(K A)} \mathrm{H}_{n}^{(1)}\left(K R_{0}\right), \tag{4.62}
\end{equation*}
$$

where $C$ is the coefficient associated with the line source in the undeformed configuration, $\mathrm{H}_{n}^{(1)}$ is an order $n$ Hankel function of the first kind, and

$$
\begin{equation*}
R_{0}=\sqrt{L\left(r_{0}^{2}+\alpha\right)} \tag{4.63}
\end{equation*}
$$

Note that in the above, the scattered field in the undeformed and deformed confirguations, respectively, are

$$
\begin{equation*}
W_{s}=\sum_{n=-\infty}^{\infty} i^{n} A_{n} \mathrm{H}_{n}^{(1)}(K R) e^{i n\left(\Theta-\Theta_{0}\right)}, \quad w_{s}=\sum_{n=-\infty}^{\infty} i^{n} a_{n} \mathrm{H}_{n}^{(1)}\left(k \sqrt{r^{2}+\alpha}\right) e^{i n\left(\theta-\theta_{0}\right)} . \tag{4.64}
\end{equation*}
$$

In the limit as $r_{0} \rightarrow \infty$, we have a propagating plane wave approaching the void from $\theta=\theta_{0}$, so when $\theta_{0}=\pi$, we have a plane wave propagating in the positive $x$-direction, as in Section 2.4. In [82], it was shown that, in this limit, the scattering coefficients are

$$
\begin{equation*}
a_{n}=A_{n}=-\frac{\mathrm{J}_{n}^{\prime}(K A)}{\mathrm{H}_{n}^{(1) \prime}(K A)}, \tag{4.65}
\end{equation*}
$$

when the incoming wave has unit magnitude. This result can be compared with the result of equation (2.323). The total solution, in the case of an incoming plane wave propagating in the positive $x$-direction, can be written as

$$
\begin{equation*}
w(r, \theta)=c \sum_{n=-\infty}^{\infty} i^{n}\left(\mathrm{~J}_{n}\left(k \sqrt{r^{2}+\alpha}\right)+a_{n} \mathrm{H}_{n}^{(1)}\left(k \sqrt{r^{2}+\alpha}\right)\right) e^{i n \theta} \tag{4.66}
\end{equation*}
$$

where $c$ is the amplitude of the incoming wave.

### 4.5.2 Mooney-Rivlin case

To the author's knowledge, it is not possible to obtain an analytical solution in the Mooney-Rivlin case for all modes. However, for $n=0$ (i.e. the mode with no dependence on $\theta$ ), equation (4.60) becomes

$$
\begin{equation*}
\left(1+\frac{m}{r^{2}}\right) \frac{d^{2} w}{d r^{2}}+\frac{1}{r}\left(1-\frac{m}{r^{2}}\right) \frac{d w}{d r}+k^{2} w=0 . \tag{4.67}
\end{equation*}
$$

If we make the substitution

$$
\begin{equation*}
s=\sqrt{r^{2}+m}, \tag{4.68}
\end{equation*}
$$

then we obtain

$$
\begin{equation*}
\frac{d^{2} w}{d s^{2}}+\frac{1}{s} \frac{d w}{d s}+k^{2} w=0 \tag{4.69}
\end{equation*}
$$

This is Bessel's equation of order 0 , hence the solution for this mode can be written as

$$
\begin{equation*}
w=C_{1} \mathrm{~J}_{0}(k s)+C_{2} \mathrm{H}_{0}^{(1)}(k s)=C_{1} \mathrm{~J}_{0}\left(k \sqrt{r^{2}+m}\right)+C_{2} \mathrm{H}_{0}^{(1)}\left(k \sqrt{r^{2}+m}\right) \tag{4.70}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants.

### 4.6 Hybrid analytical-numerical method

In this section we divide the host domain into two regions (see Figure 4.3). We assume that there exists some radius $b \gg 1$, which is large enough such that for $r>b$ the anisotropic shear moduli, $\mu_{r}(r)$ and $\mu_{\theta}(r)$, can be approximated by the constant

$$
\begin{equation*}
\mu_{r}^{\infty}(r)=\mu_{\theta}^{\infty}(r)=\frac{T \mu}{L^{2}} \tag{4.71}
\end{equation*}
$$

In other words, we treat the region $r>b$ as being homogeneous, and then use a numerical solver in order to solve for the displacement in the inhomogeneous inner region, $r<b$.

In the outer region, we will assume there is an incoming plane wave travelling in the positive $x$-direction and polarised such that the displacements are in the $z$ direction. Such a wave can be expressed as

$$
\begin{equation*}
w_{i}(r, \theta)=c e^{i k x}=c e^{i k r \cos \theta}=c \sum_{n=-\infty}^{\infty} i^{n} J_{n}(k r) e^{i n \theta} \tag{4.72}
\end{equation*}
$$



Figure 4.3: Discretisation of material into inhomogeneous region local to cavity and homogeneous region far from cavity.
where $c$ is the amplitude, and the scattered field in this region will be given by

$$
\begin{equation*}
w_{s}(r, \theta)=\sum_{n=-\infty}^{\infty} i^{n} a_{n} \mathrm{H}_{n}^{(1)}(k r) e^{i n \theta} . \tag{4.73}
\end{equation*}
$$

The total field is thus $w_{i}+w_{s}$. Here $a_{n}$ are our scattering coefficients.
We apply continuity of displacement and traction in order to determine the boundary condition for a numerical solver on $r=b$. In the following, the subscript $I$ denotes the inner solution (in the region $r<b$ ). We will assume $w_{I}(r, \theta)$ takes the form

$$
\begin{equation*}
w_{I}(r, \theta)=\sum_{n=-\infty}^{\infty} i^{n} F_{n}(r) e^{i n \theta} \tag{4.74}
\end{equation*}
$$

so that $F_{n}$ must satisfy

$$
\begin{equation*}
\frac{d}{d r}\left(\mu_{r}(r) \frac{d F_{n}}{d r}\right)+\frac{\mu_{r}(r)}{r} \frac{d F_{n}}{d r}+\left(\rho \omega^{2}-\frac{\mu_{\theta}(r) n^{2}}{r^{2}}\right) F_{n}=0 . \tag{4.75}
\end{equation*}
$$

Continuity of displacement and stress on $r=b$ gives us

$$
\begin{equation*}
w(b)=w_{I}(b), \tag{4.76}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\frac{\partial w}{\partial r}\right|_{r=b}=\left.\frac{\partial w_{I}}{\partial r}\right|_{r=b} \tag{4.77}
\end{equation*}
$$

and thus

$$
\begin{equation*}
F_{n}(b)=c \mathrm{~J}_{n}(k b)+a_{n} \mathrm{H}_{n}^{(1)}(k b), \tag{4.78}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{n}^{\prime}(b)=c k \mathrm{~J}_{n}^{\prime}(k b)+a_{n} k \mathrm{H}_{n}^{(1) \prime}(k b) \tag{4.79}
\end{equation*}
$$

The coefficients $a_{n}$ can be eliminated from the above in order to give us one boundary condition on $r=b$ :

$$
\begin{equation*}
F_{n}(b)-\frac{F_{n}^{\prime}(b) \mathrm{H}_{n}^{(1)}(k b)}{k \mathrm{H}_{n}^{(1) \prime}(k b)}=c\left(\mathrm{~J}_{n}(k b)-\frac{\mathrm{J}_{n}^{\prime}(k b)}{\mathrm{H}_{n}^{(1) \prime}(k b)} \mathrm{H}_{n}^{(1)}(k b)\right) . \tag{4.80}
\end{equation*}
$$

The other boundary condition, on $r=a$, is

$$
\begin{equation*}
F_{n}^{\prime}(a)=0, \tag{4.81}
\end{equation*}
$$

and so we can then use these to solve the ordinary differential equation using a numerical solver.

Once $F_{n}(b)$ has been found using a numerical solver, we can use the following equation, which is derived from (4.78), in order to determine our scattering coefficients, $a_{n}$ :

$$
\begin{equation*}
a_{n}=\frac{F_{n}(b)-c \mathrm{~J}_{n}(k b)}{\mathrm{H}_{n}^{(1)}(k b)} \tag{4.82}
\end{equation*}
$$

As in Chapter 3, the numerical solver used in this case was NDSolve in Mathematica 7 and the numerical method used by NDSolve was left to be automatically selected. For linear boundary value problems, NDSolve uses a Gelfand-Lokutsiyevskii chasing method [8]. NDSolve by default uses an infinity norm to measure the error for this method. Again, the AccuracyGoal and PrecisionGoal were left at their default setting of MachinePrecision/2.

### 4.7 Discretisation into $N$ layers

In this section we discretise the material local to the cavity into $N$ layers and treat each layer as a material which is homogeneous. We define $b$ to be the outer radius of
the $N^{\text {th }}$ layer and we assume that $b$ is large enough such that for $r>b$ the anisotropic shear moduli, $\mu_{r}(r)$ and $\mu_{\theta}(r)$, can be approximated by

$$
\begin{equation*}
\mu_{r}^{\infty}(r)=\mu_{\theta}^{\infty}(r)=\frac{T \mu}{L^{2}} \tag{4.83}
\end{equation*}
$$

For $r<b$, we define the inner radius, $r_{p}$, of the $p^{t h}$ layer, $D_{p}$, by

$$
\begin{equation*}
r_{p}=a+(p-1) \Delta, \tag{4.84}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta=\frac{b-a}{N} . \tag{4.85}
\end{equation*}
$$

So we see that the $p^{\text {th }}$ layer, $D_{p}$, is given by

$$
\begin{equation*}
D_{p}=\left\{r: r_{p} \leq r \leq r_{p+1}\right\}, \quad p=1, \ldots, N \tag{4.86}
\end{equation*}
$$

We shall approximate the anisotropic shear moduli in $D_{p}$ by the constant value

$$
\begin{equation*}
\mu_{r}^{p}=\mu_{r}\left(r_{p}\right)=\frac{T \mu}{L^{2}}\left(1+\frac{m}{r_{p}^{2}}\right) \tag{4.87}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{\theta}^{p}=\mu_{\theta}\left(r_{p}\right)=\frac{T \mu}{L^{2}}\left(1-\frac{m}{r_{p}^{2}+\alpha}\right), \tag{4.88}
\end{equation*}
$$

so that the partial differential equation to be solved in $D_{p}$ is

$$
\begin{equation*}
w_{r r}^{p}+\frac{1}{r} w_{r}^{p}+\frac{\mu_{\theta}^{p}}{\mu_{r}^{p}} \frac{1}{r^{2}} w_{\theta \theta}^{p}+k_{p}^{2} w^{p}=0 \tag{4.89}
\end{equation*}
$$

where $w^{p}$ is the displacement in $D_{p}$, and $k_{p}$ is the effective wavenumber in $D_{p}$, defined by

$$
\begin{equation*}
k_{p}^{2}=\frac{\rho \omega^{2}}{\mu_{r}^{p}} . \tag{4.90}
\end{equation*}
$$

We assume that $w^{p}$ can be written in the form

$$
\begin{equation*}
w^{p}(r, \theta)=\sum_{n=-\infty}^{\infty} i^{n} F_{n}^{p}(r) e^{i n \theta}, \tag{4.91}
\end{equation*}
$$

so that $F_{n}^{p}$ must satisfy

$$
\begin{equation*}
F_{n}^{p \prime \prime}+\frac{1}{r} F_{n}^{p \prime}+\left(k_{p}^{2}-\frac{\left(\gamma_{p} n\right)^{2}}{r^{2}}\right) F_{n}^{p}=0, \tag{4.92}
\end{equation*}
$$

where the prime notation denotes differentiation with respect to $r$, and

$$
\begin{equation*}
\gamma_{p}^{2}=\frac{\mu_{\theta}^{p}}{\mu_{r}^{p}} . \tag{4.93}
\end{equation*}
$$

The solution to (4.92) is

$$
\begin{equation*}
F_{n}^{p}(r)=a_{n}^{p} \mathrm{H}_{\gamma_{p} n}^{(1)}\left(k_{p} r\right)+b_{n}^{p} \mathrm{H}_{\gamma_{p} n}^{(2)}\left(k_{p} r\right), \tag{4.94}
\end{equation*}
$$

where $\mathrm{H}_{\gamma_{p} n}^{(1)}\left(k_{p} r\right)$ and $\mathrm{H}_{\gamma_{p} n}^{(2)}\left(k_{p} r\right)$ are Hankel functions of the first and second kind, respectively, of order $\gamma_{p} n$, and $a_{n}^{p}$ and $b_{n}^{p}$ are arbitrary constants, to be determined from the boundary conditions on $r_{p}$ and $r_{p+1} . \mathrm{H}_{\gamma_{p} n}^{(1)}\left(k_{p} r\right)$ is an outgoing and $\mathrm{H}_{\gamma_{p} n}^{(2)}\left(k_{p} r\right)$, an incoming solution to (4.92) due to the $e^{-i \omega t}$ time dependence.

In the outermost region, we will assume that there is an incoming plane wave which can be described as in the previous subsection, and that the scattered field will be given by

$$
\begin{equation*}
w_{s}(r, \theta)=\sum_{n=-\infty}^{\infty} i^{n} a_{n} \mathrm{H}_{n}^{(1)}(k r) e^{i n \theta} . \tag{4.95}
\end{equation*}
$$

Upon applying continuity of displacement and stress on the boundaries of the layers, we obtain the following conditions:

$$
\begin{equation*}
F^{p}\left(r_{p+1}\right)=F^{p+1}\left(r_{p+1}\right), \quad p=1, \ldots N-1, \tag{4.96}
\end{equation*}
$$

and

$$
\begin{equation*}
F^{p^{\prime}}\left(r_{p+1}\right)=F^{p+1 \prime}\left(r_{p+1}\right), \quad p=1, \ldots, N-1 . \tag{4.97}
\end{equation*}
$$

Continuity of displacement and stress on $r=b$ gives us

$$
\begin{equation*}
F_{n}(b)=c \mathrm{~J}_{n}(k b)+a_{n} \mathrm{H}_{n}^{(1)}(k b), \tag{4.98}
\end{equation*}
$$

and

$$
\begin{equation*}
F^{\prime}(b)=c k \mathrm{~J}_{n}^{\prime}(k b)+a_{n} k \mathrm{H}_{n}^{(1) \prime}(k b) . \tag{4.99}
\end{equation*}
$$

We also have

$$
\begin{equation*}
F^{1 \prime}(a)=0 . \tag{4.100}
\end{equation*}
$$

The above system of equations can then be solved in order to determine the coefficients $a_{n}^{p}, b_{n}^{p}$ and $a_{n}$.


Figure 4.4: Plot of $w(r)$ from the analytical solution in the neo-Hookean case with $\theta=0, a=1, \alpha=1, c=1$ and $L=1$ for the three prescribed values of $K A=0.2,1$ and 5 .

### 4.8 Comparison of results

### 4.8.1 Neo-Hookean material

## Displacements

In Figure 4.4, we compare the analytic solutions in the neo-Hookean case, equation (4.66), with $\theta=0, a=1, \alpha=1, c=1$ and $L=1$ for $K A=0.2,1$ and 5 . Note that the infinite sum over modal angle is truncated for $|n|>60$.

For the hybrid analytical-numerical method, a larger value of $k b$ increases the accuracy of the solution. In Figures 4.5 to 4.7 , we compare the $n=0$ analytic solutions to the corresponding solutions obtained via the hybrid analytical-numerical method with $k b=60$. Figure 4.5 corresponds to $K A=0.2$, Figure 4.6 to $K A=1$ and Figure 4.7 to $K A=5$. We observe that the hybrid analytical-numerical method works better for small values of $K A$, but the accuracy can be increased by increasing $k b$. In Figure 4.8, we show the $K A=5$ result with $k b=600$. Unfortunately, $k b$ cannot be increased indefinitely, however, due to storage limitations when solving the ordinary differential equation numerically. Therefore, it is best to choose the largest value of $k b$ for which the ordinary differential equation can be solved within a reasonable time. In Figures 4.9 to 4.11, we plot the absolute value of the maximum


Figure 4.5: Comparison of the hybrid analytical-numerical method with the analytical solution for $K A=0.2$ and $k b=60$ ( $n=0$ mode only).
error of the hybrid analytical-numerical method for the first three modes with $\theta=0$, $a=1, \alpha=1, c=1$ and $L=1$ for $K A=0.2,1$ and 5 respectively.

The $N$-layer method can also be used to plot the displacement. As the number of layers is increased, the accuracy of the solution increases. In Figure 4.12 we compare the $N$-layer solution for $\theta=0, a=1, \alpha=1, L=1, K A=0.2, k b=60, c=1$ and $N=100$ with the corresponding analytical solution. Figure 4.13 shows the same, but with $N=1000$. Note that these figures also show part of the region where $r>b$.

## Scattering coefficients

We are interested in how the pre-stress affects the scattering coefficients, $a_{n}$. It is shown in [82] that, for a neo-Hookean material, the scattering coefficients are not affected by the pre-stress, and our numerical methods confirm this. The values of the scattering coefficients predicted by the hybrid analytical-numerical method are independent of $\left(p_{\infty}-p_{a}\right) / \mu$. Unfortunately, the $N$-layer method requires too many layers to obtain a convergent value so this is not included in the figure.


Figure 4.6: Comparison of the hybrid analytical-numerical method with the analytical solution for $K A=1$ and $k b=60(n=0$ mode only).


Figure 4.7: Comparison of the hybrid analytical-numerical method with the analytical solution for $K A=5$ and $k b=60$ ( $n=0$ mode only).


Figure 4.8: Comparison of the hybrid analytical-numerical method with the analytical solution for $K A=5$ and $k b=600$ ( $n=0$ mode only).


Figure 4.9: Maximum error in $F(r)$ for $n=0$ (solid), $n=1$ (dashed) and $n=2$ (dotted) as a function of $k b$ using the hybrid analytical-numerical method for $K A=$ 0.2 .


Figure 4.10: Maximum error in $F(r)$ for $n=0$ (solid), $n=1$ (dashed) and $n=2$ (dotted) as a function of $k b$ using the hybrid analytical-numerical method for $K A=1$.


Figure 4.11: Maximum error in $F(r)$ for $n=0$ (solid), $n=1$ (dashed) and $n=2$ (dotted) as a function of $k b$ using the hybrid analytical-numerical method for $K A=5$.


Figure 4.12: Comparison of the $N$-layer method with the analytical solution for $K A=0.2, k b=60$ and $N=100$ ( $n=0$ mode only).


Figure 4.13: Comparison of the $N$-layer method with the analytical solution for $K A=0.2, k b=60$ and $N=1000$ ( $n=0$ mode only).


Figure 4.14: Comparison of neo-Hookean and Mooney-Rivlin results for $K A=0.2$ ( $n=0$ mode only).

### 4.8.2 Mooney-Rivlin material

## Displacements

Here we compare the displacements in a neo-Hookean material with the displacements in a Mooney-Rivlin material. In Figures 4.14 to 4.16 , we compare the $n=0$ solutions obtained via the hybrid analytical-numerical method for $S_{1}=1, S_{1}=0.8$ and $S_{1}=$ 0.2 , with $k b=600, \theta=0, a=1, \alpha=1, L=1$ and $c=1$. Figure 4.14 corresponds to $K A=0.2$, Figure 4.15 to $K A=1$ and Figure 4.16 to $K A=5$. We observe that the effect of Mooney-Rivlin material behaviour (as opposed to neo-Hookean) is greater at higher frequencies.

## Scattering coefficients

In Figure 4.17, we plot the absolute value of the scattering coefficients $a_{0}, a_{1}$ and $a_{2}$ as a function of $\left(p_{\infty}-p_{a}\right) / S \mu$ for $A=\sqrt{6}, K A=1, k b=600$ using the hybrid analytical-numerical method for $S_{1}=0.8$. We observe that for this value of $S_{1}$, increasing $\left(p_{\infty}-p_{a}\right) / S \mu$ decreases the magnitude of the scattering coefficients.

In Figure 4.18 we plot the absolute value of the scattering coefficients $a_{0}, a_{1}$ and $a_{2}$ as a function of $K a$ for $A=\sqrt{6}, K A=1, k b=600$ using the hybrid analyticalnumerical method for $S_{1}=0.8$ along with the scattering coefficients in unstressed


Figure 4.15: Comparison of neo-Hookean and Mooney-Rivlin results for $K A=1$ ( $n=0$ mode only)


Figure 4.16: Comparison of neo-Hookean and Mooney-Rivlin results for $K A=5$ ( $n=0$ mode only).


Figure 4.17: $a_{0}$ (solid), $a_{1}$ (long dashes) and $a_{2}$ (short dashes) as a function of ( $p_{\infty}-$ $\left.p_{a}\right) / S \mu$ for $A=\sqrt{6}, K A=1, k b=600, L=1$ using the hybrid analytical-numerical method for a Mooney-Rivlin material with $S_{1}=0.8$.
materials with the same void radius $a$. In the legend, the scattering coefficients with the superscript $u$ are unstressed and those without are pre-stressed. We observe that a pre-stressed material with a given cavity radius $a$ after the large deformation behaves more like it would have prior to the large deformation (i.e. an unstressed material with cavity radius $A$ ) than an unstressed material of deformed cavity radius $a$.

### 4.9 Conclusions

In this chapter, we have considered the canonical problem of the effect of pre-stress on the scattering of antiplane waves from a single cylindrical cavity in a Mooney-Rivlin material. The pre-stress consists of a uniform longitudinal stretch and hydrostatic pressures imposed on the inner surface of the cavity, and at infinity, thus altering the radii. Importantly, the latter generates an inhomogeneous deformation.

The theory of small-on-large was used to derive the incremental equation in the pre-stressed configuration. It was then discussed that this equation was difficult to


Figure 4.18: $a_{0}$ (solid), $a_{1}$ (long dashes) and $a_{2}$ (short dashes) as a function of $K a$ for $A=\sqrt{6}, K A=1, k b=600, L=1$ using the hybrid analytical-numerical method for a Mooney-Rivlin material with $S_{1}=0.8$ and comparison with the scattering coefficients in unstressed materials with the same cavity radius.
solve due to the spatial dependence of its coefficients. In Section 4.5.2, we examined the fact that, for a neo-Hookean material, an analytical solution for the scattering coefficients is available and that they are completely unaffected by pre-stress, whereas, for a Mooney-Rivlin material, this is not the case.

In Section 4.6, we presented a hybrid analytical-numerical scheme to determine the scattering coefficients in the Mooney-Rivlin case by discretising the material into an inhomogeneous region close $(r<b)$ to the cavity and a homogenous region far $(r>b)$ from the cavity. The governing equations were solved numerically in the inner region and matched to an analytical solution in the outer region.

In Section 4.7 the inner region was descretised into $N$ layers and each was treated as homogeneous. This allowed an approximate solution to be found in each layer. We expect the overall solution to increase in accuracy as the number of layers increases, but unfortunately, this also increases the required computation power and time taken to reach a solution.

Finally, in Section 4.8, we compared the displacement fields and scattering coefficients for neo-Hookean and Mooney-Rivlin materials over a range of values of $k A$. It was shown that, unlike in the neo-Hookean case, the scattering coefficients for a Mooney-Rivlin material are dependent on the pre-stress. They do, however, take values very different to those for a cavity of equal size in an unstressed material, so it is, therefore, important to take account of the pre-stress when calculating scattering cofficients in these problems.

Potential areas of further work would be a study into the effect of the elastic parameter $S_{2}$ on the stated results, and an investigation of whether the same trends could be observed for other choices of strain energy function.

## Chapter 5

## Employing pre-stress to generate finite cloaks for antiplane elastic

## waves

### 5.1 Overview

In Chapter 2, we discussed how anisotropy can be utilised to create cloaks for certain types of elastic waves, specifically antiplane waves. We also briefly mentioned that pre-stress can be used to generate the necessary anisotropy to acheive this effect. In this chapter, we investigate in detail the potential use of pre-stress to generate cloaks in neo-Hookean and Mooney-Rivlin materials. We observe that neo-Hookean materials are optimal for this type of cloaking, but that Mooney-Rivlin materials can still be used to generate "near cloaks".

### 5.2 Neo-Hookean cloak

The work in this section builds on the work in parts of Chapter 4 and also appears in [84]. Note that Figure 2 from [84] contains an error which is corrected in Figure 5.2.

In this section, it is shown that nonlinear pre-stress of neo-Hookean hyperelastic
materials can be used as a mechanism to generate finite cloaks and thus theoretically render objects near-invisible to incoming antiplane elastic waves. This approach appears to negate the requirement for special cloaking metamaterials with inhomogeneous and anisotropic material properties in this case. These properties are induced naturally by virtue of the pre-stress. This appears to provide a mechanism for broadband cloaking since dispersive effects due to metamaterial microstructure will not arise.

In [81], a method to generate elastic cloaks was proposed which used the notion of nonlinear pre-stress. This was possible due to the fact that the antiplane wave field scattered from a cylindrical cavity is invariant under pre-stress for an incompressible neo-Hookean material. Scattering coefficients in the deformed configuration depend only on the initial cavity radius, $R_{1}$, and therefore, provided that this is small compared with the incident wavelength, scattering from the inflated cavity of radius $r_{1}$ will be negligible regardless of the relative size of $r_{1}$ and the incident wavelength. Therefore, we can conclude that an object placed inside the inflated cavity region would be nearly undetectable (i.e. cloaked) upon choosing $R_{1}$ appropriately. In [81], the pre-stress affected the entire elastic domain however, and therefore its influence was felt by both the source and receiver. In this chapter, we show how this theory may be adapted in order to create a finite cloak by means of an additional deformation taking the form of an axial stretch.

With reference to Figure 5.1, let us consider an elastic material within which is located a cylindrical cavity of radius $R_{2}$. Let us assume that the density of this medium is $\rho_{0}$ and its axial shear modulus (corresponding to shearing on planes parallel to the axis of the cylindrical cavity) is $\mu_{0}$. Additionally, we take a cylindrical annulus of isotropic incompressible neo-Hookean material with associated shear modulus $\mu$ and density $\rho$ and with inner and outer radii $R_{1}$ and $R_{2}$, respectively, with $R_{1} \ll R_{2}$. The exact nature of this latter relationship will be described shortly. We shall consider deformations of the cylindrical annulus in order that it can act as an elastodynamic cloak to incoming antiplane elastic waves. We deform the material so that its inner radius is significantly increased (to $r_{1}$ ) but its outer radius $R_{2}$ remains


Figure 5.1: The incompressible neo-Hookean cylindrical annulus is pre-stressed as depicted on the right. This annulus then creates a cloak when slotted into a cylindrical cavity in an unbounded elastic medium, as illustrated on the left.
unchanged. The deformed cylindrical annulus can then slot into the existing cylindrical cavity region within the unbounded (unstressed) domain. We choose $\mu$ and $\rho$ so that subsequent waves satisfy the necessary continuity conditions on $r=R_{2}$.

The constitutive behaviour of an incompressible neo-Hookean material is described by the strain energy function [73]

$$
\begin{equation*}
\mathcal{W}=\frac{\mu}{2}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right) \tag{5.1}
\end{equation*}
$$

where $\lambda_{j}, j=r, \theta, z$ are the radial, azimuthal and axial principal stretches of the large deformation. We consider the initial deformation of the cylindrical annulus domain as depicted in Figure 5.1. Since the material is incompressible and $R_{2}$ is required to be fixed, the deformation is induced either by applying a uniform axial stretch $L$ or a radial pressure difference $p_{o}-p_{i}$, where $p_{o}$ and $p_{i}$ denote the pressures applied to the outer and inner face of the cylindrical annulus, respectively. The ensuing deformation is described via the relations

$$
\begin{equation*}
R=R(r), \quad \Theta=\theta, \quad Z=\frac{z}{L} \tag{5.2}
\end{equation*}
$$

where $(R, \Theta, Z)$ and $(r, \theta, z)$ are cylindrical polar coordinates in the undeformed and deformed configurations. Note the convention introduced in equation (5.2), i.e., that
upper case variables correspond to the undeformed configuration whilst lower case corresponds to the deformed configuration. This is analagous to the notation used for untransformed and transformed configurations in equation (2.436).

The principal stretches for this deformation are

$$
\begin{equation*}
\lambda_{r}=\frac{d r}{d R}=\frac{1}{R^{\prime}(r)}, \quad \lambda_{\theta}=\frac{r}{R(r)}, \quad \lambda_{z}=L \tag{5.3}
\end{equation*}
$$

For an incompressible material $\lambda_{r} \lambda_{\theta} \lambda_{z}=1$, implying

$$
\begin{equation*}
R(r)=\sqrt{L\left(r^{2}+\alpha\right)} \tag{5.4}
\end{equation*}
$$

where $\alpha=R_{2}^{2}\left(L^{-1}-1\right)$ is a constant determined by imposing that the outer wall of the cylindrical annulus remains fixed, i.e., $R\left(R_{2}\right)=R_{2}$. The deformation defined by equation (5.4) is easily inverted to obtain $r(R)$. Given incompressibility and the fixed outer wall of the annulus, in order to induce this deformation we may either (i) prescribe the axial stretch $L$ which then determines the deformed inner radius $r_{1}$ and the radial pressure difference required to maintain the deformation or (ii) prescribe the radial pressure difference which then determine the deformed inner radius $r_{1}$ and the axial stretch $L$.

We shall discuss the radial pressure difference shortly, but either way, we can obtain $L$ and thus feed this into equation (5.4). Imposing the requirement that $R\left(r_{1}\right)=R_{1}$ and using the form of $\alpha$ gives rise to the useful relation

$$
\begin{equation*}
L=\frac{R_{2}^{2}-R_{1}^{2}}{R_{2}^{2}-r_{1}^{2}} \tag{5.5}
\end{equation*}
$$

The Cauchy stress for an incompressible material is [73]

$$
\begin{equation*}
\boldsymbol{T}=\boldsymbol{F} \frac{d W}{d \boldsymbol{F}}+Q \boldsymbol{I} \tag{5.6}
\end{equation*}
$$

where $W$, in this case, is the neo-Hookean strain energy function introduced in equation (5.1), $\boldsymbol{F}$ is the deformation gradient tensor, $\boldsymbol{I}$ is the identity tensor, and $Q$ is the scalar Lagrange multiplier associated with the incompressibility constraint.

Only diagonal components of the Cauchy stress are non-zero, being given by (no sum on the indices)

$$
\begin{equation*}
T_{j j}=\mu_{j}(r)+Q \tag{5.7}
\end{equation*}
$$

for $j=r, \theta, z$, where

$$
\begin{equation*}
\mu_{r}(r)=\frac{\mu^{2}}{L^{2}} \frac{1}{\mu_{\theta}(r)}=\frac{\mu}{L}\left(\frac{r^{2}+\alpha}{r^{2}}\right), \quad \mu_{z}=L^{2} \mu \tag{5.8}
\end{equation*}
$$

The second and third of the static equations of equilibrium $\operatorname{div} \boldsymbol{T}=\mathbf{0}$ (where div signifies the divergence operator in the deformed configuration) merely yield $Q=$ $Q(r)$. The remaining equation

$$
\begin{equation*}
\frac{\partial T_{r r}}{\partial r}+\frac{1}{r}\left(T_{r r}-T_{\theta \theta}\right)=0, \tag{5.9}
\end{equation*}
$$

can be integrated using equations (5.7) and (5.8) to obtain $Q(r)$.
Writing $\left.T_{r r}\right|_{r=R_{2}}=-p_{o},\left.T_{r r}\right|_{r=r_{1}}=-p_{i}$, we find

$$
\begin{equation*}
\frac{L\left(p_{i}-p_{o}\right)}{\mu}=\frac{1}{2 L}\left(1-\frac{R_{1}^{2}}{r_{1}^{2}}\right)+\log \left(\frac{r_{1}}{R_{1}}\right) . \tag{5.10}
\end{equation*}
$$

Given $L$ and thus $r_{1}$ via equation (5.5) this equation prescribes the required pressure difference.

Now assume that the cylindrical annulus has been pre-stressed in an appropriate manner and slotted into the unbounded elastic material with perfect bonding at $r=$ $R_{2}$. We consider wave propagation in this medium given a time-harmonic antiplane line source located at $\left(R_{0}, \Theta_{0}\right)$ with $R_{0}>R_{2}$. In $r>R_{2}$, the antiplane wave with corresponding displacement which we shall denote by $w(r, \theta)$, is again governed by equation (2.435). In the region $r_{1} \leq r \leq R_{2}$, the wave satisfies a different equation since this annulus region has been pre-stressed according to the deformation defined by equations (5.2) and (5.4). We can obtain the governing equation using the theory of small-on-large [73]. It was shown in [81] that the wave in this region satisfies equation (2.437) but now with $\mu_{r}(r)$ and $\mu_{\theta}(r)$ defined by equation (5.8) and with $d(r)=\rho$, where $\rho$ is the (constant) density of the cloaking material. Note, in particular, that the density is homogeneous inside the cloak region.

Let us introduce the identity mapping for $r>R_{2}$ and

$$
\begin{equation*}
R^{2}=L\left(r^{2}+\alpha\right), \quad \Theta=\theta, \quad \text { for } \quad r_{1} \leq r \leq R_{2}, \tag{5.11}
\end{equation*}
$$

which corresponds to the actual physical deformation (see equation (5.4)). Finally, define $W(R, \Theta)=w(r(R), \theta(\Theta))$. It is then straightforward to show that the equation governing wave propagation in the entire domain $R \geq R_{1}$ is equation (2.435),
provided that we choose $\mu=L \mu_{0}$ and $\rho=L \rho_{0}$. These relations ensure that the wavenumbers in the exterior and cloak regions are the same and they also maintain continuity of traction on $R=R_{2}$. Furthermore, since equation (5.11) corresponds to the actual deformation, the inner radius $r_{1}$ maps back to $R_{1}$. Therefore, with the appropriate choice of cloak material properties, the scattering problem in the undeformed configuration is equivalent to that in the deformed configuration. We can, therefore, solve the equation in the undeformed configuration and then map back to the deformed configuration to find the physical solution. Decomposing the solution into incident and scattered parts $W=W_{i}+W_{s}$, we have $W_{i}=\frac{C}{4 i \mu_{0}} \mathrm{H}_{0}(K S)$, where we have defined the wavenumber $K$ via $K^{2}=\rho_{0} \frac{\omega^{2}}{\mu_{0}}$ and $S=\sqrt{\left(X-X_{0}\right)^{2}+\left(Y-Y_{0}\right)^{2}}$. Here, $\mathrm{H}_{n}=\mathrm{H}_{n}^{(1)}$ is the Hankel function of the first kind of order $n$. The scattered field is written in the form [81]

$$
\begin{equation*}
W_{s}(R, \Theta)=\sum_{n=-\infty}^{\infty}(-i)^{n} a_{n} \mathrm{H}_{n}(K R) e^{i n\left(\Theta-\Theta_{0}\right)} . \tag{5.12}
\end{equation*}
$$

Satisfaction of the traction free boundary condition on $R=R_{1}$ gives $a_{n}$. We want the wave field with respect to the deformed configuration, so we map back in order to find $w=w_{i}+w_{s}$. The incident wave is most conveniently determined by using Graf's addition theorem in order to distinguish between the regions $r<R_{0}$ and $r>R_{0}$, as was described in [81]. The incident and scattered fields are then, respectively,

$$
\begin{align*}
& w_{i}(r, \theta)=\frac{C}{4 i \mu_{0}} \sum_{n=-\infty}^{\infty} e^{i n\left(\theta-\theta_{0}\right)} \\
& \times \begin{cases}\mathrm{H}_{n}\left(K R_{0}\right) \mathrm{J}_{n}\left(K \sqrt{L\left(r^{2}+\alpha\right)}\right), & r_{1} \leq r \leq R_{2}, \\
\mathrm{H}_{n}\left(K R_{0}\right) \mathrm{J}_{n}(K r), & R_{2} \leq r \leq R_{0}, \\
\mathrm{H}_{n}(K r) \mathrm{J}_{n}\left(K R_{0}\right), & r>R_{0},\end{cases}  \tag{5.13}\\
& w_{s}(r, \theta)=-\frac{C}{4 i \mu_{0}} \sum_{n=-\infty}^{\infty} e^{i n\left(\theta-\theta_{0}\right)} \frac{\mathrm{J}_{n}^{\prime}\left(K R_{1}\right)}{\mathrm{H}_{n}^{\prime}\left(K R_{1}\right)} \mathrm{H}_{n}\left(K R_{0}\right) \\
& \times \begin{cases}\mathrm{H}_{n}\left(K \sqrt{L\left(r^{2}+\alpha\right)}\right), & r_{1} \leq r \leq R_{2}, \\
\mathrm{H}_{n}(K r), & r \geq R_{2} .\end{cases} \tag{5.14}
\end{align*}
$$

The key to cloaking is to ensure that the scattered field is small compared with the incident field, i.e., $a_{n} \ll 1$. Note from equation (5.14) that $a_{n}$ are solely dependent


Figure 5.2: Cloaking of antiplane shear waves. Line source is located at $K r=K R_{0}=$ $8 \pi, \Theta_{0}=0$, shown as a white circle. Total displacement field is plotted. Upper left: a region of (nondimensionalised) radius $K r_{1}=2 \pi$ is cloaked using a classic linear elastic cloak $g(R)=r_{1}+R\left(\frac{R_{2}-r_{1}}{R_{2}}\right)$ in $2 \pi \leq K r \leq 4 \pi$. Upper right: scattering from a cavity of radius $K R_{1}=\frac{2 \pi}{20}$ in an unstressed medium. Lower left: a "pre-stress" cloak in $2 \pi \leq K r \leq 4 \pi$ generated from an annulus with initial inner radius $K R_{1}=\frac{2 \pi}{20}$. Lower right: scattering from a cavity with radius $K R_{1}=2 \pi$ in an unstressed medium. Scattering and the shadow region presence in the latter is significant, as compared with that for an equivalent sized cavity for the "pre-stress" cloak.
on the initial annulus inner radius $R_{1}$ (and source distance $R_{0}$ ) but are independent of the deformed inner radius $r_{1}$. Therefore, we must choose $R_{1}$ such that $K R_{1} \ll 1$ which will ensure negligible scattering regardless of the size of $r_{1}$. We illustrate with some examples in Figure 5.2, showing that the "pre-stress" cloak appears to work well. In the figure, the "pre-stress" cloak is in the bottom left.

In conclusion, we have shown how a finite cloak for antiplane elastic waves can be generated by employing nonlinear pre-stress of an incompressible neo-Hookean hyperelastic material. The performance of the cloak is limited only by the size of the initial radius of the cylindrical cavity inside the annulus region. The anisotropic, inhomogeneous material moduli in the cloaking region, defined by equation (5.8), are induced naturally by the pre-stress and therefore exotic metamaterials are not
required. Dispersive effects, which naturally arise in metamaterials due to their inherent inhomogeneity at some length scale, will not be present in the pre-stress context and we also note that the density of the cloak is homogeneous. In order to acheive the required pre-stress, a radial pressure difference is required across the cylindrical annulus. It would be inconvenient to prescribe $p_{o}$ on the outer face. However, since we only need a pressure difference we can prescribe $p_{i}$ with $p_{o}=0$, ensuring the prescribed deformation and eliminating this difficulty. The incompressible neo-Hookean model is an approximation to reality, holding in general for rubber-like materials and moderate deformations. If the material is not neo-Hookean, invariance of the scattering coefficients is not guaranteed in general and therefore similar exact results will not hold. However, it is of interest to ascertain whether scattering from inflated cavities in other hyperelastic pre-stressed media is still significantly reduced as compared with an equivalent sized cavity in an unstressed medium. In the next section, we investigate the cloaking properties of Mooney-Rivlin materials.

Finally, we remark that one of the fundamental advantages of the pre-stress approach is that pre-stress generates equations with incremental moduli (analogies of the elastic moduli) which do not possess the minor symmetries. Therefore, this approach can be used for cloaking in the more general elastodynamic setting, where classical linear elastic materials cannot be used [69].

### 5.3 Mooney-Rivlin cloak

In this section, we investigate the effectiveness of Mooney-Rivlin cloaks. The methodology is exactly the same as in the previous section, except now we use the MooneyRivlin strain energy function instead of the neo-Hookean strain energy function:

$$
\begin{equation*}
W=\frac{\mu}{2}\left(S_{1}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right)+S_{2}\left(\lambda_{r}^{2} \lambda_{\theta}^{2}+\lambda_{r}^{2} \lambda_{z}^{2}+\lambda_{\theta}^{2} \lambda_{z}^{2}-3\right)\right), \tag{5.15}
\end{equation*}
$$

which leads to the following forms for the anistropic shear moduli:

$$
\begin{equation*}
\mu_{r}(r)=\frac{T \mu}{L^{2}}\left(1+\frac{m}{r^{2}}\right), \tag{5.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{\theta}(r)=\frac{T \mu}{L^{2}}\left(1-\frac{m}{r^{2}+\alpha}\right), \tag{5.17}
\end{equation*}
$$

where

$$
\begin{equation*}
m=\frac{\alpha L S_{1}}{T} \tag{5.18}
\end{equation*}
$$

and

$$
\begin{equation*}
T=1+(L-1) S_{1} . \tag{5.19}
\end{equation*}
$$

These forms for $\mu_{r}(r)$ and $\mu_{\theta}(r)$ lead to a slightly modified version of equation (5.10):

$$
\begin{equation*}
\frac{L\left(p_{i}-p_{o}\right)}{S \mu}=\frac{1}{2 L}\left(1-\frac{R_{1}^{2}}{r_{1}^{2}}\right)+\log \left(\frac{r_{1}}{R_{1}}\right), \tag{5.20}
\end{equation*}
$$

where

$$
\begin{equation*}
S=\frac{S_{1}}{L}+L S_{2} \tag{5.21}
\end{equation*}
$$

We also obtain a modified governing ordinary differential equation for $w(r, \theta)$ in the cloaking region:

$$
\begin{equation*}
\left(1+\frac{m}{r^{2}}\right) \frac{\partial^{2} w}{\partial r^{2}}+\frac{1}{r}\left(1-\frac{m}{r^{2}}\right) \frac{\partial w}{\partial r}+\frac{r^{2}+\alpha-m}{r^{2}\left(r^{2}+\alpha\right)} \frac{\partial^{2} w}{\partial \theta^{2}}+k^{2} w=0 \tag{5.22}
\end{equation*}
$$

where

$$
\begin{equation*}
k^{2}=\frac{L^{2} K^{2}}{T} \tag{5.23}
\end{equation*}
$$

which, to the author's knowledge, cannot be solved analytically, and so a numerical solver is used in order to produce the data for Figure 5.3, where the effect of a Mooney-Rivlin cloak on an incoming plane wave propagating the positive $x$-direction is plotted. For this figure, $S_{1}=0.8$, and the same values of $K r_{1}$ and $K R_{1}$ were selected as in the previous section (i.e. $K r_{1}=2 \pi$ and $K R_{1}=\pi / 10$ ).

It can be observed that the Mooney-Rivlin cloak is not perfect as there is a significant shadow region as well as some scattering in all directions. By comparison with Figure 5.4, however, it does appear that there is less scattering for a MooneyRivlin cloak than for an unstressed medium with $K r_{1}=2 \pi$. We quantify the amount of scattering in each case via the scattering cross-section (see [108], [55] and [69]).

The scattering cross-section is defined by

$$
\begin{equation*}
\gamma=\frac{2}{K r_{1}} \sum_{n=0}^{\infty} e_{n}\left|a_{n}\right|^{2}, \tag{5.24}
\end{equation*}
$$



Figure 5.3: A Mooney-Rivlin cloak with $S_{1}=0.8, K r_{1}=2 \pi$ and $K R_{1}=\pi / 10$.


Figure 5.4: Scattering of SH waves from a cylindrical cavity of radius $2 \pi$.


Figure 5.5: The scattering cross-section for horizontally polarised shear waves scattered from a cylindrical cavity in a stress-free medium as a function of $K r_{1}$ with lines corresponding to the value of $\gamma$ for $K r_{1}=2 \pi$ in an unstressed medium (solid) and in a pre-stressed medium with $K R_{1}=\pi / 10$ (dashed).
where

$$
\begin{array}{rll}
e_{n}=1 & \text { if } & n=0, \\
=2 & \text { if } & n \geq 1 . \tag{5.25}
\end{array}
$$

Using this definition, we determine the scattering cross-section in the stress free case to be 1.71605 and, in the case of the Mooney-Rivlin cloak to be 1.28866. This confirms our obervations from Figures 5.3 and 5.4 - that there is less scattering in the cloaked case.

In Figure 5.5 we plot the scattering cross-section for horizontally polarised shear waves scattered from a cylindrical cavity in a stress-free medium as a function of $K r_{1}$, with horizontal lines corresponding to the value of $\gamma$ corresponding to $K r_{1}=2 \pi$ in an unstressed medium (solid) and in a pre-stressed medium with $K R_{1}=\pi / 10$ (dashed). We can see that the dashed line crosses the plotted function at a much lower value of $K r_{1}$. In fact, this value can be determined to be $K r_{1}=1.72460$ (compared with $K r_{1}=6.28319$ for the unstressed medium).

We conclude that, whilst not as effective as a neo-Hookean cloak (whose scattering


Figure 5.6: A Mooney-Rivlin cloak with $S_{1}=0.9, K r_{1}=2 \pi$ and $K R_{1}=\pi / 10$.
cross-section for a cloak with $K r_{1}=2 \pi$ and $K R_{1}=\pi / 10$ is 0.00536 ), a Mooney-Rivlin cloak still significantly reduces the amount of energy scattered from a cylindrical cavity in the cloaked region.

In Figure 5.6 we plot a Mooney-Rivlin cloak with $S_{1}=0.9$. It can be observed that, in this case, the cloak is more effective, but still not as effective as a neo-Hookean cloak. The scattering cross-section in this case is 0.61282 , which corresponds to an unstressed cavity with $K r_{1}=0.64608$.

In Figure 5.7 we plot a Mooney-Rivlin cloak with $S_{1}=0.2$. It can be observed that, in this case, the cloak is not very effective. In fact, its scattering cross-section is 6.12203, which indicates that there is more scattering in this case than in the stressfree case. This indicates that materials which are strongly dependent on the second strain invariant are unsuitable for cloaking.

In Figure 5.8, we plot scattering cross-sections for various various values of $S_{1}$ with $K R_{1}=\pi / 10$. The solid line corresponds to $S_{1}=0.8$, the dashed line corresponds to $S_{1}=0.9$ and the dotted line corresponds to $S_{1}=0.2$. The dot-dashed line corresponds to a neo-Hookean material. We note that, for Mooney-Rivlin materials, as expected, the greater the value of $S_{1}$, the better the cloak.


Figure 5.7: A Mooney-Rivlin cloak with $S_{1}=0.2, K r_{1}=2 \pi$ and $K R_{1}=\pi / 10$.


Figure 5.8: Scattering cross-sections for various values of $S_{1}$ with $K R_{1}=\pi / 10$. The solid line corresponds to $S_{1}=0.8$, the dashed line corresponds to $S_{1}=0.9$ and the dotted line corresponds to $S_{1}=0.2$. The dot-dashed line corresponds to a neo-Hookean material.


Figure 5.9: Scattering cross-sections for various values of $S_{1}$ with $K R_{1}=\pi / 10$, compared with scattering cross-section in an unstressed material. The solid line corresponds to an unstressed material, the dashed line corresponds to $S_{1}=0.8$ and the dotted line corresponds to $S_{1}=0.2$.

In Figure 5.9, we compare the scattering cross-sections of Mooney-Rivlin prestress cloaks with $S_{1}=0.8$ and $S_{1}=0.2$ with the scattering cross-section of an unstressed material. We observe that the cloak with $S_{1}=0.8$ performs better than an unstressed material over the whole range of $K r_{1}$ considered, whereas the cloak with $S_{1}=0.2$ performs better over a small range of $K r_{1}$, but then performs worse than an unstressed material for larger $K r_{1}$.

### 5.4 Conclusions

We observe that, for cloaks with $K R_{1}=\pi / 10$ and $K r_{1}=2 \pi$, as $S_{1}$ tends from 1 towards 0 (in other words, as the material becomes less dependent on the first strain invariant, and more dependent on the second strain invariant), there is more scattering from the cloaking region. Hence, an ideal cloak will be dependent on the first strain invariant only. This explains why a neo-Hookean material is a good material for cloaking purposes in this context.

For materials which are strongly dependent on the second strain invariant the scattering cross-section for larger values of $K r_{1}$ is greater than the scattering crosssection for an unstressed material, hence these materials are unsuitable for this type of pre-stress cloaking.

Potential areas of further work would be an investigation into pre-stress cloaking using other strain energy functions and a comparison of the pre-stress cloaking method with metamaterial cloaking.

## Chapter 6

## Scattering of shear waves from a spherical cavity in a pre-stressed host medium

### 6.1 Overview

In Chapter 2, we studied the scattering of dilatational and shear waves from a spherical cavity in an unstressed linear elastic material. In this chapter we shall consider the scattering of shear waves from a spherical cavity in a pre-stressed host medium. A spherical void is located in an unbounded, nonlinear-elastic material. Pressure is applied on the inner surface, and at infinity, modifying the radius of the cavity and creating an inhomogeneous deformation field in the region close to the cavity. We aim to show the effect that this pre-stress has on the scattering of small-amplitude shear waves through the medium. We use the theory of small-on-large to determine the linear wave equations that govern incremental waves in a spherical coordinate system in the context of the applied pre-stress. We emphasise that the inhomogeneous deformation makes the coefficients of the governing ODEs spatially dependent. We will show that if the pressure at infinity is greater than that on the surface of the cavity then the resulting inhomogeneous stress region acts to magnify the scattering of one component of shear wave and reduce the scattering of (infinitely long) dilatational
waves.
In Section 6.2 we use the condition of incompressibility to determine the form of the applied radial deformation. The static equations of equilibrium are determined and they are used to determine the effect of the applied pressures on the radii. In Section 6.3 we consider the propagation of small-amplitude time-harmonic waves through the finitely-deformed medium and give the relevant governing equations. In Section 6.4 we discuss the governing ODEs, which are hard to solve due to the spatial dependence of their coefficients, as mentioned above, and describe the numerical method used to solve them. In Section 6.5 we use the numerical method to determine the effect of the pre-stress on the scattering coefficients and plot the scattering crosssections for the two components of the scattered shear waves.

### 6.2 Initial finite (static) deformation

Consider an unbounded, nonlinear-elastic medium. A spherical void of initial radius $A$ is located at the origin. We will assume that the host material is isotropic and hyperelastic so that its constitutive behaviour may be described by a strain energy function, $W$, which may be written as a function of either the principal stretches $\lambda_{j}$, or the strain invariants $i_{j}$ or $I_{j}$ (see [73], [34], [38] or Section 2.2.4). Pressure will be applied on the inner surface of the void, and at infinity, which will have the effect of deforming the radius of the void to $a$. Since we are working in curvilinear coordinates, it will be convenient to use the tensorial notation of [73].

When we impose the above deformation, it is described by

$$
\begin{equation*}
R=R(r), \quad \Theta=\theta, \quad \Phi=\phi, \tag{6.1}
\end{equation*}
$$

where $(R, \Theta, \Phi)$ and $(r, \theta, \phi)$ are spherical polar coordinates in the undeformed and deformed configurations respectively. Here, $R$ and $r$ measure the radial distance, $\Theta$ and $\theta$ measure the inclination angle and $\Phi$ and $\phi$ measure the azimuthal angle. Note the convention introduced in (6.1) above, i.e. that upper case variables correspond
to the undeformed configuration whilst lower case corresponds to the deformed configuration. It will be convenient for us to derive equations in terms of coordinates in the deformed configuration. Position vectors with respect to a Cartesian basis, in the undeformed (upper case) and deformed (lower case) configurations are

$$
\boldsymbol{X}=\left(\begin{array}{c}
R \sin \Theta \cos \Phi  \tag{6.2}\\
R \sin \Theta \sin \Phi \\
R \cos \Theta
\end{array}\right)=\left(\begin{array}{c}
R(r) \sin \theta \cos \phi \\
R(r) \sin \theta \sin \phi \\
R(r) \cos \theta
\end{array}\right), \quad \boldsymbol{x}=\left(\begin{array}{c}
r \sin \theta \cos \phi \\
r \sin \theta \sin \phi \\
r \cos \theta
\end{array}\right)
$$

The deformation gradient tensor is given by $\boldsymbol{F}=\operatorname{Grad} \boldsymbol{x},[73]$, where we use Grad to represent the gradient operator with respect to the undeformed configuration.

In our case, we have

$$
\boldsymbol{F}=\left(\begin{array}{ccc}
\lambda_{r} & 0 & 0  \tag{6.3}\\
0 & \lambda_{\theta} & 0 \\
0 & 0 & \lambda_{\phi}
\end{array}\right)=\left(\begin{array}{ccc}
\frac{1}{R^{\prime}(r)} & 0 & 0 \\
0 & \frac{r}{R(r)} & 0 \\
0 & 0 & \frac{r}{R(r)}
\end{array}\right)
$$

where $\lambda_{r}, \lambda_{\theta}$ and $\lambda_{\phi}$ are the principal stretches in the radial, circumferential and azimuthal directions respectively.

For an incompressible material, we must have $J=\operatorname{det} \boldsymbol{F}=1$, and so

$$
\begin{equation*}
\lambda_{r} \lambda_{\theta} \lambda_{\phi}=\frac{r^{2}}{R^{2} R^{\prime}}=1 . \tag{6.4}
\end{equation*}
$$

Solving the above, we obtain

$$
\begin{equation*}
R(r)=\left(r^{3}+\alpha\right)^{\frac{1}{3}} \tag{6.5}
\end{equation*}
$$

where $\alpha$ is a constant defined by

$$
\begin{equation*}
\alpha=A^{3}-a^{3} . \tag{6.6}
\end{equation*}
$$

As given in equation (2.81), the Cauchy stress tensor for an incompressible material is

$$
\begin{equation*}
\boldsymbol{T}=\boldsymbol{F} \frac{\partial W}{\partial \boldsymbol{F}}+Q \boldsymbol{I} \tag{6.7}
\end{equation*}
$$

where $W$ is the strain energy function of the material under consideration, $\boldsymbol{I}$ is the identity tensor and $Q$ is a Lagrange multiplier associated with the incompressibility constraint and referred to as an arbitrary hydrostatic pressure.

Equation (6.7) can be written in index notation as

$$
\begin{equation*}
T_{i j}=F_{i \alpha} \frac{\partial W}{\partial F_{j \alpha}}+Q \delta_{i j}, \tag{6.8}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta. We note that since $\boldsymbol{F}$ is diagonal, so is $\boldsymbol{T}$. In our case, we have

$$
\boldsymbol{T}=\left(\begin{array}{ccc}
\lambda_{r} \frac{\partial W}{\partial \lambda_{r}}+Q & 0 & 0  \tag{6.9}\\
0 & \lambda_{\theta} \frac{\partial W}{\partial \lambda_{\theta}}+Q & 0 \\
0 & 0 & \lambda_{\phi} \frac{\partial W}{\partial \lambda_{\phi}}+Q
\end{array}\right)
$$

If we neglect body forces, then the static equations of equilibrium are given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{T}=\mathbf{0} \tag{6.10}
\end{equation*}
$$

where as usual div to represent the divergence operator in the deformed configuration. For our deformation, the only equation not trivially satisfied is the radial equation:

$$
\begin{equation*}
r \frac{d}{d r} T_{r r}+2\left(T_{r r}-T_{\theta \theta}\right)=0 \tag{6.11}
\end{equation*}
$$

where we have used the fact that $T_{\theta \theta}=T_{\phi \phi}$.
If we assume that a hydrostatic pressure, $p_{a}$, is applied to the inner surface and a pressure, $p_{\infty}$, is applied at infinity, then we also have the following boundary conditions:

$$
\begin{equation*}
T_{r r}(a)=-p_{a}, \quad T_{r r} \rightarrow-p_{\infty} \text { as } r \rightarrow \infty . \tag{6.12}
\end{equation*}
$$

The radial equation of equilibrium can be used to determine the constant $\alpha$ in $R(r)$ given in (6.5). This is, however, dependent on our choice of strain energy function. In Chapter 4.5 of [34], six classes of compressible material are presented for which an analytic solution for $R(r)$ may be found. Unfortunately, however, to the author's knowledge, none of these strain energy functions has been shown to exhibit results which agree with experiments on rubber. Therefore, we shall use the incompressible neo-Hookean and Mooney-Rivlin models.

### 6.2.1 Specific strain energy functions

## Neo-Hookean strain energy function

Here we present some results for a neo-Hookean material. The neo-Hookean strain energy function is given by:

$$
\begin{equation*}
W=\frac{\mu}{2}\left(I_{1}-3\right)=\frac{\mu}{2}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right), \tag{6.13}
\end{equation*}
$$

where $\mu$ is the shear modulus of the material under consideration. Upon substituting the neo-Hookean strain energy function into equation (6.9), we obtain

$$
\begin{gather*}
T_{r r}=\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}} \mu+Q,  \tag{6.14}\\
T_{\theta \theta}=T_{\phi \phi}=\frac{r^{2}}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}} \mu+Q . \tag{6.15}
\end{gather*}
$$

We can then substitute the above into (6.11) in order to determine $Q$. Upon doing so, we obtain

$$
\begin{equation*}
\frac{d Q}{d r}=\frac{2 \alpha^{2} \mu}{r^{5}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}, \tag{6.16}
\end{equation*}
$$

which gives

$$
\begin{equation*}
Q=\frac{\mu}{2 r^{4}}\left(3 r^{3}-\alpha\right)\left(r^{3}+\alpha\right)^{\frac{1}{3}}+Q_{0} \tag{6.17}
\end{equation*}
$$

where $Q_{0}$ is a constant to be determined from the boundary conditions.
Applying the boundary condition at infinity, we obtain

$$
\begin{equation*}
Q_{0}=-\frac{5 \mu+2 p_{\infty}}{2}, \tag{6.18}
\end{equation*}
$$

and using the boundary condition on $r=a$, we obtain the following condition on $\alpha$ :

$$
\begin{equation*}
\frac{\left(5 a^{3}+\alpha\right)\left(a^{3}+\alpha\right)^{\frac{1}{3}}}{2 a^{4}}-\frac{5}{2}=\frac{p_{\infty}-p_{a}}{\mu} . \tag{6.19}
\end{equation*}
$$

Via equation (6.6), we can eliminate $\alpha$ from the above in order to obtain

$$
\begin{equation*}
\frac{\left(4 a^{3}+A^{3}\right) A}{2 a^{4}}-\frac{5}{2}=\frac{p_{\infty}-p_{a}}{\mu}, \tag{6.20}
\end{equation*}
$$

which can we rewritten as

$$
\begin{equation*}
\frac{\left(4 \bar{a}^{3}+1\right)}{2 \bar{a}^{4}}-\frac{5}{2}=\frac{p_{\infty}-p_{a}}{\mu}, \tag{6.21}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{a}=\frac{a}{A} . \tag{6.22}
\end{equation*}
$$

## Mooney-Rivlin strain energy function

Here we present some results for a Mooney-Rivlin material. The Mooney-Rivlin strain energy function is given by:

$$
\begin{align*}
W & =\frac{\mu}{2}\left(S_{1}\left(I_{1}-3\right)+S_{2}\left(I_{2}-3\right)\right) \\
& =\frac{\mu}{2}\left(S_{1}\left(\lambda_{r}^{2}+\lambda_{\theta}^{2}+\lambda_{z}^{2}-3\right)+S_{2}\left(\lambda_{r}^{2} \lambda_{\theta}^{2}+\lambda_{r}^{2} \lambda_{z}^{2}+\lambda_{\theta}^{2} \lambda_{z}^{2}-3\right)\right) \tag{6.23}
\end{align*}
$$

where $\mu$ is the shear modulus of the material under consideration, and $S_{1}$ and $S_{2}$ are two constants which sum to one. This strain energy function yields

$$
\begin{equation*}
T_{r r}=\left(S_{1} \frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}}+2 S_{2} \frac{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}{r^{2}}\right) \mu+Q \tag{6.24}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{\theta \theta}=T_{\phi \phi}=\left(S_{1} \frac{r^{2}}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+S_{2}\left(\frac{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}{r^{2}}+\frac{r^{4}}{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}\right)\right) \mu+Q . \tag{6.25}
\end{equation*}
$$

We can then substitute the above into (6.11) in order to determine $Q$. Upon doing so, we obtain

$$
\begin{equation*}
\frac{d Q}{d r}=\frac{2 \alpha^{2} \mu}{r^{7}}\left(S_{1} \frac{r^{2}}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+S_{2} \frac{r^{4}}{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}\right) \tag{6.26}
\end{equation*}
$$

which gives

$$
\begin{equation*}
Q=\left(\frac{S_{1}}{2 r^{4}}\left(3 r^{3}-\alpha\right)\left(r^{3}+\alpha\right)^{\frac{1}{3}}-\frac{S_{2}}{r^{2}\left(r^{3}+\alpha\right)^{\frac{1}{3}}}\left(3 r^{3}+\alpha\right)\right) \mu+Q_{0} \tag{6.27}
\end{equation*}
$$

where $Q_{0}$ is a constant to be determined from the boundary conditions.
Upon applying the boundary condition at infinity, we obtain

$$
\begin{equation*}
Q_{0}=\frac{\left(2 S_{2}-5 S_{1}\right) \mu-2 p_{\infty}}{2} \tag{6.28}
\end{equation*}
$$

and so the boundary condition on $r=a$ gives the following condition on $\alpha$ :

$$
\begin{equation*}
\left(\frac{\left(5 a^{3}+\alpha\right)\left(a^{3}+\alpha\right)^{\frac{1}{3}}}{2 a^{4}}-\frac{5}{2}\right) S_{1}+\left(\frac{\alpha-a^{3}}{a^{2}\left(a^{3}+\alpha\right)^{\frac{1}{3}}}+1\right) S_{2}=\frac{p_{\infty}-p_{a}}{\mu} . \tag{6.29}
\end{equation*}
$$

Using equation (6.6), we can eliminate $\alpha$ from the above in order to obtain

$$
\begin{equation*}
\left(\frac{\left(4 a^{3}+A^{3}\right) A}{2 a^{4}}-\frac{5}{2}\right) S_{1}+\left(\frac{A^{3}-2 a^{3}}{a^{2} A}+1\right) S_{2}=\frac{p_{\infty}-p_{a}}{\mu} \tag{6.30}
\end{equation*}
$$

which can we rewritten as

$$
\begin{equation*}
\left(\frac{\left(4 \bar{a}^{3}+1\right)}{2 \bar{a}^{4}}-\frac{5}{2}\right) S_{1}+\left(\frac{1}{\bar{a}}-2 \bar{a}+1\right) S_{2}=\frac{p_{\infty}-p_{a}}{\mu}, \tag{6.31}
\end{equation*}
$$

where again

$$
\begin{equation*}
\bar{a}=\frac{a}{A} . \tag{6.32}
\end{equation*}
$$

### 6.3 Incremental deformations

We now consider the propagation of small-amplitude, time-harmonic shear waves through an incompressible medium. We use the theory of small-on-large, i.e. linearisation about a non-linear deformation state [73]. The total displacement field may be represented by

$$
\begin{equation*}
\bar{U}=\boldsymbol{U}+\boldsymbol{u} . \tag{6.33}
\end{equation*}
$$

where $\boldsymbol{U}$ is the displacement field derived from the finite deformation (6.1) and $\boldsymbol{u}$ is the incremental displacement. As mentioned, the incident field is a plane harmonic shear wave which we will assume is propagating in the postive $z$-direction and is polarised such that the displacements are in the $x$ direction (see Figure 6.1). This incremental displacement takes the form

$$
\begin{equation*}
\boldsymbol{u}=\Re\left\{(u(r, \theta, \phi), v(r, \theta, \phi), w(r, \theta, \phi)) e^{-i \omega t}\right\}, \tag{6.34}
\end{equation*}
$$

where $\omega$ is the frequency of the wave, and $u, v$ and $w$ are the displacements in the radial, polar and azimuthal directions, respectively. We will also assume that $|\boldsymbol{u}| \ll|\boldsymbol{U}|$.

It can be shown that the equations of motion governing $\boldsymbol{u}$, in the absence of body forces, are given by

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\zeta}=\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}} \tag{6.35}
\end{equation*}
$$

where $\rho$ is the density of the pre-stressed body, and $\boldsymbol{\zeta}$ is the so-called push forward of the incremental nominal stress tensor (see equation (2.130)), which can be expressed as

$$
\begin{equation*}
\boldsymbol{\zeta}=\boldsymbol{M}: \boldsymbol{\gamma}+q \boldsymbol{I}-Q \boldsymbol{\gamma} \tag{6.36}
\end{equation*}
$$



Figure 6.1: Shear waves incident on a spherical cavity.
where $\boldsymbol{\gamma}=\operatorname{grad} \boldsymbol{u}, q$ is the perturbation to the Lagrange multiplier $Q$, and $\boldsymbol{M}$ is a fourth order tensor, whose only non-zero components are given by

$$
\begin{align*}
M_{i i j j} & =J^{-1} \lambda_{i} \lambda_{j} W_{i j},  \tag{6.37}\\
M_{i j i j} & =\lambda_{i}^{2}\left(\lambda_{i} W_{i}-\lambda_{j} W_{j}\right) /\left(\lambda_{i}^{2}-\lambda_{j}^{2}\right), \quad \lambda_{i} \neq \lambda_{j}, \quad i \neq j,  \tag{6.38}\\
M_{i j i j} & =\frac{1}{2}\left(M_{i i i i}-M_{i i j j}+\lambda_{i} W_{i}\right), \quad \lambda_{i}=\lambda_{j}, \quad i \neq j,  \tag{6.39}\\
M_{j i i j} & =M_{i j j i}=M_{i j i j}-J^{-1} \lambda_{i} W_{i}, \quad i \neq j, \tag{6.40}
\end{align*}
$$

where $W_{i j}=\partial^{2} W / \partial \lambda_{i} \partial \lambda_{j}$, etc.
In our case, we have

$$
\gamma=\left(\begin{array}{ccc}
\frac{\partial u}{\partial r} & \frac{1}{r}\left(\frac{\partial u}{\partial \theta}-v\right) & \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi}-\frac{w}{r}  \tag{6.41}\\
\frac{\partial v}{\partial r} & \frac{1}{r}\left(\frac{\partial v}{\partial \theta}+u\right) & \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi}-\frac{w \cot \theta}{r} \\
\frac{\partial w}{\partial r} & \frac{1}{r} \frac{\partial w}{\partial \theta} & \frac{u}{r}+\frac{v \cot \theta}{r}+\frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi}
\end{array}\right) e^{-i \omega t}
$$

and (6.35) can be written in spherical coordinates as follows:

$$
\begin{align*}
& \frac{\partial \zeta_{11}}{\partial r}+\frac{1}{r} \frac{\partial \zeta_{21}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \zeta_{31}}{\partial \phi}+\frac{1}{r}\left(2 \zeta_{11}-\zeta_{22}-\zeta_{33}\right)+\frac{\cot \theta}{r} \zeta_{21}=\rho \frac{\partial^{2} u_{1}}{\partial t^{2}} \\
& \frac{\partial \zeta_{12}}{\partial r}+\frac{1}{r} \frac{\partial \zeta_{22}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \zeta_{32}}{\partial \phi}+\frac{1}{r}\left(2 \zeta_{12}+\zeta_{21}\right)+\frac{\cot \theta}{r}\left(\zeta_{22}-\zeta_{33}\right)=\rho \frac{\partial^{2} u_{2}}{\partial t^{2}}, \\
& \frac{\partial \zeta_{13}}{\partial r}+\frac{1}{r} \frac{\partial \zeta_{23}}{\partial \theta}+\frac{1}{r \sin \theta} \frac{\partial \zeta_{33}}{\partial \phi}+\frac{1}{r}\left(2 \zeta_{13}+\zeta_{31}\right)+\frac{\cot \theta}{r}\left(\zeta_{23}+\zeta_{32}\right)=\rho \frac{\partial^{2} u_{3}}{\partial t^{2}} . \tag{6.42}
\end{align*}
$$

For our problem, (6.42) can be written in full as

$$
\begin{align*}
& \quad\left(M_{1111}-Q\right) u_{r r}+\left(r M_{1111}^{\prime}+2 M_{1111}-\left(r Q^{\prime}+2 Q\right)\right) \frac{u_{r}}{r}+2\left(r M_{2211}^{\prime}+M_{2211}-M_{2222}-\right. \\
& \left.M_{2233}+Q\right) \frac{u}{r^{2}}+M_{2121} \frac{u_{\theta \theta}}{r^{2}}+\cot \theta M_{2121} \frac{u_{\theta}}{r^{2}}+\cot \theta\left(M_{2211}+M_{2112}-Q\right) \frac{v_{r}}{r}+\left(M_{2211}+M_{2112}-\right. \\
& Q) \frac{w_{r \theta}}{r \sin \theta}+\left(r M_{2211}^{\prime}+M_{2211}-M_{2222}-M_{2233}-M_{2121}+Q\right) \frac{v_{\theta}}{r^{2}}+\left(M_{2211}+M_{2112}-Q\right) \frac{v_{r \theta}}{r}+ \\
& \cot \theta\left(r M_{2211}^{\prime}+M_{2211}-M_{2222}-M_{2233}-M_{2121}+Q\right) \frac{v}{r^{2}}+\frac{M_{2121}}{\sin ^{2} \theta} \frac{u_{\phi \phi}}{r^{2}}+\left(r M_{2211}^{\prime}+M_{2211}-\right. \\
& \left.M_{2222}-M_{2233}-M_{2121}+Q\right) \frac{w_{\phi}}{r^{2} \sin \theta}+\frac{\partial q}{\partial r}=-\rho \omega^{2} u, \tag{6.43}
\end{align*}
$$

$$
\begin{aligned}
& M_{1212} v_{r r}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{v_{r}}{r}-\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+\cot ^{2} \theta M_{2222}-\right. \\
& Q \cot ^{2} \theta-\left(r Q^{\prime}+Q\right) \frac{v}{r^{2}}+\left(M_{2222}-Q\right) \frac{v_{\theta \theta}}{r^{2}}+\cot \theta\left(M_{2222}-Q\right) \frac{v_{\theta}}{r^{2}}+\left(M_{2112}+M_{2211}-Q\right) \frac{u_{r \theta}}{r}+ \\
& \left(r M_{2112}^{\prime}+M_{2112}+M_{2222}+M_{2233}+M_{2121}-\left(r Q^{\prime}+2 Q\right)\right) \frac{u_{\theta}}{r^{2}}-\left(M_{2222}+M_{3232}-Q\right) \cot \theta \frac{w_{\phi}}{r^{2} \sin \theta} \\
& +\left(M_{2233}+M_{2332}-Q\right) \frac{w_{\theta \phi}}{r^{2} \sin \theta}+M_{3232} \frac{v_{\phi \phi}}{r^{2} \sin ^{2} \theta}+\frac{1}{r} \frac{\partial q}{\partial \theta}=-\rho \omega^{2} v, \\
& M_{1212} w_{r r}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{w_{r}}{r}-\left(r M_{2112}^{\prime}+M_{2112}-M_{2332}+M_{2121}+\cot ^{2} \theta M_{3232}-\right. \\
& \left.r Q^{\prime}\right) \frac{w}{r^{2}}+\left(M_{2222}-Q\right) \frac{w_{\phi \phi}}{r^{2} \sin ^{2} \theta}+M_{2323} \frac{w_{\theta}}{r^{2}}+\left(M_{2112}+M_{2211}-Q\right) \frac{u_{r \phi}}{r \sin \theta}+\left(r M_{2112}^{\prime}+M_{2112}+\right. \\
& M_{2222}+M_{2233}+M_{2121}-\left(r Q^{\prime}+2 Q\right) \frac{u_{\phi}}{r^{2} \sin \theta}+M_{2323} \frac{w_{\theta \theta}}{r^{2}}+\left(M_{2233}+M_{2332}-Q\right) \frac{v_{\theta \phi}}{r^{2} \sin \theta}+ \\
& \left(M_{2222}+M_{3232}-Q\right) \cot \theta \frac{v_{\phi}}{r^{2} \sin \theta}+\frac{1}{r \sin \theta} \frac{\partial q}{\partial \phi}=-\rho \omega^{2} w, \quad(6.45)
\end{aligned}
$$

where subscripts on $u, v$ and $w$ denote partial differentiation and $M_{i j k l}^{\prime}=d M_{i j k l} / d r$.
We now expand $\boldsymbol{u}$ in terms of the vector spherical harmonics, $\boldsymbol{Y}_{l m}, \boldsymbol{\Psi}_{l m}$ and $\boldsymbol{\Phi}_{l m}$, which are defined as follows:

$$
\begin{equation*}
\boldsymbol{Y}_{l m}=Y_{l}^{m} \boldsymbol{e}_{r}, \quad \boldsymbol{\Psi}_{l m}=r \nabla Y_{l}^{m}, \quad \boldsymbol{\Phi}_{l m}=\boldsymbol{r} \times \nabla Y_{l}^{m} \tag{6.46}
\end{equation*}
$$

where $\boldsymbol{e}_{r}$ is a unit vector in the radial direction, $\boldsymbol{r}$ is a position vector, and $Y_{l}^{m}$ is a scalar spherical harmonic, defined as follows:

$$
\begin{equation*}
Y_{l}^{m}=Y_{l}^{m}(\theta, \phi)=P_{l}^{m}(\cos \theta) e^{i m \phi} \tag{6.47}
\end{equation*}
$$

where $P_{l}^{m}$ is an associated Legendre polynomial of degree $l$ and order $m$.
The expansion is as follows:

$$
\begin{equation*}
\boldsymbol{u}=\Re\left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(f_{l}(r) \boldsymbol{Y}_{l m}+g_{l}(r) \boldsymbol{\Psi}_{l m}+h_{l}(r) \boldsymbol{\Phi}_{l m}\right) e^{-i \omega t}\right), \tag{6.48}
\end{equation*}
$$

so that $u, v$ and $w$ can be expressed as

$$
\begin{gather*}
u(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l}(r) P_{l}^{m}(\cos \theta) e^{i m \phi}  \tag{6.49}\\
v(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(g_{l}(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+h_{l}(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}\right), \tag{6.50}
\end{gather*}
$$

$$
\begin{equation*}
w(r, \theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l}\left(g_{l}(r) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}-h_{l}(r) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}\right) . \tag{6.51}
\end{equation*}
$$

We also expand $q$ in terms of a scalar spherical harmonic as follows:

$$
\begin{equation*}
q=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{l}(r) Y_{l}^{m}(\theta, \phi)=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{l}(r) P_{l}^{m}(\cos \theta) e^{i m \phi} . \tag{6.52}
\end{equation*}
$$

Upon making the above expansions, and exploiting incompressibility, equations (6.43) - (6.45) reduce to:

$$
\begin{align*}
& \left(M_{1111} f^{\prime \prime}+\left(r M_{1111}^{\prime}+2 M_{1111}-r Q^{\prime}\right) \frac{f^{\prime}}{r}+\left(2 r M_{2211}^{\prime}+2 M_{2211}-2 M_{2222}-2 M_{2233}-\right.\right. \\
& \left.l(l+1) M_{2121}\right) \frac{f}{r^{2}}-l(l+1)\left(\left(M_{2211}+M_{2112}\right) \frac{g^{\prime}}{r}+\left(r M_{2211}^{\prime}+M_{2211}-M_{2121}-M_{2222}-\right.\right. \\
& \left.\left.\left.M_{2233}\right) \frac{g}{r^{2}}\right)+\rho \omega^{2} f+B^{\prime}(r)\right) P_{l}^{m}(\cos \theta) e^{i m \phi}=0,  \tag{6.53}\\
& \left(M_{1212} g^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{g^{\prime}}{r}-\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+(l(l+1)-\right.\right. \\
& \text { 1) } \left.M_{2222}-r Q^{\prime}\right) \frac{g}{r^{2}}+\left(M_{2112}+M_{2211}\right) \frac{f^{\prime}}{r}+\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+M_{2222}\right. \\
& \left.\left.-r Q^{\prime}\right) \frac{f}{r^{2}}+\rho \omega^{2} g+\frac{B(r)}{r}\right) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+\left(M_{1212} h^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{h^{\prime}}{r}-\right. \\
& \left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}-M_{2222}+l(l+1) M_{3232}-r Q^{\prime}\right) \frac{h}{r^{2}}+ \\
& \left.\rho \omega^{2} h\right) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}=0, \quad(6.54)  \tag{6.54}\\
& \left(M_{1212} g^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{g^{\prime}}{r}-\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+(l(l+1)-\right.\right. \\
& \text { 1) } \left.M_{2222}-r Q^{\prime}\right) \frac{g}{r^{2}}+\left(M_{2112}+M_{2211}\right) \frac{f^{\prime}}{r}+\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+M_{2222}\right. \\
& \left.\left.-r Q^{\prime}\right) \frac{f}{r^{2}}+\rho \omega^{2} g+\frac{B(r)}{r}\right) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}+\left(M_{1212} h^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{h^{\prime}}{r}-\right. \\
& \left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}-M_{2222}+l(l+1) M_{3232}-r Q^{\prime}\right) \frac{h}{r^{2}}+ \\
& \left.\rho \omega^{2} h\right) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}=0, \quad(6.55) \tag{6.55}
\end{align*}
$$

where we have dropped the subscript $l$ on $f(r), g(r), h(r)$ and $B(r)$.

Due to the orthogonality of $\frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}$ and $\frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}$, the above equations can be separated as follows:

$$
\begin{array}{r}
M_{1111} f^{\prime \prime}+\left(r M_{1111}^{\prime}+2 M_{1111}-r Q^{\prime}\right) \frac{f^{\prime}}{r}+\left(2 r M_{2211}^{\prime}+2 M_{2211}-2 M_{2222}-2 M_{2233}-\right. \\
\left.l(l+1) M_{2121}\right) \frac{f}{r^{2}}-l(l+1)\left(\left(M_{2211}+M_{2112}\right) \frac{g^{\prime}}{r}+\left(r M_{2211}^{\prime}+M_{2211}-M_{2121}-M_{2222}-\right.\right. \\
\left.\left.M_{2233}\right) \frac{g}{r^{2}}\right)+\rho \omega^{2} f+B^{\prime}(r)=0, \tag{6.56}
\end{array}
$$

$$
M_{1212} g^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{g^{\prime}}{r}-\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+(l(l+1)-\right.
$$

$$
\text { 1) } \left.M_{2222}-r Q^{\prime}\right) \frac{g}{r^{2}}+\left(M_{2112}+M_{2211}\right) \frac{f^{\prime}}{r}+\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}+M_{2222}\right.
$$

$$
\begin{equation*}
\left.-r Q^{\prime}\right) \frac{f}{r^{2}}+\rho \omega^{2} g+\frac{B(r)}{r}=0 \tag{6.57}
\end{equation*}
$$

$$
\begin{array}{r}
M_{1212} h^{\prime \prime}+\left(r M_{1212}^{\prime}+2 M_{1212}\right) \frac{h^{\prime}}{r}-\left(r M_{2112}^{\prime}+M_{2112}+M_{2233}+M_{2121}-M_{2222}+l(l+1) M_{3232}\right. \\
\left.-r Q^{\prime}\right) \frac{h}{r^{2}}+\rho \omega^{2} h=0, \quad \tag{6.58}
\end{array}
$$

Note that the incompressibility condition, $\operatorname{div} \boldsymbol{u}=0$, gives us a condition on $f$ and $g:$

$$
\begin{align*}
\operatorname{div} \boldsymbol{u}=0 & \Rightarrow \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(v \sin \theta)+\frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi}=0  \tag{6.59}\\
& \Rightarrow \frac{\partial f}{\partial r}+\frac{2}{r} f-\frac{l(l+1)}{r} g=0  \tag{6.60}\\
& \Rightarrow g(r)=\frac{1}{l(l+1)}\left(r f^{\prime}(r)+2 f(r)\right) . \tag{6.61}
\end{align*}
$$

Using this condition, we can eliminate $g$ from (6.56) and (6.57) as follows:

$$
\begin{align*}
& \left(M_{1111}+M_{2211}+M_{2112}\right) f^{\prime \prime}+\left(r M_{1111}^{\prime}+2 M_{1111}-r M_{2211}^{\prime}-4 M_{2211}-3 M_{2112}+M_{2121}+\right. \\
& \left.\left.M_{2222}+M_{2233}-r Q^{\prime}\right) \frac{f^{\prime}}{r}+(2-l(l+1)) M_{2121}\right) \frac{f}{r^{2}}+\rho \omega^{2} f+B^{\prime}(r)=0, \tag{6.62}
\end{align*}
$$

$$
\begin{gather*}
r M_{1212} f^{(3)}+\left(r M_{1212}^{\prime}+6 M_{1212}\right) f^{\prime \prime}-\left(r M_{2112}^{\prime}+(1-l(l+1)) M_{2112}-3 r M_{1212}^{\prime}-6 M_{1212}+\right. \\
\left.M_{2233}+M_{2121}+(l(l+1)-1) M_{2222}-l(l+1) M_{2211}-r Q^{\prime}\right) \frac{f^{\prime}}{r}-(2-l(l+1))\left(r M_{2112}^{\prime}+M_{2112}+\right. \\
\left.M_{2233}+M_{2121}-M_{2222}-r Q^{\prime}\right) \frac{f}{r^{2}}+\rho \omega^{2} r f^{\prime}+2 \rho \omega^{2} f+l(l+1) \frac{B(r)}{r}=0 . \tag{6.63}
\end{gather*}
$$

### 6.3.1 Neo-Hookean material

If we use the neo-Hookean strain energy function, then the components of the tensor $\boldsymbol{M}$ can be explicitly evaluated:

$$
\begin{gather*}
M_{1111}=M_{1212}=\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}} \mu,  \tag{6.64}\\
M_{1122}=M_{2211}=M_{1133}=M_{3311}=M_{2233}=M_{3322}=M_{1221}=M_{2112}=M_{2332} \\
=M_{3223}=0,  \tag{6.65}\\
M_{2222}=M_{3333}=M_{2121}=M_{2323}=M_{3232}=\frac{r^{2}}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}} \mu, \tag{6.66}
\end{gather*}
$$

and equations (6.62), (6.63) and (6.58) simplify to

$$
\begin{gather*}
\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}} f^{\prime \prime}+\frac{4\left(r^{6}-\alpha^{2}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{2}{3}}} f^{\prime}+\left(\frac{2-l(l+1)}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+k^{2}\right) f+\frac{B^{\prime}(r)}{\mu}=0  \tag{6.67}\\
\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{3}} f^{(3)}+\frac{2\left(3 r^{6}+4 r^{3} \alpha+\alpha^{2}\right)}{r^{4}\left(r^{3}+\alpha\right)^{\frac{2}{3}}} f^{\prime \prime}+\left(\frac{r^{6}(6-l(l+1))-4 \alpha^{2}}{r^{5}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+r k^{2}\right) f^{\prime}+ \\
2\left(\frac{(2-l(l+1)) \alpha^{2}}{\left.r^{6}\left(r^{3}+\alpha\right)^{\frac{2}{3}}+k^{2}\right) f+l(l+1) \frac{B(r)}{\mu r}=0,}\right.  \tag{6.68}\\
\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}} h^{\prime \prime}+\frac{2\left(r^{6}-\alpha^{2}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{2}{3}}} h^{\prime}+\left(\frac{2 \alpha^{2}-l(l+1) r^{6}}{r^{6}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+k^{2}\right) h=0 \tag{6.69}
\end{gather*}
$$

where

$$
\begin{equation*}
k^{2}=\frac{\rho \omega^{2}}{\mu} . \tag{6.70}
\end{equation*}
$$

$B(r)$ can be eliminated from (6.67) and (6.68) in order to obtain

$$
\begin{align*}
& \frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{2}} f^{(4)}+8\left(r^{3}+\alpha\right)^{\frac{1}{3}} f^{(3)}+ \\
& \quad\left(\frac{2(6-l(l+1)) r^{6}-2(2+l(l+1)) r^{3} \alpha-(10+l(l+1)) \alpha^{2}}{r^{4}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+r^{2} k^{2}\right) f^{\prime \prime}+ \\
& 2\left(l(l+1) \frac{\left(-2 r^{9}-3 r^{6} \alpha+r^{3} \alpha^{2}+\alpha^{3}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+\frac{2\left(3 r^{6} \alpha+7 r^{3} \alpha^{2}+5 \alpha^{3}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+2 r k^{2}\right) f^{\prime}+ \\
& \quad(l(l+1)-2)\left(\frac{2\left(7 r^{3} \alpha^{2}+5 \alpha^{3}\right)}{r^{6}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+\frac{l(l+1)}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}-k^{2}\right) f=0 . \tag{6.71}
\end{align*}
$$

We now have two ordinary differential equations ((6.71) and (6.69)) in two unknowns ( $f$ and $h$ ).

### 6.3.2 Boundary conditions

It was shown in equation (2.163) that, for an incompressible material, on $r=a$ the following boundary condition must be satisfied:

$$
\begin{equation*}
\boldsymbol{\zeta}^{T} \boldsymbol{n}=p_{a} \boldsymbol{\gamma}^{T} \boldsymbol{n} \tag{6.72}
\end{equation*}
$$

where $p_{a}$ is the pressure applied on $r=a$ and $\boldsymbol{n}$ is the outer unit normal to the boundary. In our case, $\boldsymbol{n}=\boldsymbol{e}_{r}$.

In the case considered here, we obtain the following:

$$
\begin{align*}
&\left(\left(M_{1111}-Q-p_{a}\right) f^{\prime}(r)-M_{1122} l(l+1) \frac{g(r)}{r}+\right. 2 M_{1122} \frac{f(r)}{r}+ \\
&B(r))\left.P_{l}^{m}(\cos \theta) e^{i m \phi}\right|_{r=a}=0, \tag{6.73}
\end{align*}
$$

and

$$
\begin{align*}
& \left(M_{1212} g^{\prime}(r)+\left(M_{1221}-Q-p_{a}\right) \frac{f(r)-g(r)}{r}\right) \frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}+ \\
& \left.\quad\left(M_{1212} h^{\prime}(r)-\left(M_{1221}-Q-p_{a}\right) \frac{h(r)}{r}\right) \frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}\right|_{r=a}=0 . \tag{6.74}
\end{align*}
$$

Using the relationship between $f$ and $g$, and the orthogonality of $\frac{d}{d \theta}\left(P_{l}^{m}(\cos \theta)\right) e^{i m \phi}$ and $\frac{P_{l}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}$, the above equations can be simplified as follows:

$$
\begin{gather*}
\left(M_{1111}-M_{1122}-Q-p_{a}\right) f^{\prime}(r)+\left.B(r)\right|_{r=a}=0,  \tag{6.75}\\
M_{1212}\left(r f^{\prime \prime}(r)+3 f^{\prime}(r)\right)+\left.\left(M_{1221}-Q-p_{a}\right)\left(l(l+1) \frac{f(r)}{r}-f^{\prime}(r)-2 \frac{f(r)}{r}\right)\right|_{r=a}=0,  \tag{6.76}\\
M_{1212} h^{\prime}(r)-\left.\left(M_{1221}-Q-p_{a}\right) \frac{h(r)}{r}\right|_{r=a}=0 . \tag{6.77}
\end{gather*}
$$

### 6.4 Hybrid analytical-numerical method

In this section we divide the host domain into two regions (see Figure 6.4). We assume that there exists some radius $b$, which is large enough such that for $r>b$ the quantities in equations (6.64) and (6.66) can be approximated by

$$
\begin{equation*}
M_{1111}=M_{1212}=M_{2222}=M_{3333}=M_{2121}=M_{2323}=M_{3232}=\mu \tag{6.78}
\end{equation*}
$$



Figure 6.2: Discretisation of material into inhomogeneous region local to void and homogeneous region far from void.

In other words, we treat the region $r>b$ as being homogeneous, and then use a numerical solver in order to analyse the inhomogeneous inner region, $r<b$.

Upon using the above approximation, equations (6.62) and (6.63) reduce, for $r>b$, to

$$
\begin{equation*}
f^{\prime \prime}+\frac{4}{r} f^{\prime}+\left(\frac{2-l(l+1)}{r^{2}}+k^{2}\right) f+\frac{B^{\prime}(r)}{\mu}=0 \tag{6.79}
\end{equation*}
$$

and

$$
\begin{equation*}
r f^{(3)}+6 f^{\prime \prime}+\left(\frac{6-l(l+1)}{r}+r k^{2}\right) f^{\prime}+2 k^{2} f+\frac{B(r)}{\mu r}=0 \tag{6.80}
\end{equation*}
$$

and equation (6.58) reduces to

$$
\begin{equation*}
h^{\prime \prime}+\frac{2}{r} h^{\prime}+\left(k^{2}-\frac{l(l+1)}{r^{2}}\right) h=0 . \tag{6.81}
\end{equation*}
$$

Once again we can eliminate $B(r)$ from equations (6.79) and (6.80) in order to obtain

$$
\begin{align*}
& r^{2} f^{(4)}+8 r f^{(3)}+\left(2(6-l(l+1))+r^{2} k^{2}\right) f^{\prime \prime}+4\left(r k^{2}-\frac{l(l+1)}{r}\right) f^{\prime}+ \\
&(l(l+1)-2)\left(\frac{l(l+1)}{r^{2}}-k^{2}\right) f=0 . \tag{6.82}
\end{align*}
$$

The general solution of this equation can be written as

$$
\begin{equation*}
f_{l}(r)=C_{l}^{(1)} \frac{j_{l}(k r)}{r}+C_{l}^{(2)} \frac{h_{l}(k r)}{r}+C_{l}^{(3)} r^{-(l+2)}+C_{l}^{(4)} r^{l-1} \tag{6.83}
\end{equation*}
$$

where $C_{l}^{(1)}, C_{l}^{(2)}, C_{l}^{(3)}$ and $C_{l}^{(4)}$ are constants.
The third and fourth terms in the above are algebraic, but in fact correspond to spherical Bessel functions of the second and first kind, respectively, in the limit as the wavelength tends to infinity (see Appendix C). The infinite wavelength is due to the fact that as a material tends towards incompressibility, the wavenumber for P -waves tends to 0 . Therefore, the third and fourth terms in the above can be interpreted as infinite wavelength scattered P-waves which have been excited by the incoming shear waves. These are required in order to satisfy the boundary conditions. We immediately choose $C_{l}^{(4)}=0$ since for $l>1$, the fourth term is unbounded as $r \rightarrow \infty$.

The general solution of equation (6.81) can be written as

$$
\begin{equation*}
h_{l}(r)=D_{l}^{(1)} j_{l}(k r)+D_{l}^{(2)} h_{l}(k r), \tag{6.84}
\end{equation*}
$$

where $D_{l}^{(1)}$ and $D_{l}^{(2)}$ are constants.
In the outer region, we will assume there is an incoming plane wave propagating in the positive $z$-direction and polarised such that the displacement is in the $x$-direction. In [29] it is given that the displacement vector for such a wave may be written as

$$
\begin{equation*}
\boldsymbol{u}_{i}=\sum_{n=1}^{\infty} \frac{(n+1) i^{n}}{n(n+1)}\left(\boldsymbol{M}_{o 1 n}^{1}(\boldsymbol{r})-i \boldsymbol{N}_{e 1 n}^{1}(\boldsymbol{r})\right) e^{-i \omega t}, \tag{6.85}
\end{equation*}
$$

where $\boldsymbol{M}$ and $\boldsymbol{N}$ are defined, as in [62] and [29], by

$$
\begin{equation*}
\boldsymbol{M}_{\sigma m n}^{1}=(n(n+1))^{\frac{1}{2}} \boldsymbol{C}_{m n}^{\sigma} j_{n}(k r), \tag{6.86}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{N}_{\sigma m n}^{1}=n(n+1) \boldsymbol{P}_{m n}^{\sigma} \frac{1}{k r} j_{n}(k r)+(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{1}{k r} \frac{d}{d r}\left(r j_{n}(k r)\right), \tag{6.87}
\end{equation*}
$$

in which the label $\sigma$ is either $e$ (even) or $o$ (odd) and designates whether the even (real) or odd (imaginary) part of the azimuthal function is to be employed,

$$
\begin{equation*}
\boldsymbol{P}_{m n}=\boldsymbol{Y}_{n m}=\boldsymbol{e}_{r} Y_{n}^{m}(\theta, \phi), \tag{6.88}
\end{equation*}
$$

$$
\begin{align*}
\boldsymbol{B}_{m n} & =\frac{(n(n+1))^{\frac{1}{2}}}{(2 n+1) \sin \theta}\left(\boldsymbol{e}_{\theta}\left(\frac{n-m+1}{n+1} Y_{n+1}^{m}-\frac{n+m}{n} Y_{n-1}^{m}\right)+\boldsymbol{e}_{\phi} \frac{m(2 n+1)}{n(n+1)} i Y_{n}^{m}\right) \\
& =(n(n+1))^{-\frac{1}{2}}\left(\boldsymbol{e}_{\theta} \frac{d}{d \theta}\left(P_{n}^{m}(\cos \theta)\right) e^{i m \phi}+\boldsymbol{e}_{\phi} \frac{P_{n}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}\right), \\
& =(n(n+1))^{-\frac{1}{2}} \boldsymbol{\Psi}_{n m} \\
& =\boldsymbol{e}_{r} \times \boldsymbol{C}_{m n}, \tag{6.89}
\end{align*}
$$

and

$$
\begin{align*}
\boldsymbol{C}_{m n} & =\frac{(n(n+1))^{\frac{1}{2}}}{(2 n+1) \sin \theta}\left(\boldsymbol{e}_{\theta} \frac{m(2 n+1)}{n(n+1)} i Y_{n}^{m}-\boldsymbol{e}_{\phi}\left(\frac{n-m+1}{n+1} Y_{n+1}^{m}-\frac{n+m}{n} Y_{n-1}^{m}\right)\right) \\
& =(n(n+1))^{-\frac{1}{2}}\left(\boldsymbol{e}_{\theta} \frac{P_{n}^{m}(\cos \theta)}{\sin \theta} i m e^{i m \phi}-\boldsymbol{e}_{\phi} \frac{d}{d \theta}\left(P_{n}^{m}(\cos \theta)\right) e^{i m \phi}\right) \\
& =(n(n+1))^{-\frac{1}{2}} \boldsymbol{\Phi}_{n m} . \tag{6.90}
\end{align*}
$$

Using the above, we can rewrite $\boldsymbol{M}$ and $\boldsymbol{N}$ as

$$
\begin{equation*}
\boldsymbol{M}_{\sigma m n}^{1}=j_{n}(k r) \boldsymbol{\Phi}_{n m} \tag{6.91}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{N}_{\sigma m n}^{1}=n(n+1) \frac{j_{n}(k r)}{k r} \boldsymbol{Y}_{n m}+\frac{1}{k r} \frac{d}{d r}\left(r j_{n}(k r)\right) \boldsymbol{\Psi}_{n m} . \tag{6.92}
\end{equation*}
$$

Therefore, by refering to (6.48), we see that the incoming wave has the following components:

$$
\begin{gather*}
f_{l}^{i}(r)=\left\{\begin{array}{cc}
-(2 l+1) i^{l+1} \frac{j_{l}(k r)}{k r} & l \geq 1 \\
0 & l \leq 0
\end{array},\right.  \tag{6.93}\\
g_{l}^{i}(r)=\left\{\begin{array}{cc}
-\frac{(2 l+1) i^{l+1}}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r j_{l}(k r)\right) & l \geq 1 \\
0 & l \leq 0
\end{array},\right.  \tag{6.94}\\
h_{l}^{i}(r)=\left\{\begin{array}{cc}
\frac{(2 l+1) i^{l}}{l(l+1)} j_{l}(k r) & l \geq 1 \\
0 & l \leq 0
\end{array}\right. \tag{6.95}
\end{gather*}
$$

Note that $f_{l}^{i}(r)$ and $g_{l}^{i}(r)$ satisfy

$$
\begin{equation*}
\frac{\partial f_{l}^{i}}{\partial r}+\frac{2}{r} f_{l}^{i}-\frac{l(l+1)}{r} g_{l}^{i}=0, \tag{6.96}
\end{equation*}
$$

as required.

We now consider the scattered part of the wave. The shear portion of the scattered field can be written, similarly to the above, as

$$
\begin{equation*}
\boldsymbol{u}_{s s}=\sum_{n=1}^{\infty} \frac{(2 n+1) i^{n}}{n(n+1)}\left(a_{n} \boldsymbol{M}_{o 1 n}^{3}(\boldsymbol{r})-i b_{n} \boldsymbol{N}_{e 1 n}^{3}(\boldsymbol{r})\right) e^{-i \omega t}, \tag{6.97}
\end{equation*}
$$

where $a_{n}$ and $b_{n}$ are scattering coefficients,

$$
\begin{equation*}
\boldsymbol{M}_{\sigma m n}^{3}=(n(n+1))^{\frac{1}{2}} \boldsymbol{C}_{m n}^{\sigma} h_{n}^{(1)}(k r), \tag{6.98}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{N}_{\sigma m n}^{3}=n(n+1) \boldsymbol{P}_{m n}^{\sigma} \frac{1}{k r} h_{n}^{(1)}(k r)+(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{1}{k r} \frac{d}{d r}\left(r h_{n}^{(1)}(k r)\right) . \tag{6.99}
\end{equation*}
$$

This corresponds to the following forms for $f_{l}^{s s}(r), g_{l}^{s s}(r)$ and $h_{l}^{s s}(r)$ :

$$
\begin{gather*}
f_{l}^{s s}(r)=\left\{\begin{array}{cc}
-b_{l}(2 l+1) i^{l+1} \frac{h_{l}^{(1)}(k r)}{k r} & l \geq 1 \\
0 & l \leq 0
\end{array},\right.  \tag{6.100}\\
g_{l}^{s s}(r)=\left\{\begin{array}{cc}
-b_{l} \frac{(2 l+1) i^{l+1}}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r h_{l}^{(1)}(k r)\right) & l \geq 1 \\
0 & l \leq 0
\end{array},\right.  \tag{6.101}\\
h_{l}^{s s}(r)=\left\{\begin{array}{cc}
a_{l} \frac{(2 l+1) i^{l}}{l(l+1)} h_{l}^{(1)}(k r) & l \geq 1 \\
0 & l \leq 0
\end{array},\right. \tag{6.102}
\end{gather*}
$$

and we can show that $f_{l}^{s s}(r)$ and $g_{l}^{s s}(r)$ satisfy

$$
\begin{equation*}
\frac{\partial f_{l}^{s s}}{\partial r}+\frac{2}{r} f_{l}^{s s}-\frac{l(l+1)}{r} g_{l}^{s s}=0, \tag{6.103}
\end{equation*}
$$

as required. The part of the scattered field which takes the form of a compressional wave with infinite wavelength can be written as

$$
\begin{equation*}
\boldsymbol{u}_{s c}=\sum_{n=1}^{\infty} \frac{(2 n+1) i^{n}}{n(n+1)} d_{n} \hat{\boldsymbol{L}}_{e 1 n}^{3}(\boldsymbol{r}) e^{-i \omega t}, \tag{6.104}
\end{equation*}
$$

where $d_{n}$ is a scattering coefficient, and

$$
\begin{equation*}
\hat{\boldsymbol{L}}_{e 1 n}^{3}=i n(n+1) \boldsymbol{P}_{m n}^{\sigma} \frac{r^{-(n+2)}}{k}-i n(n(n+1))^{\frac{1}{2}} \boldsymbol{B}_{m n}^{\sigma} \frac{r^{-(n+2)}}{k} . \tag{6.105}
\end{equation*}
$$

This gives the following forms for $f_{l}^{s c}(r), g_{l}^{s c}(r)$ and $h_{l}^{s c}(r)$ :

$$
f_{l}^{s c}(r)=\left\{\begin{array}{cc}
d_{l} \frac{(2 l+1) i^{l+1}}{k} r^{-(l+2)} & l \geq 1  \tag{6.106}\\
0 & l \leq 0
\end{array},\right.
$$

$$
g_{l}^{s c}(r)=\left\{\begin{array}{cc}
-d_{l} \frac{(2 l+1) i^{l+1}}{k(l+1)} r^{-(l+2)} & l \geq 1 \\
0 & l \leq 0  \tag{6.108}\\
h_{l}^{s c}(r)=0
\end{array},\right.
$$

Note that $f_{l}^{s c}(r)$ and $g_{l}^{s c}(r)$ satisfy

$$
\begin{equation*}
\frac{\partial f_{l}^{s c}}{\partial r}+\frac{2}{r} f_{l}^{s c}-\frac{l(l+1)}{r} g_{l}^{s c}=0 \tag{6.109}
\end{equation*}
$$

as required.
The total scattered field is given by

$$
\begin{equation*}
\boldsymbol{u}_{s}=\boldsymbol{u}_{s s}+\boldsymbol{u}_{s c}, \tag{6.110}
\end{equation*}
$$

and the total field is given by

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{i}+\boldsymbol{u}_{s} . \tag{6.111}
\end{equation*}
$$

Therefore, we have

$$
\begin{gather*}
f_{l}(r)=-(2 l+1) i^{l+1} \frac{j_{l}(k r)}{k r}-b_{l}(2 l+1) i^{l+1} \frac{h_{l}^{(1)}(k r)}{k r}-d_{l} \frac{(2 l+1) i^{l+1}}{k} r^{-(l+2)},  \tag{6.112}\\
g_{l}(r)=-\frac{(2 l+1) i^{l+1}}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r j_{l}(k r)\right)-b_{l} \frac{(2 l+1) i^{l+1}}{l(l+1)} \frac{1}{k r} \frac{d}{d r}\left(r h_{l}^{(1)}(k r)\right)+ \\
d_{l} \frac{(2 l+1) i^{l+1}}{k(l+1)} r^{-(l+2)},  \tag{6.113}\\
h_{l}(r)=\frac{(2 l+1) i^{l}}{l(l+1)} j_{l}(k r)+a_{l} \frac{(2 l+1) i^{l}}{l(l+1)} h_{l}^{(1)}(k r), \tag{6.114}
\end{gather*}
$$

for $l \geq 1$, and

$$
\begin{equation*}
f_{l}(r)=g_{l}(r)=h_{l}(r)=0, \tag{6.115}
\end{equation*}
$$

for $l \leq 0$.
We now apply continuity of displacement and traction in order to determine the boundary conditions for a numerical solver on $r=b$. In the following, the superscript $I$ denotes the inner solutions (in the region $r<b$ ). We will assume that

$$
\begin{equation*}
f_{l}^{I}(r)=-\frac{(2 l+1) i^{l+1}}{k} F_{l}(r) \tag{6.116}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{l}^{I}(r)=\frac{(2 l+1) i^{l}}{l(l+1)} H_{l}(r) . \tag{6.117}
\end{equation*}
$$

We note that $F_{l}(r)$ satisfies

$$
\begin{align*}
& \frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{2}} F_{l}^{(4)}+8\left(r^{3}+\alpha\right)^{\frac{1}{3}} F_{l}^{(3)}+ \\
& \quad\left(\frac{2(6-l(l+1)) r^{6}-2(2+l(l+1)) r^{3} \alpha-(10+l(l+1)) \alpha^{2}}{r^{4}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+r^{2} k^{2}\right) F_{l}^{\prime \prime}+ \\
& 2\left(l(l+1) \frac{\left(-2 r^{9}-3 r^{6} \alpha+r^{3} \alpha^{2}+\alpha^{3}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+\frac{2\left(3 r^{6} \alpha+7 r^{3} \alpha^{2}+5 \alpha^{3}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+2 r k^{2}\right) F_{l}^{\prime}+ \\
& \quad(l(l+1)-2)\left(\frac{2\left(7 r^{3} \alpha^{2}+5 \alpha^{3}\right)}{r^{6}\left(r^{3}+\alpha\right)^{\frac{5}{3}}}+\frac{l(l+1)}{\left(r^{3}+\alpha\right)^{\frac{2}{3}}}-k^{2}\right) F_{l}=0, \quad(6.1 \tag{6.118}
\end{align*}
$$

and $H_{l}(r)$ satisfies

$$
\begin{equation*}
\frac{\left(r^{3}+\alpha\right)^{\frac{4}{3}}}{r^{4}} H_{l}^{\prime \prime}+\frac{2\left(r^{6}-\alpha^{2}\right)}{r^{5}\left(r^{3}+\alpha\right)^{\frac{2}{3}}} H_{l}^{\prime}+\left(\frac{2 \alpha^{2}-l(l+1) r^{6}}{r^{6}\left(r^{3}+\alpha\right)^{\frac{2}{3}}}+k^{2}\right) H_{l}=0 . \tag{6.119}
\end{equation*}
$$

Continuity of displacement and traction on $r=b$ gives us

$$
\begin{align*}
f_{l}(b) & =f_{l}^{I}(b),  \tag{6.120}\\
f_{l}^{\prime}(b) & =f_{l}^{I \prime}(b),  \tag{6.121}\\
f_{l}^{\prime \prime}(b) & =f_{l}^{I \prime \prime}(b),  \tag{6.122}\\
f_{l}^{(3)}(b) & =f_{l}^{I(3)}(b),  \tag{6.123}\\
h_{l}(b) & =h_{l}^{I}(b), \tag{6.124}
\end{align*}
$$

and

$$
\begin{equation*}
h_{l}^{\prime}(b)=h_{l}^{I I}(b), \tag{6.125}
\end{equation*}
$$

and thus

$$
\begin{gather*}
F_{l}(b)=\frac{j_{l}(k b)}{b}+b_{l} \frac{h_{l}^{(1)}(k b)}{b}+d_{l} b^{-(l+2)},  \tag{6.126}\\
F_{l}^{\prime}(b)=k \frac{j_{l}^{\prime}(k b)}{b}-\frac{j_{l}(k b)}{b^{2}}+b_{l}\left(k \frac{h_{l}^{(1) \prime}(k b)}{b}-\frac{h_{l}^{(1)}(k b)}{b^{2}}\right)-d_{l}(l+2) b^{-l(l+3)},  \tag{6.127}\\
F_{l}^{\prime \prime}(b)=k^{2} \frac{j_{l}^{\prime \prime}(k b)}{b}-2 k \frac{j_{l}^{\prime}(k b)}{b^{2}}+2 \frac{j_{l}(k b)}{b^{3}}+ \\
b_{l}\left(k^{2} \frac{h_{l}^{(1) \prime \prime}(k b)}{b}-2 k \frac{h_{l}^{(1) \prime}(k b)}{b^{2}}+2 \frac{h_{l}^{(1)}(k b)}{b^{3}}\right)+d_{l}(l+2)(l+3) b^{-(l+4)}, \tag{6.128}
\end{gather*}
$$

and

$$
\begin{align*}
& F_{l}^{(3)}(b)= k^{3} \frac{j_{l}^{(3)}(k b)}{b}-3 k^{2} \frac{j_{l}^{\prime \prime}(k b)}{b^{2}}+6 k \frac{j_{l}^{\prime}(k b)}{b^{3}}-6 \frac{j_{l}(k b)}{b^{4}}+ \\
& b_{l}\left(k^{3} \frac{h_{l}^{(1)(3)}(k b)}{b}-3 k^{2} \frac{h_{l}^{(1) \prime \prime}(k b)}{b^{2}}+6 k \frac{h_{l}^{(1) \prime}(k b)}{b^{3}}-6 \frac{h_{l}^{(1)}(k b)}{b^{4}}\right)- \\
& d_{l}(l+2)(l+3)(l+4) b^{-(l+5)} \tag{6.129}
\end{align*}
$$

with

$$
\begin{equation*}
H_{l}(b)=j_{l}(k b)+a_{l} h_{l}^{(1)}(k b), \tag{6.130}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{l}^{\prime}(b)=k j_{l}^{\prime}(k b)+a_{l} k H_{l}^{(1) \prime}(k b) . \tag{6.131}
\end{equation*}
$$

The scattering coefficients $a_{l}, b_{l}$ and $d_{l}$ can be eliminated from the above equations to give us two boundary conditions on $F_{l}(r)$ and one boundary condition on $H_{l}(r)$ on $r=b$. The other boundary conditions are

$$
\begin{gather*}
2 \frac{\left(a^{3}+\alpha\right)^{\frac{4}{3}}}{a^{4}} F_{l}^{\prime}(a)+B(a)=0,  \tag{6.132}\\
F_{l}^{\prime \prime}(a)+2 \frac{F_{l}^{\prime}(a)}{a}+(l(l+1)-2) \frac{F_{l}(a)}{a^{2}}=0, \tag{6.133}
\end{gather*}
$$

and

$$
\begin{equation*}
H_{l}^{\prime}(a)-\frac{H_{l}(a)}{a}=0 \tag{6.134}
\end{equation*}
$$

and so we can then use these to solve the two ordinary differential equations using a numerical solver.

Once $F_{l}(b), F_{l}^{\prime}(b)$ and $H_{l}(b)$ have been evaluated numerically, we can use equations (6.126) and (6.127) in order to determine the scattering coefficients $b_{l}$ and $d_{l}$, and equation (6.130) in order to determine the scattering coefficients $a_{l}$.

As in previous chapters, the numerical solver used in this case was NDSolve in Mathematica 7. The numerical method was, again, automatically selected by NDSolve, and since the problem considered here is a boundary value problem, the Gelfand-Lokutsiyevskii chasing method [8] was used. The AccuracyGoal and PrecisionGoal were both set to 25 digits, and the WorkingPrecision was set to 50 .


Figure 6.3: $\left|a_{1}\right|$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$.

### 6.5 Results

Using the hybrid analytical-numerical method described above, we can plot the effect of the pre-stress on the scattering coefficients. In Figure 6.3, we give the absolute value of the scattering coefficient $a_{1}$ as a function of $k A$ for various values of $\left(p_{\infty}-\right.$ $\left.p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$. These values of $\left(p_{\infty}-p_{a}\right) / \mu$ have been chosen as they corresponds to $\alpha=0$ and $\alpha= \pm 1$.

In Figure 6.4, we plot $\left|a_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

In Figure 6.5, we plot the scattering cross-section $\gamma_{a}$, defined as

$$
\begin{equation*}
\gamma_{a}=\frac{2}{(k a)^{2}} \sum_{l=1}^{\infty}(2 l+1)\left|a_{l}\right|^{2}, \tag{6.135}
\end{equation*}
$$

as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$. The above infinite sum was truncated at a


Figure 6.4: $\left|a_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.
value of $L$ such that the magnitude of $a_{l}$ for $l \geq L$ was so small that its contribution to $\gamma_{a}$ could not be observed in the figures given. For the range of $\left(p_{\infty}-p_{a}\right) / \mu$ and $k A$ considered here, this value was $L=5$ and this truncation number was used in Figures 6.5 and 6.6.

In Figure 6.6, we plot the scattering cross-section $\gamma_{a}$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

We observe that the magnitude of $a_{1}$ is strongly dependent on the (nondimensionalised) pressure difference $\left(p_{\infty}-p_{a}\right) / \mu$. This dependence is not always monotonic as can be seen in Figure 6.4. However, the overall trend appears to be that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, so does $\left|a_{1}\right|$. This is, perhaps, counterintuitive as it means that the scattering of this mode decreases even though the radius of the cavity increases.

The scattering cross-section $\gamma_{a}$ is also strongly dependent on $\left(p_{\infty}-p_{a}\right) / \mu$, as is illustrated in Figure 6.6. Note that the scattering cross-section is at a minimum at a different value of $\left(p_{\infty}-p_{a}\right) / \mu$ for different values of $k A$.


Figure 6.5: The scattering cross-section $\gamma_{a}$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=$ -0.5 .


Figure 6.6: The scattering cross-section $\gamma_{a}$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.


Figure 6.7: $\left|b_{1}\right|$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$.

In Figure 6.7, we plot the absolute value of the scattering coefficient $b_{1}$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$.

In Figure 6.8, we plot $\left|b_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

In Figure 6.9, we plot the scattering cross-section $\gamma_{b}$, defined as

$$
\begin{equation*}
\gamma_{b}=\frac{2}{(k a)^{2}} \sum_{l=1}^{\infty}(2 l+1)\left|b_{l}\right|^{2}, \tag{6.136}
\end{equation*}
$$

as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$. As with $\gamma_{a}$, the infinite sum in $\gamma_{b}$ was truncated at a value of $L$ such that the magnitude of $b_{l}$ for $l \geq L$ was so small that its contribution could not be observed in Figures 6.9 and 6.10. For the range of $\left(p_{\infty}-p_{a}\right) / \mu$ and $k A$ considered here, this value was $L=5$.


Figure 6.8: $\left|b_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.


Figure 6.9: The scattering cross-section $\gamma_{b}$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=$ -0.5 .


Figure 6.10: The scattering cross-section $\gamma_{b}$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

In Figure 6.10, we plot the scattering cross-section $\gamma_{b}$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

We observe that, as for $a_{1}$, the magnitude of $b_{1}$ is strongly dependent on the (nondimensionalised) pressure difference $\left(p_{\infty}-p_{a}\right) / \mu$. The overall trend appears to be that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, so does $\left|b_{1}\right|$. This trend is borne out by the dependence of the scattering cross-section $\gamma_{b}$ on $\left(p_{\infty}-p_{a}\right) / \mu$, as is illustrated in Figure 6.10.

In Figure 6.11, we plot the absolute value of the scattering coefficient $d_{1}$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$.

In Figure 6.12, we plot $\left|d_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.

In general, the scattering cross section for dilatational waves scattered from a


Figure 6.11: $\left|d_{1}\right|$ as a function of $k A$ for various values of $\left(p_{\infty}-p_{a}\right) / \mu$. The solid line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=0$, the dashed line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=1.3$ and the dotted line corresponds to $\left(p_{\infty}-p_{a}\right) / \mu=-0.5$.


Figure 6.12: $\left|d_{1}\right|$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The solid line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$.
sphere is given by [29]:

$$
\begin{equation*}
\gamma_{d}=\frac{2 k a}{(K a)^{3}} \sum_{l=1}^{\infty} \frac{2 l+1}{l(l+1)}\left|d_{l}\right|^{2}, \tag{6.137}
\end{equation*}
$$

where $K$ is the wavenumber of the dilatational wave. In our case, however, we effectively have $K=0$ and, therefore, the above expression is undefined. Hence, in Figure 6.13, we instead plot the absolute value of the displacement on $r=b$ associated with the scattered compressional (long) wave

$$
\begin{equation*}
\beta=\left|\sum_{l=1}^{N} f_{l}^{s c}(b)\right| \tag{6.138}
\end{equation*}
$$

where $f_{l}^{s c}(r)$ is given in (6.106), as a function of the pressure difference $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. In the figure, the dot-dashed lines correspond to $k A=1$, the dashed linesto $k A=2$ and the dotted lines to $k A=0.5$; the black lines correspond to $N=1$, the blue lines to $N=2$, and the red lines to $N=3$; the value of $b$ selected was 100 . We observe that the $N=1$ term is dominant, and the other terms provide higher order corrections, which become more significant for larger values of $\left(p_{\infty}-p_{a}\right) / \mu$. We observe that the overall trend for all values of $k A$ is that the magnitude of the displacement on $r=b$ associated with the scattered compressional (long) wave decreases as $\left(p_{\infty}-p_{a}\right) / \mu$ increases. This appears to indicate that a greater pressure at infinity than on $r=a$ leads to reduced dilatational wave scattering, and a greater pressure on $r=a$ than at infinity leads to enhanced dilatational wave scattering.

### 6.6 Conclusions

In this chapter we have studied the problem of shear wave scattering from a spherical cavity in a pre-stressed Mooney-Rivlin material. The pre-stress consists of hydrostatic pressures imposed on the inner surface of the cavity, and at infinity. Importantly, this generates an inhomogeneous deformation in the host domain.

The theory of small-on-large was used to derive the incremental equations in the pre-stressed configuration. It was then discussed that these equations are difficult


Figure 6.13: $\beta$ as a function of $\left(p_{\infty}-p_{a}\right) / \mu$ for various values of $k A$. The dot-dashed line corresponds to $k A=1$, the dashed line corresponds to $k A=2$ and the dotted line corresponds to $k A=0.5$. The black lines correspond to $N=1$, the blue lines to $N=2$, and the red lines to $N=3 ; b=100$.
to solve due to the spatial depdence of their coefficients. In Section 6.4 we presented a numerical scheme to analyse the governing equations, and used it to plot the scattering coefficients and cross-sections of the scattered waves in Section 6.5.

The dependence of the scattering cross-sections $\gamma_{a}$ and $\gamma_{b}$ on the pre-stress is displayed in Figures 6.6, and 6.10, respectively. We observe that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, $\gamma_{b}$ increases, i.e. the pre-stress has the effect of increasing the scattering of this type of shear wave for $\left(p_{\infty}-p_{a}\right) / \mu>0$ and decreasing the scattering for $\left(p_{\infty}-p_{a}\right) / \mu<0$. This may appear initially counterintuitive, as the scattering of this type of shear wave is increasing despite the fact that the size of the cavity is decreasing; however, the inhomogeneous region around the cavity is clearly enhancing the scattering in this situation. The dependence of $\gamma_{a}$ on $\left(p_{\infty}-p_{a}\right) / \mu$ is non-monotonic, and takes a different minimum value for each value of $k A$ that has been analysed. Further work could be undertaken to establish the reason for the non-monotonicity.

As discussed above, the scattering cross-section is undefined for dilatational waves as their wavelength is infinite in this incompressible limit. However, the effect of the
pre-stress on the magnitude of the corresponding displacement, $\beta$ (see eqref6.beta), is shown to be that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, $\beta$ decreases, and vice versa. This appears to indicate that a greater pressure at infinity than on $r=a$ leads to increased shear wave scattering and reduced dilatational wave scattering, and a greater pressure on $r=a$ than at infinity leads to enhanced dilatational wave scattering and reduced shear wave scattering.

The dependence of the scattering coefficients and cross-sections on the pre-stress could potentially be used to modify the magnitudes of specific types of scattered waves from a cavity of a given size. Potential areas of further work would be a study of the effect of the elastic parameter $S_{2}$ (of the Mooney-Rivlin strain energy function) on the stated results, and an investigation of whether the behaviour of the scattered waves would be similar for other choices of strain energy function.

## Chapter 7

## Conclusions

### 7.1 Summary of results

In this thesis we have considered the effect of pre-stress on the propagation and scattering of waves through nonlinear elastic media. A key feature of this work is that inhomogeneous pre-stress has been considered, whereas most previous work in this area has involved homogeneous pre-stress. We have observed that this type of pre-stress can significantly alter the propagation characteristics of the waves under consideration.

In Chapter 3 we studied the problem of torsional wave propagation in a prestressed, Mooney-Rivlin, annular cylinder. The pre-stress consisted of a uniform longitudinal stretch and hydrostatic pressures imposed on the inner and outer surfaces of the cylinder, thus altering the radii. Importantly, the latter generated an inhomogeneous deformation in the host domain.

The theory of small-on-large was used to derive the incremental equation in the pre-stressed configuration. It was then discussed that this equation was difficult to solve due to the spatial dependence of its coefficients and the singular limit of the equation in the case of zero pre-stress. In Section 3.4.4 we presented a LiouvilleGreen approximation to the solution of the ODE and discussed when we expect this approximation to be accurate. It was shown that for $\hat{\alpha}>0.087$ ( $\hat{\alpha}$ is defined in equation (3.49)), we expect the Liouville-Green approximation to be good for an
annular cylinder of any size.
We noted that a positive value of $\alpha$ (which corresponds to $p_{\text {out }}>p_{\text {in }}$ ) causes the roots of the dispersion curves to be spaced further apart, whilst a negative value of $\alpha$ (which corresponds to $p_{\text {out }}<p_{\text {in }}$ ) causes them to be spaced more closely. In the neo-Hookean case (i.e. when $S_{1}=1, S_{2}=0$ ), $\alpha$ does not affect the gradients of the dispersion curves, whereas in the Mooney-Rivlin model, a positive value of $\alpha$ decreases their gradients and a negative value increases them. We also noted that for $L>1$ (corresponding to a longitudinal stretch) the cut-on frequencies move closer together and the dispersion curves are less steep, whilst for $L<1$ (corresponding to a longitudinal compression) the cut-on frequencies move further apart and the dispersion curves are steeper.

In Chapter 4 we investigated the effect of a longitudinal stretch and pressure, applied both to the inner surface of a cylindrical cavity and at infinity, on the propagation and scattering of horizontally polarised shear waves in an unbounded medium. It was shown that, for certain parameter values, the scattering coefficients obtained in a pre-stressed medium are closer to those that would be obtained in the undeformed configuration than those that would be obtained in the deformed configuration if the pre-stress were neglected. This result was utilised in Chapter 5 where the cloaking of a cylindrical cavity from horizontally polarised shear waves was examined. It was shown that, for cloaks with $K R_{1}=\pi / 10$ and $K r_{1}=2 \pi$, neo-Hookean materials are optimal. A stonger dependence of the strain energy function on the second strain invariant (for a Mooney-Rivlin material) led to a less efficient cloak.

We observed that, for cloaks with $K R_{1}=\pi / 10$ and $K r_{1}=2 \pi$, as $S_{1}$ tends from 1 towards 0 (in other words, as the material becomes less dependent on the first strain invariant, and more dependent on the second strain invariant), there is more scattering from the cloaking region. Hence an ideal cloak will be dependent on the first strain invariant only. This explains why a neo-Hookean material is a good material for cloaking purposes in this context.

For materials which are strongly dependent on the second strain invariant the
scattering cross-section for larger values of $K r_{1}$ is greater than the scattering crosssection for an unstressed material; hence these materials are unsuitable for this type of pre-stress cloaking.

Finally, in Chapter 6 we studied the effect of pressure, applied both to the inner surface of a spherical cavity and at infinity, on the propagation and scattering of shear waves in an unbounded medium. It was shown that the scattering coefficients and cross-sections for this problem are strongly dependent on the pre-stress considered. We observed that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, the scattering cross-section $\gamma_{b}$ increases. This corresponds to the pre-stress having the effect of increasing the scattering of this type of shear wave for $\left(p_{\infty}-p_{a}\right) / \mu>0$ and decreasing the scattering for $\left(p_{\infty}-p_{a}\right) / \mu<$ 0 . This is initially counterintuitive as the scattering of this type of shear wave is increasing despite the fact that the size of the cavity is decreasing, but indicates that the inhomogeneous deformation around the cavity actually enhances the scattering in this case. The dependence of $\gamma_{a}$ on $\left(p_{\infty}-p_{a}\right) / \mu$ is non-monotonic, and takes a different minimum value for each value of $k A$ that has been analysed.

We noted that the scattering cross-section for the part of the solution corresponding to the dilatational waves is undefined as they have infinite wavelength. However, the effect of the pre-stress on the magnitude of the corresponding displacement, $\beta$ (see (6.138)), is shown to be that as $\left(p_{\infty}-p_{a}\right) / \mu$ increases, $\beta$ decreases, and vice versa. This appears to indicate that a greater pressure at infinity than on $r=a$ leads to enhanced shear wave scattering and diminished dilatational wave scattering, whereas a greater pressure on $r=a$ than at infinity leads to the opposite effect.

### 7.2 Implications for materials modelling and applications

The canonical problems considered in this thesis show that it is important to consider the underlying stress and not just geometrical effects when determining how waves propagate and are scattered by single cavities in pre-stressed materials. Therefore,
the pre-stress must be taken into account in any homogenisation scheme which aims to deal with scattering from multiple cavities in a pre-stressed material. If one were to only consider geometrical changes then inaccurate results would be obtained. This finding has implications in various areas of application. In particular, in the nondestructive evaluation of engineering materials, or in the detection of tumours, we have shown that the presence of pre-stress will greatly affect the far field response of a material. Hence, predictions of crack sizes in engineering materials, or tumour sizes in biological imaging could potentially be extremely inaccurate.

In two papers by Varadan et al. [105], [104], a theory for the multiple scattering of elastic waves by cylinders of arbitrary cross-section is formulated in terms of the scattering coefficients of each individual scatterer. The results in this thesis could potentially be used to modify this theory to take account of pre-stress. Hence, by building upon the work in this thesis, it may be possible to construct a general theory of multiple scattering in arbitrarily pre-stressed materials.

## Appendix A

## WKB approximation

The Wentzel-Kramers-Brillouin (or WKB) approximation is a method for finding approximate solutions to linear ordinary differential equations with spatially varying coefficients. The method is typically used for approximating the solution of a differential equation whose highest derivative is multiplied by a small parameter, $\epsilon$ say, and is particularly useful in high frequency wave problems. The method of approximation is as follows.

Given an equation of the form:

$$
\begin{equation*}
\epsilon \frac{d^{n} y}{d x^{n}}+a(x) \frac{d^{n-1} y}{d x^{x-1}}+\ldots+k(x) \frac{d y}{d x}+m(x) y=0 \tag{A.1}
\end{equation*}
$$

we use an ansatz of the form

$$
\begin{equation*}
y(x) \sim \exp \left(\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^{n} S_{n}(x)\right) \tag{A.2}
\end{equation*}
$$

in the limit $\delta \rightarrow 0$. Substitution of the above ansatz into the differential equation, cancelling out the exponential terms and equating orders of the coefficients allows one to solve for an arbitrary number of terms $S_{n}(x)$ in the expansion. It also permits a choice of $\delta=\epsilon^{m}$ for some $m \in \mathbb{R}$.

In Chapter 3, we are interested in solving an equation of the form

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}+q(x) y(x)=0 \tag{A.3}
\end{equation*}
$$

where $q(x)$ is a real, continously differentiable function.

If we make the substitution

$$
\begin{equation*}
X=\epsilon x \tag{A.4}
\end{equation*}
$$

then the above equations becomes

$$
\begin{equation*}
\epsilon^{2} \frac{d^{2} Y}{d X^{2}}+Q(X) Y(X)=0 \tag{A.5}
\end{equation*}
$$

where

$$
\begin{equation*}
Y(X)=y\left(\frac{X}{\epsilon}\right) \tag{A.6}
\end{equation*}
$$

and

$$
\begin{equation*}
Q(X)=q\left(\frac{X}{\epsilon}\right) \tag{A.7}
\end{equation*}
$$

so we can use the above WKB approximation, and we expect it to improve as $\epsilon \rightarrow 0$ when $Q(X)$ is continously differentiable. Upon doing so, we obtain

$$
\begin{equation*}
\epsilon^{2}\left(\frac{1}{\delta^{2}}\left(\sum_{n=0}^{\infty} \delta^{n} S_{n}^{\prime}\right)^{2}+\frac{1}{\delta} \sum_{n=0}^{\infty} \delta^{n} S_{n}^{\prime \prime}\right)=-Q(X) \tag{A.8}
\end{equation*}
$$

To leading order the above can be approximated as

$$
\begin{equation*}
\frac{\epsilon^{2}}{\delta^{2}} S_{0}^{\prime 2}+\frac{2 \epsilon^{2}}{\delta} S_{0}^{\prime} S_{1}^{\prime}+\frac{\epsilon^{2}}{\delta} S_{0}^{\prime \prime}=-Q(X) \tag{A.9}
\end{equation*}
$$

In the limit $\delta \rightarrow 0$, the dominant balance is given by

$$
\begin{equation*}
\frac{\epsilon^{2}}{\delta^{2}} S_{0}^{\prime 2} \sim-Q(X) \tag{A.10}
\end{equation*}
$$

So $\delta$ is proportional to $\epsilon$. Setting them equal and comparing powers renders, at leading order,

$$
\begin{equation*}
S_{0}^{\prime 2}=-Q(X), \tag{A.11}
\end{equation*}
$$

and so,

$$
\begin{equation*}
S_{0}(X)= \pm i \int \sqrt{Q(X)} d X \tag{A.12}
\end{equation*}
$$

This is equivalent to $\pm i \xi$, where the definition of $\xi$ is given in (3.54).
At first order, we have

$$
\begin{equation*}
2 S_{0}^{\prime} S_{1}^{\prime}+S_{0}^{\prime \prime}=0 \tag{A.13}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
2( \pm i \sqrt{Q(X)}) S_{1}^{\prime} \pm \frac{i Q^{\prime}(X)}{2 \sqrt{Q(X)}}=0 \tag{A.14}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
S_{1}^{\prime}=-\frac{Q^{\prime}(X)}{4 Q(X)} \tag{A.15}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
S_{1}=-\frac{1}{4} \log (Q(X))=\log (Q(X))^{-\frac{1}{4}} \tag{A.16}
\end{equation*}
$$

Finally, substituting the derived expressions for $S_{0}, S_{1}$ and $\delta$ into (A.2), we obtain

$$
\begin{align*}
Y(X) & =\exp \left(\frac{1}{\epsilon} S_{0}(X)+S_{1}(X)+O(\epsilon)\right) \\
& =(Q(X))^{-\frac{1}{4}} \exp \left( \pm \frac{i}{\epsilon} \int \sqrt{Q(X)} d X+O(\epsilon)\right)  \tag{A.17}\\
& =(q(x))^{-\frac{1}{4}} \exp \left( \pm i \int \sqrt{q(x)} d x+O(\epsilon)\right)
\end{align*}
$$

By comparing this with (3.57), we note that the Liouville-Green solution presented in Chapter 3 can be seen as a first-order WKB approximation.

## Appendix B

## Elliptic integral representation of

$\xi(\varrho)$

Consider the following integral representation of $\xi(r)$ as defined in 3.4.4,

$$
\begin{equation*}
\xi(r)=\int^{r} \sqrt{q(\varrho)} d \varrho=\int^{r} \frac{\sqrt{4 \beta^{2}\left(\varrho^{6}+a_{2} \varrho^{4}+a_{1} \varrho^{2}+a_{0}\right)}}{2 \varrho\left(\varrho^{2}+\alpha\right)} d \varrho \tag{B.1}
\end{equation*}
$$

with $a_{2}=\left[4 \alpha\left(\beta^{2}+\delta\right)-3\right] /\left(4 \beta^{2}\right), a_{1}=2 \alpha(2 \alpha \delta-3) /\left(4 \beta^{2}\right), a_{0}=\alpha^{2} /\left(4 \beta^{2}\right)$ and noting that $\alpha, \beta^{2}, \delta \in \mathbb{R}$ with $\delta \geq 0$. If we make a change of integration variable via $z=\varrho^{2}$ (B.1) can be recast, after rationalizing the numerator, as

$$
\begin{equation*}
\int^{r} \sqrt{q(\rho)} d \rho=\frac{|\beta|}{2} \int^{\zeta} \frac{R(z)}{\sqrt{\Omega(z)}} d z \tag{B.2}
\end{equation*}
$$

with $\zeta=r^{2}$ and with $R(z)$ and $\sqrt{\Omega(z)}$ defined as

$$
\begin{equation*}
R(z)=\frac{\left(z+z_{1}\right)\left(z+z_{2}\right)\left(z+z_{3}\right)}{z(z+\alpha)}, \quad \sqrt{\Omega(z)}=\sqrt{\left(z+z_{1}\right)\left(z+z_{2}\right)\left(z+z_{3}\right)} . \tag{B.3}
\end{equation*}
$$

Here $-z_{1},-z_{2}$, and $-z_{3}$ are the roots of the cubic equation with real coefficients $\Omega(z)=z^{3}+a_{2} z^{2}+a_{1} z+a_{0}$. The integrand of (B.2) is a rational function of $z$ and $\sqrt{\Omega(z)}$. For repeated roots of $\Omega(z)$, the integral can be expressed in terms of elementary functions. For non-repeated roots, the integral can be expressed in terms of three incomplete elliptic integrals as follows

$$
\begin{align*}
& F(\phi \mid m)=\int_{0}^{\phi} \frac{d \theta}{\left(1-m \sin ^{2} \theta\right)^{1 / 2}}, \quad E(\phi \mid m)=\int_{0}^{\phi}\left(1-m \sin ^{2} \theta\right)^{1 / 2} d \theta \\
& \Pi(n ; \phi \mid m)=\int_{0}^{\phi} \frac{d \theta}{\left(1-n \sin ^{2} \theta\right)\left(1-m \sin ^{2} \theta\right)^{1 / 2}} \tag{B.4}
\end{align*}
$$

These Legendre integrals are known, respectively, as elliptic integrals of the first, second and third kinds. We shall use here elliptic integrals in the most general form, with $\phi$ defined as the real or complex amplitude, $m$ the real or complex modulus and $n$ the real or complex parameter. Additionally, we assume $1-\sin ^{2} \phi \in \mathbb{C} \backslash(-\infty, 0]$ and $1-m \sin ^{2} \phi \in \mathbb{C} \backslash(-\infty, 0]$, except that one of them may be 0 and $1-n \sin ^{2} \phi \in \mathbb{C} \backslash\{0\}$ (see [77]).

Using polynomial long division and partial fraction expansion, we may rewrite $R(z)$ as

$$
\begin{equation*}
R(z)=\mathcal{C}_{1}+z+\frac{\mathcal{C}_{2}}{z}+\frac{\mathcal{C}_{3}}{(z+\alpha)} \tag{B.5}
\end{equation*}
$$

with $\mathcal{C}_{1}=E_{1}-\alpha, \mathcal{C}_{2}=E_{3} / \alpha$, and $\mathcal{C}_{3}=\left(\alpha^{3}-E_{1} \alpha^{2}+E_{2} \alpha-E_{3}\right) / \alpha$ and the so called elementary symmetric polynomials $E_{1}=z_{1}+z_{2}+z_{3}, E_{2}=z_{1} z_{2}+z_{1} z_{3}+z_{2} z_{3}$, and $E_{3}=z_{1} z_{2} z_{3}$. Hence, $\xi(\zeta)$ can be expressed in four parts by

$$
\begin{array}{ll}
I_{1}(\zeta)=\frac{|\beta|}{2} \mathcal{C}_{1} \int^{\zeta} \frac{1}{\sqrt{\Omega(z)}} d z, & I_{2}(\zeta)=\frac{|\beta|}{2} \int^{\zeta} \frac{z}{\sqrt{\Omega(z)}} d z \\
I_{3}(\zeta)=\frac{|\beta|}{2} \mathcal{C}_{2} \int^{\zeta} \frac{1}{z \sqrt{\Omega(z)}} d z, & I_{4}(\zeta)=\frac{|\beta|}{2} \mathcal{C}_{3} \int^{\zeta} \frac{1}{(z+\alpha) \sqrt{\Omega(z)}} d z \tag{B.6}
\end{array}
$$

Notice that $\xi(\zeta)$ is defined here in terms of anti-derivatives, hence we will be using indefinite integrals, with no lower limits, as opposed to the elliptic integrals which are definite integrals. Given the definite integral with e.g. $0<\phi_{1}<\phi$

$$
\begin{equation*}
\int_{\phi_{1}}^{\phi}(\cdot) d \theta=\int_{0}^{\phi}(\cdot) d \theta-\int_{0}^{\phi_{1}}(\cdot) d \theta \tag{B.7}
\end{equation*}
$$

its anti-derivative will be defined here by neglecting the second integral on the right hand side of (B.7).

## B. 1 Integration of $I_{1}$

The cubic polynomial $\Omega(z)$ can be transformed into a bi-quadratic one using the substitution $z=t^{2}-z_{1}$. Traditionally $z_{1}$ is chosen as the real root, however this is not a restriction here, since we shall use elliptic integrals in their most general form (i.e. with complex arguments). Hence, defining $d=1 / \sqrt{z_{3}-z_{1}}$, and after two
additional transformations, $d^{2} t^{2}+1=u^{-2}$ and $u=\sin \theta, I_{1}$ becomes

$$
\begin{align*}
I_{1} & =\frac{|\beta|}{2} \mathcal{C}_{1} \int^{\tau} \frac{2 c d}{\sqrt{\left(1+c^{2} t^{2}\right)\left(1+d^{2} t^{2}\right)}} d t=-(2 d) \frac{|\beta|}{2} \mathcal{C}_{1} \int^{v} \frac{d u}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}} \\
& =-(2 d) \frac{|\beta|}{2} \mathcal{C}_{1} \int_{0}^{\phi} \frac{d \theta}{\sqrt{1-m \sin ^{2} \theta}}=-(2 d) \frac{|\beta|}{2} \mathcal{C}_{1} F(\phi \mid m), \tag{B.8}
\end{align*}
$$

where $I_{1}(\varrho)$ is expressed in terms of the real or complex modulus $m$ and the real or complex amplitude $\phi$ defined hereafter as

$$
\begin{equation*}
m=1-\frac{d^{2}}{c^{2}}, \quad \phi=\arcsin \left\{\frac{1}{\left[1+d^{2}\left(\varrho^{2}+z_{1}\right)\right]^{1 / 2}}\right\} \tag{B.9}
\end{equation*}
$$

with $c=1 / \sqrt{z_{2}-z_{1}}$ and $d=1 / \sqrt{z_{3}-z_{1}}$.

## B. 2 Integration of $I_{2}$

Using $z=t^{2}-z_{1}$ we can rewrite $I_{2}(\tau)=I_{21}(\tau)+I_{22}(\tau)$ as

$$
\begin{equation*}
I_{21}=\frac{|\beta|}{2} \int^{\tau} \frac{-2 c d z_{1}}{\sqrt{\left(1+c^{2} t^{2}\right)\left(1+d^{2} t^{2}\right)}} d t, \quad I_{22}=\frac{|\beta|}{2} \int^{\tau} \frac{2 c d t^{2}}{\sqrt{\left(1+c^{2} t^{2}\right)\left(1+d^{2} t^{2}\right)}} d t \tag{B.10}
\end{equation*}
$$

Hence $I_{21}$ can be easily derived, following B.1, as

$$
\begin{equation*}
I_{21}(\varrho)=\left(2 d z_{1}\right) \frac{|\beta|}{2} F(\phi \mid m) . \tag{B.11}
\end{equation*}
$$

The derivation of $I_{22}$ is considerably more lengthy. Using $d^{2} t^{2}+1=u^{-2}$ we may write

$$
\begin{equation*}
I_{22}=-\left(\frac{2}{d}\right) \frac{|\beta|}{2}\left\{\int^{v} \frac{d u}{u^{2} \sqrt{1-m u^{2}} \sqrt{1-u^{2}}}-\int^{v} \frac{d u}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}}\right\} \tag{B.12}
\end{equation*}
$$

Now using the following relations

$$
\begin{gather*}
\frac{d}{d u}\left(\frac{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}}{u}\right)=\frac{m u^{2}}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}}-\frac{1}{u^{2} \sqrt{1-m u^{2}} \sqrt{1-u^{2}}}  \tag{B.13}\\
\int^{v} \frac{m u^{2}}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}} d u=\int^{v} \frac{d u}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}}-\int^{v} \frac{\sqrt{1-m u^{2}}}{\sqrt{1-u^{2}}} d u  \tag{B.14}\\
\int^{v} \frac{\sqrt{1-m u^{2}}}{\sqrt{1-u^{2}}} d u=\int_{0}^{\phi} \sqrt{1-m \sin ^{2} \theta} d \theta=E(\phi \mid m) \tag{B.15}
\end{gather*}
$$

we may finally express $I_{2}(\varrho)$, after using relations (B.13)-(B.15) to obtain $I_{22}$, as

$$
\begin{equation*}
I_{2}=\frac{|\beta|}{2}\left\{\frac{2}{d} \frac{\sqrt{1-m \sin ^{2} \phi} \sqrt{1-\sin ^{2} \phi}}{\sin \phi}+\frac{2}{d} E(\phi \mid m)+2 d z_{1} F(\phi \mid m)\right\} . \tag{B.16}
\end{equation*}
$$

## B. 3 Integration of $I_{3}$ and $I_{4}$

Using $z=t^{2}-z_{1}$ and $d^{2} t^{2}+1=u^{-2}$ we may recast $I_{3}$ as

$$
\begin{align*}
I_{3} & =\frac{|\beta|}{2} \mathcal{C}_{2} \int^{\tau} \frac{2 c d}{\left(t^{2}-z_{1}\right) \sqrt{\left(1+c^{2} t^{2}\right)\left(1+d^{2} t^{2}\right)}} d t \\
& =\frac{2 d^{3}}{\left(z_{1} d^{2}+1\right)} \frac{|\beta|}{2} \mathcal{C}_{2} \int^{v} \frac{u^{2}}{\left(u^{2}-\nu^{2}\right)} \frac{1}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}} d u, \tag{B.17}
\end{align*}
$$

with $\nu^{2}=1 /\left(z_{1} d^{2}+1\right)$. Using $u^{2} /\left(u^{2}-\nu^{2}\right)=1+\nu^{2} /\left(u^{2}-\nu^{2}\right)$ and then $u=\sin \theta$ we obtain

$$
\begin{align*}
I_{3} & =\frac{2 d^{3}}{\left(z_{1} d^{2}+1\right)} \frac{|\beta|}{2} \mathcal{C}_{2}\left\{\int^{v} \frac{d u}{\sqrt{1-m u^{2}} \sqrt{1-u^{2}}}+\int^{v} \frac{\nu^{2} d u}{\left(u^{2}-\nu^{2}\right) \sqrt{1-m u^{2}} \sqrt{1-u^{2}}}\right\} \\
& =\frac{2 d^{3}}{\left(z_{1} d^{2}+1\right)} \frac{|\beta|}{2} \mathcal{C}_{2}\left\{\int_{0}^{\phi} \frac{d \theta}{\sqrt{1-m \sin ^{2} \theta}}-\int_{0}^{\phi} \frac{d \theta}{\left(1-n_{1} \sin ^{2} \theta\right) \sqrt{1-m \sin ^{2} \theta}}\right\} \\
& =\frac{2 d^{3}}{\left(z_{1} d^{2}+1\right)} \frac{|\beta|}{2} \mathcal{C}_{2}\left\{F(\phi \mid m)-\Pi\left(n_{1} ; m \mid \phi\right)\right\} \tag{B.18}
\end{align*}
$$

with $n_{1}=\left(z_{1} d^{2}+1\right)$. The derivation of $I_{4}$ is obtained following the same steps as in $I_{3}$, hence

$$
\begin{equation*}
I_{4}=\frac{2 d^{3}}{\left(\left(z_{1}-\alpha\right) d^{2}+1\right)} \frac{|\beta|}{2} \mathcal{C}_{3}\left\{F(\phi \mid m)-\Pi\left(n_{2} ; m \mid \phi\right)\right\} \tag{B.19}
\end{equation*}
$$

with $n_{2}=\left(\left(z_{1}-\alpha\right) d^{2}+1\right)$.

Adding $I_{1}, I_{2}, I_{3}$, and $I_{4}$ we can finally obtain an integral representation for the independent variable $\xi$ in the Liouville-Green transformation

$$
\begin{gather*}
\xi(\varrho)=\frac{|\beta|}{2}\left\{\frac{2}{d} \frac{\sqrt{1-m \sin ^{2} \phi} \sqrt{1-\sin ^{2} \phi}}{\sin \phi}+\frac{2}{d} E(\phi \mid m)+\mathcal{D}_{1} F(\phi \mid m)+\right. \\
\left.\mathcal{D}_{2} \Pi\left(n_{1} ; \phi \mid m\right)+\mathcal{D}_{3} \Pi\left(n_{2} ; \phi \mid m\right)\right\}, \tag{B.20}
\end{gather*}
$$

with constants $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ defined as

$$
\begin{equation*}
\mathcal{D}_{1}=2 d\left(z_{1}-\mathcal{C}_{1}\right)+\mathcal{C}_{2} \frac{2 d^{3}}{n_{1}}+\mathcal{C}_{3} \frac{2 b^{3}}{n_{2}}, \quad \mathcal{D}_{2}=-\mathcal{C}_{2} \frac{2 d^{3}}{n_{1}}, \quad \mathcal{D}_{3}=-\mathcal{C}_{3} \frac{2 d^{3}}{n_{2}} \tag{B.21}
\end{equation*}
$$

## Appendix C

## Asymptotic behaviour of Bessel functions with small argument

Upon solving the linear elastic wave equation in spherical coordinates, the radial component of the dilatational part of the solution can be expressed as

$$
\begin{equation*}
f_{l}(r)=A_{l} \frac{d}{d r}\left(j_{l}(k r)\right)+B_{l} \frac{d}{d r}\left(y_{l}(k r)\right), \tag{C.1}
\end{equation*}
$$

where $A_{l}$ and $B_{l}$ are constants, $j_{l}(k r)$ and $y_{l}(k r)$ are the spherical Bessel functions, of order $l$, of the first and second kind, respectively and

$$
\begin{equation*}
k^{2}=\frac{\rho \omega^{2}}{\lambda+2 \mu}, \tag{C.2}
\end{equation*}
$$

where $\rho$ is the density of the material under consideration, $\omega$ is the frequency of the wave, and $\lambda$ and $\mu$ are the Lamé coefficients. As we approach the incompressible limit for an elastic material, $\lambda \rightarrow \infty$ and, therefore, $k \rightarrow 0$. Note that in this limit, the wavelength, $l \sim 1 / k \rightarrow \infty$.

The spherical Bessel functions can be related to the Bessel functions via the following expressions:

$$
\begin{align*}
j_{n}(z) & =\sqrt{\frac{\pi}{2 z}} \mathrm{~J}_{n+\frac{1}{2}}(z)  \tag{C.3}\\
y_{n}(z) & =\sqrt{\frac{\pi}{2 z}} \mathrm{Y}_{n+\frac{1}{2}}(z) \tag{C.4}
\end{align*}
$$

where $\mathrm{J}_{n+\frac{1}{2}}(z)$ and $\mathrm{Y}_{n+\frac{1}{2}}(z)$ are the Bessel functions, of order $n+\frac{1}{2}$, of the first and second kind, respectively.

When the argument, $z$, of the Bessel functions tends to 0 (as is the case as we approach incompressibility), we have the following results from [3]:

$$
\begin{gather*}
\mathrm{J}_{\nu}(z) \sim\left(\frac{z}{2}\right)^{\nu} \Gamma(\nu+1), \quad(\nu \neq-1,-2,-3, \ldots),  \tag{C.5}\\
\mathrm{Y}_{\nu}(z) \sim-\frac{1}{\pi} \Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad(\Re(\nu)>0) \tag{C.6}
\end{gather*}
$$

where $\Gamma$ is the gamma function.
Therefore, in the same limit, the spherical Bessel functions take on the following form:

$$
\begin{align*}
j_{n}(z) & \sim \alpha_{n} z^{n}, \quad\left(n \neq-\frac{3}{2},-\frac{5}{2},-\frac{7}{2}, \ldots\right),  \tag{C.7}\\
y_{n}(z) & \sim \beta_{n} z^{-(n+1)}, \quad\left(\Re(n)>-\frac{1}{2}\right), \tag{C.8}
\end{align*}
$$

where

$$
\begin{equation*}
\alpha_{n}=\frac{\sqrt{\pi}}{2^{n} \Gamma(n+3 / 2)}, \tag{C.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta_{n}=-\frac{2^{n} \Gamma(n+1 / 2)}{\sqrt{\pi}} . \tag{C.10}
\end{equation*}
$$

Hence, in the incompressible limit, equation (C.1) can be expressed as

$$
\begin{equation*}
f_{l}(r)=\hat{A}_{l} r^{l-1}+\hat{B}_{l} r^{-(l+2)} \tag{C.11}
\end{equation*}
$$

where

$$
\begin{gather*}
\hat{A}_{l}=l A_{l} \alpha_{l} k^{l},  \tag{C.12}\\
\hat{B}_{l}=-(l+1) B_{l} \beta_{l} k^{-(l+1)}, \tag{C.13}
\end{gather*}
$$

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