Heat Transfer

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Thomas Rodgers

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Chapter 1

Introduction to Heat Transfer

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1.1 Introduction

Heat transfer is one of the most useful things you will ever learn as a process engineer, for the following reasons:

- Many chemical reactions are carried out at high temperatures; attaining and maintaining these temperatures for optimum operation requires a knowledge of heat transfer.
- Energy comprises a major economic cost in the processing industries and in domestic situations, with energy often lost as heat (the lowest form of energy); energy conservation and recovery requires an understanding of heat transfer principles.
- Heat transfer is a good example of transport phenomena (of which the other two are mass transfer and momentum transfer), the basis of chemical engineering; a good understanding of heat transfer eases the understanding of these other transfer processes and of rate processes generally.

Heat transfer is also a relatively easy subject to understand, conceptually, and one that is very familiar – in fact, the one subject in chemical engineering that we probably allude to every day. However, the downside of a subject that is conceptually easy to understand is that the theory for it is therefore well developed mathematically. So to become expert at it, you need to become skillful at difficult maths.

Heat transfer is about transfer of energy, and you probably already know the following facts:

- Unit of energy is the Joule.
- Energy is conserved (First law of Thermodynamics).
- Heat can only flow from a hotter material to a colder material (Second law of Thermodynamics).

This knowledge will actually form the basis for this course. Firstly we will consider Energy balances briefly, as energy balances are the foundation of heat transfer. Then we will consider Heat Transfer, i.e. the mechanisms by which heat is transferred from a hotter to a colder body, and how to calculate the rate at which this happens. Then we will finish the course by considering the Applications of Heat Transfer theory to some specific examples of industrial relevance, including heat losses from pipes, insulation, and heating up batch vessels.

1.2 Forms of energy

Forms of energy include:

Kinetic energy – energy arising from motion. This is important if the system is rapidly moving, such as a bullet. Most processes are fairly stationary, so the kinetic energies involved are negligible and can be ignored in the energy balance. But this might not be true if, for example, a stream enters or leaves the system with high velocity, such as a jet from a nozzle.

Potential energy – energy arising from being moved against gravity. Most processes occur at or near the earthâĂŹs surface, so potential energy is not a major consideration in an energy balance. However, when liquids are pumped to reasonable heights above the ground, the energy requirement to pump them may be substantial, and certainly is the energy that you would need to consider in sizing the pump motor.

Internal energy – the energy of molecular motion (translation, vibration and rotation) and of intermolecular attraction and repulsion This is related to enthalpy, which we will talk about later, and is usually obtained from tables.

Heat and Work – In many ways these are the forms of energy most familiar to us. In an important sense, heat and work are different from the other forms of energy described above, in that they are energy in transit, i.e. energy being transferred from one body to another. Possibly this is why they seem familiar to us. Looking at a brick, it is not evident that it contains internal energy, but if you drop it on your foot, the work it does on your foot is felt quite evidently.

1.2.1 Heat

Heat is the most familiar form of energy. We know that a stove feels hot and ice feels cold. To describe this familiar phenomenon more carefully, what happens is that when we touch a stove, heat flows from the stove to our hand, and it therefore feels hot (relative to our hand). When we touch an ice cube, heat flows from our hand to the ice cube, and it feels cold (relative to our hand). From these familiar notions we can formulate two important ideas:

- Heat is a form of energy which flows, or as we often say, is transferred from one object to another or between a system and its surroundings. Because of this heat flow, one object loses some energy, and the other object gains this energy. When we hold an ice cube, heat flows from our hand to the ice cube. So our hand loses some of its energy content, as shown by the decrease in its temperature. Conversely, the energy content of the ice cube increases, as shown by the fact that the ice cube melts. So heat is a form of energy in transit, a form which flows or is transferred as a result of a temperature difference.
- Secondly, in order for there to be a flow of heat, there must be a temperature difference or gradient (heat, like water, will only flow “downhill”).

From these two ideas, we can define heat as “the form of energy which flows from one object or system to another as the result of a temperature difference”. And it is one of the laws of thermodynamics, and something that we know from our everyday experience, that the direction of the flow is from hotter bodies to colder bodies.

1.2.2 Enthalpy

We noted earlier the concept of Internal Energy, that is, the energy that a material possesses as a result of the motion and attractions and repulsions of its molecules. This energy depends on the composition of the material and its state, which is determined by the temperature and pressure. Related to the internal energy is the Enthalpy, h. The properties of enthalpy are as follows:

- For a given material, at constant pressure, the enthalpy depends only on the material's temperature and physical state (i.e. liquid, solid, vapour) So, for example, water at 100 °C has less energy and less enthalpy than steam at 100 °C.
- At constant temperature and physical state, the change of enthalpy with pressure is zero for ideal gases and small for liquids and solids.

This means that, for liquids, if you know the enthalpy at a given temperature and the corresponding vapour pressure then this is close enough for other pressures. So tables often give enthalpy data at a particular temperature and the corresponding vapour pressure.

There are no absolute values for enthalpy. Instead, the enthalpy of a substance is given a value of zero at some arbitrary datum point, and all other enthalpies are quoted relative to this reference point. For many substances the reference datum is set at 25 °C and 1 atm pressure, with the substance in its physical state normal to those conditions. But other datum points can be taken – for example, for water, the datum point at which the enthalpy of liquid water is zero is often taken to be the triple point (the point at which solid, liquid and gaseous water can coexist) which occurs at a conveniently close to zero temperature of 0.01 °C, and its equilibrium vapour pressure of 611.2 Pa.

1.3 Specific heat capacity

Which weighs more, a kg of water or a kg of air?

Okay, then which will require more energy to heat it up?

When you heat a material up, its enthalpy increases as the temperature increases. How much energy (or enthalpy) does it take to raise the temperature of a material by, say, 1 °C? This depends on the material. For water, for example, it takes about 4180 J to raise the temperature of 1 kg by 1 °C, while for air, it takes only about 1005 J (less than a quarter) to achieve the same temperature rise.

The amount of energy required to heat up a kilogram of a material by 1 °C is called its specific heat capacity, C_p ("specific" refers to the fact that we are dealing with a kilogram – if we were talking about the energy to heat up a mole of material, we would use the molar heat capacity). The units of specific heat capacity are $\mathrm{J\,kg^{-1}\,K^{-1}}$ i.e. the amount of energy in Joules to raise the temperature of 1 kg by 1 K.

Specific heat capacity is clearly just the slope of the graph of enthalpy against temperature (provided the material does not change its phase).

$$\frac{\mathrm{d}\,h}{\mathrm{d}\,T} = C_p$$

Specific heat capacity depends slightly on the temperature. When calculating the energy required to achieve a particular temperature change, use the specific heat capacity at the midpoint temperature (but not if there is a phase change involved! – in this case, the energy requirements above and below the phase change must be calculated separately). Or, for a more accurate calculation, look up the enthalpy of the material at the two temperatures, and subtract one from the other.

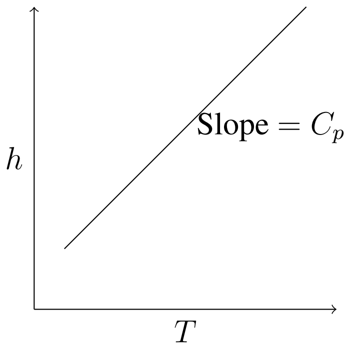


Figure 1.1: Specific heat capacity.

1.4 Rate of Energy Uptake

Specific heat capacity is the amount of energy required to raise the temperature of 1 kg of material by 1°C; it has units of $\mathrm{J\,kg^{-1}\,K^{-1}}$. Looking at the units, if we multiply specific heat capacity by a temperature change and a mass flow rate, we will therefore get units of $\mathrm{J\,s^{-1}}$, i.e. Watts, or rate of heat transfer. The symbol for rate of heat transfer is $\dot{Q}$ (the dot indicates a flow rate):

$$\begin{aligned}\dot{Q} &= \dot{m}C_p\left(T_{\text{out}} - T_{\text{in}}\right)\\ \mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}} &= \frac{\mathrm{kg}}{\mathrm{s}}\cdot\frac{\mathrm{J}}{\mathrm{kg\,K}}\cdot\mathrm{K}\end{aligned}$$

So if we know the heat transfer rate, mass flow rate and specific heat capacity, we can calculate the temperature change of a material from this equation.

The above equation is one of the most useful equations you'll learn in this course. There is no need to memorise it, as you can work it out logically just by considering the units of specific heat capacity. But you do need to understand it – and by understanding it, naturally you'll remember it.

For a batch operation the equation is much the same but without the dots, indicating the total quantity of energy required to change the temperature of a given mass:

$$\begin{aligned}Q &= mC_p\left(T_{\text{final}} - T_{\text{initial}}\right)\\ \mathrm{J} &= \mathrm{kg}\cdot\frac{\mathrm{J}}{\mathrm{kg\,K}}\cdot\mathrm{K}\end{aligned}$$

Alternatively, the rate of change of temperature can be related to the heat transfer rate:

$$\begin{aligned}\dot{Q} &= mC_p\frac{\mathrm{d}T}{\mathrm{d}t}\\ \mathrm{W} &= \mathrm{kg}\cdot\frac{\mathrm{J}}{\mathrm{kg\,K}}\cdot\frac{\mathrm{K}}{\mathrm{s}}\end{aligned}$$

1.5 Latent heat of vaporisation/condensation (h_{fg}) and latent heat of fusion/melting (h_{sf})

Water at atmospheric pressure boils when it reaches 100 °C, to form steam at 100 °C. Both are at the same temperature, but steam evidently contains more energy than water.

To put it another way, it takes energy to convert water into steam, and the question is, how much? The amount of energy required to convert water into steam is called the latent heat of vaporisation. Similarly, when steam gives up this energy and condenses into water, this is called the latent heat of condensation. The magnitude of the energy change is the same whether we are evaporating water or condensing steam. For water/steam at 100 °C and atmospheric pressure, the latent heat of vaporisation/condensation is 2257 kJ kg^{-1}. Comparing this with the enthalpy difference between water at 0 and 100 °C of 419.2 kJ kg^{-1}, we can see that it takes a lot more energy to convert water into steam than it does to heat water from 0 to 100 °C. So, for example, if we were drying a product, we would be much better, in energy terms, to remove as much water as possible mechanically (e.g. by squeezing or filtering) before evaporating off the rest. (Water can evaporate even below 100 °C, and still requires energy to do so. During drying, this energy comes from the air. At 25 °C, the latent heat of vaporisation is 2442 kJ kg^{-1}, slightly greater than at 100 °C.)

Similarly, it takes energy to melt ice, and water turning to ice gives up energy. This is called the latent energy of melting or fusion, and has a magnitude of 333.5 kJ kg^{-1} at 0 °C.

Figure 1.2 shows the change in enthalpy with temperature for water, as it changes from ice to liquid water to steam, showing that the largest change arises due to the latent heat of vaporisation.

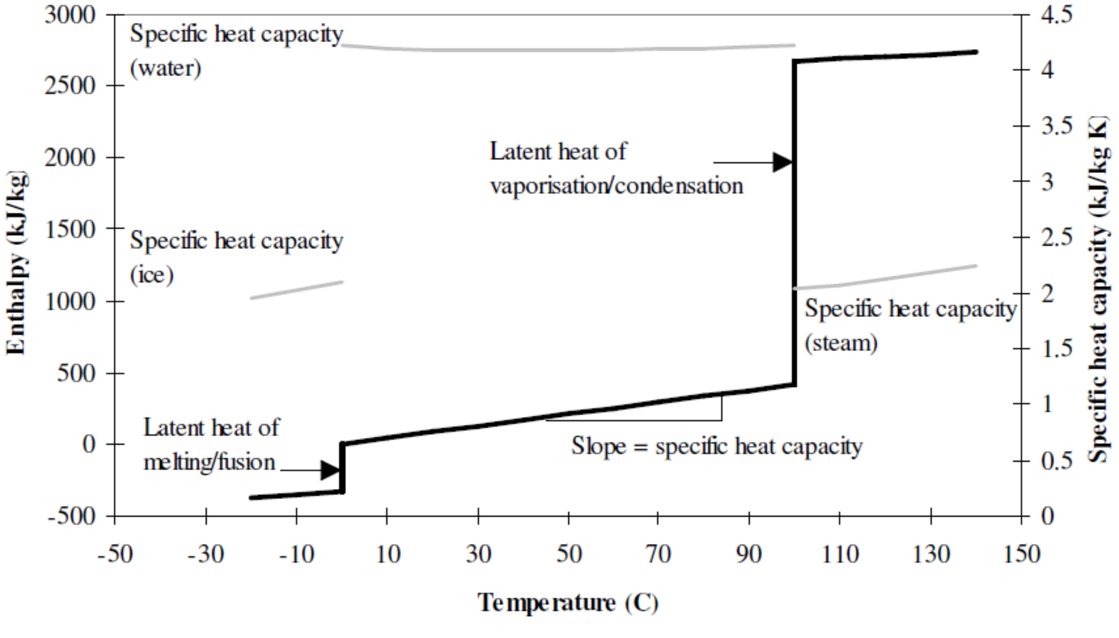


Figure 1.2: Variation with temperature of enthalpy and specific heat capacity of water and steam at atmospheric pressure.

1.6 Thermophysical Properties of Water, Steam, and Air

The most commonly encountered materials in the process industries are water, steam and air, and if you know how to read enthalpy tables for these, you will be able to do it for other substances. Also, it is important as engineers that you remember some of the important thermophysical properties of water, steam, and air, so that you can do rough calculations in your head or on an envelope. Table 1.1 shows typical values of important thermophysical properties for water, steam and air. Note that values are given for different temperatures, either 0, 25, or 100 °C. The shaded values you ought to remember, at least roughly – the others are there for reference. The density of air is not usually tabulated as it is easily calculated from the ideal gas law. Thermal conductivities and diffusivities are given for reference – we will introduce these later. Dynamic viscosities are also given, just for interest.

Table 1.1: Typical values of important thermophysical properties of water, ice, steam, and air.

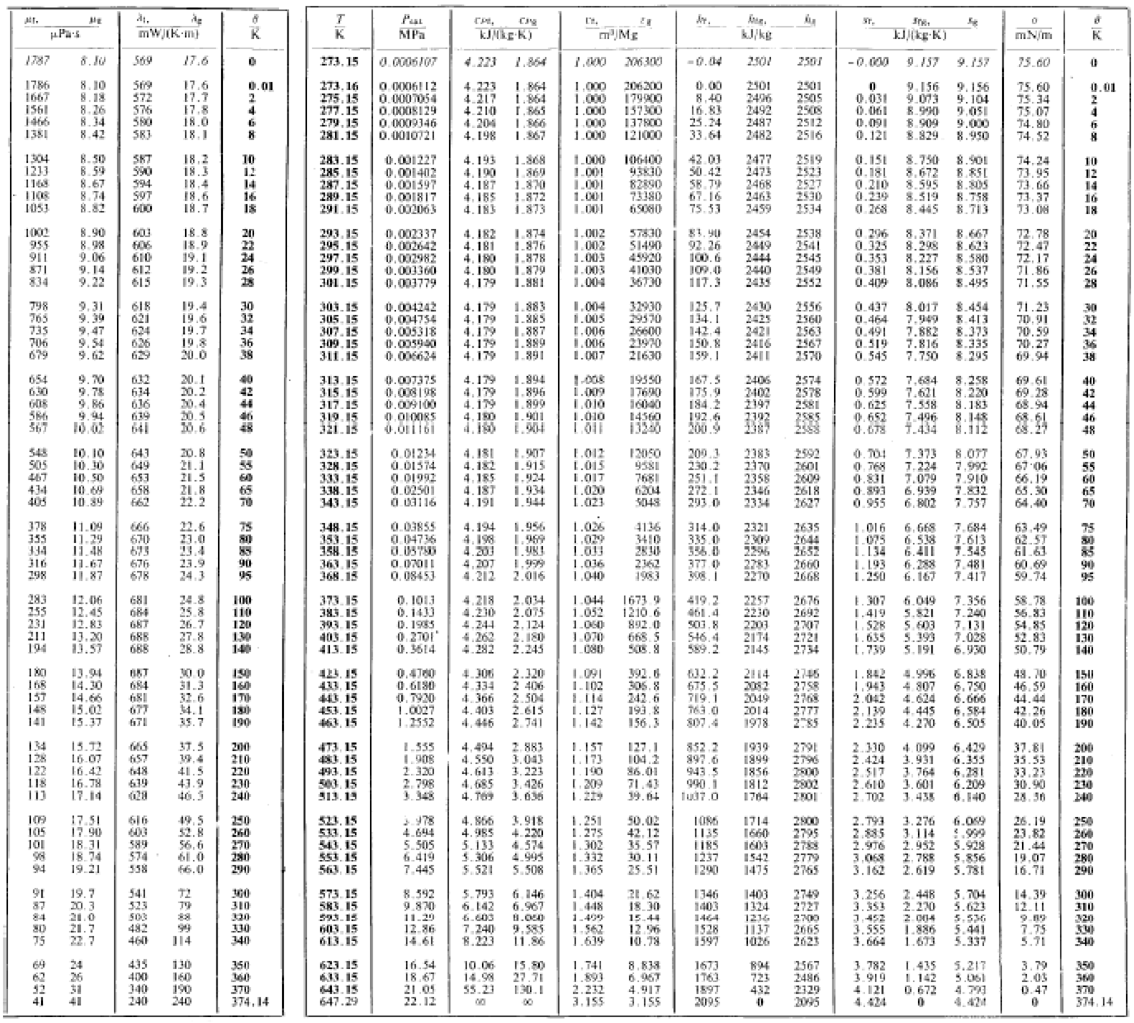
Property	Water	Ice	Steam	Air
Specific heat capacity, C_p (kJ kg^{-1} K^{-1})	4.180^{a}	2.101^{c}	2.034^{b}	1.005^{a}
Latent heat of vaporisation/-condensation, h_{lg} (kJ kg^{-1})	2257^{b}		2257^{b}	
Latent heat of fusion/melting, h_{sl} (kJ kg^{-1})	333.5^{c}	333.5^{c}		
Density, ρ (kg m^{3})	997^{a}– 958^{b}	917^{c}	0.6^{b}	$1.186^{a,*}$
Thermal conductivity, λ or k (W m^{-1} K^{-1})	0.611^{a}	2.240^{c}	0.0248^{b}	0.025^{a}
Viscosity, μ (Pa s)	891×10^{-6a}		12.06×10^{-6b}	18.3×10^{-6a}
Thermal diffusivity, α (m^{2} s^{-1})	0.147×10^{-6a}	1.17×10^{-6c}	20.3×10^{-6b}	21.0×10^{-6a}
Dynamic viscosity, v (m^{2} s^{-1}) (= momentum diffusivity)	0.894×10^{-6a}		20.1×10^{-6b}	15.4×10^{-6a}
Ratio $\frac{\text{momentum diffusivity}}{\text{thermal diffusivity}}$	6.1^{a}		0.99^{b}	0.73^{a}

^{a} 25 °C, ^{b} 100 °C, ^{c} 0 °C, * From ideal gas law.
Shaded values you ought to remember, at least approximately.

Figure 1.3 presents an abridged steam table, which list the important thermophysical properties of liquid water and steam at selected temperatures and the corresponding saturated vapour pressure.

Example: What is the energy requirement to raise the temperature of 60 kg of water from 10 °C to 80 °C? Calculate the answer in two ways – by using the specific heat capacity at the midpoint temperature (45 °C), and by subtracting the enthalpy of water at 10 °C from the enthalpy of water at 80 °C. Why are the two answers different? Which answer is more accurate?

Example: What is the energy requirement to convert 1 kg of water at 10 °C into steam at 150 °C? Calculate your answer by adding the energy required to raise the water from 10 °C to 100 °C (using the specific heat capacity at the midpoint temperature of 55 °C), adding the latent heat of vaporisation, the adding the energy to raise the steam from 100 °C to 150 °C (again using the appropriate mid-point temperature to look up the specific heat capacity). Then calculate the energy requirement by subtracting the enthalpy of water at 10 °C from the enthalpy of steam at 150 °C. How do the two answers compare?



μ_l	μ_g	λ_l	λ_g	θ
μPa·s		mW/(K·m)		K
1787	*8.10*	*569*	*17.6*	**0**
1786	8.10	569	17.6	**0.01**
1667	8.18	572	17.7	**2**
1561	8.26	576	17.8	**4**
1466	8.34	580	18.0	**6**
1381	8.42	583	18.1	**8**
1304	8.50	587	18.2	**10**
1233	8.59	590	18.3	**12**
1168	8.67	594	18.4	**14**
1108	8.74	597	18.6	**16**
1053	8.82	600	18.7	**18**
1002	8.90	603	18.8	**20**
955	8.98	606	18.9	**22**
911	9.06	610	19.1	**24**
871	9.14	612	19.2	**26**
834	9.22	615	19.3	**28**
798	9.31	618	19.4	**30**
765	9.39	621	19.6	**32**
735	9.47	624	19.7	**34**
706	9.54	626	19.8	**36**
679	9.62	629	20.0	**38**
654	9.70	632	20.1	**40**
630	9.78	634	20.2	**42**
608	9.86	636	20.4	**44**
586	9.94	639	20.5	**46**
567	10.02	641	20.6	**48**
548	10.10	643	20.8	**50**
505	10.30	649	21.1	**55**
467	10.50	653	21.5	**60**
434	10.69	658	21.8	**65**
405	10.89	662	22.2	**70**
378	11.09	666	22.6	**75**
355	11.29	670	23.0	**80**
334	11.48	673	23.4	**85**
316	11.67	676	23.9	**90**
298	11.87	678	24.3	**95**
283	12.06	681	24.8	**100**
255	12.45	684	25.8	**110**
231	12.83	687	26.7	**120**
211	13.20	688	27.8	**130**
194	13.57	688	28.8	**140**
180	13.94	687	30.0	**150**
168	14.30	684	31.3	**160**
157	14.66	681	32.6	**170**
148	15.02	677	34.1	**180**
141	15.37	671	35.7	**190**
134	15.72	665	37.5	**200**
128	16.07	657	39.4	**210**
122	16.42	648	41.5	**220**
118	16.78	639	43.9	**230**
113	17.14	628	46.3	**240**
109	17.51	616	49.5	**250**
105	17.90	603	52.8	**260**
101	18.31	589	56.6	**270**
98	18.74	574	61.0	**280**
94	19.21	558	66.0	**290**
91	19.7	541	72	**300**
87	20.3	523	79	**310**
84	21.0	503	88	**320**
80	21.7	482	99	**330**
75	22.7	460	114	**340**
69	24	435	130	**350**
62	26	400	160	**360**
52	31	340	190	**370**
41	41	240	240	374.14

T	P_{sat}	c_{pl}	c_{pg}	v_l	v_g	h_l	h_{lg}	h_g	s_l	s_{lg}	s_g	σ	θ
K	MPa	kJ/(kg·K)		m³/Mg		kJ/kg			kJ/(kg·K)			mN/m	K
273.15	*0.0006107*	*4.223*	*1.864*	*1.000*	*206300*	*−0.04*	*2501*	*2501*	*−0.000*	*9.157*	*9.157*	*75.60*	**0**
273.16	0.0006112	4.223	1.864	1.000	206200	0.00	2501	2501	0	9.156	9.156	75.60	**0.01**
275.15	0.0007054	4.217	1.864	1.000	179900	8.40	2496	2505	0.031	9.073	9.104	75.34	**2**
277.15	0.0008129	4.210	1.865	1.000	157300	16.83	2492	2508	0.061	8.990	9.051	75.07	**4**
279.15	0.0009346	4.206	1.866	1.000	137800	25.24	2487	2512	0.091	8.909	9.000	74.80	**6**
281.15	0.0010721	4.198	1.867	1.000	121000	33.64	2482	2516	0.121	8.829	8.950	74.52	**8**
283.15	0.001227	4.193	1.868	1.000	106400	42.03	2477	2519	0.151	8.750	8.901	74.24	**10**
285.15	0.001402	4.190	1.869	1.001	93830	50.42	2473	2523	0.181	8.672	8.851	73.95	**12**
287.15	0.001597	4.187	1.870	1.001	82890	58.79	2468	2527	0.210	8.595	8.805	73.66	**14**
289.15	0.001817	4.185	1.872	1.001	73380	67.16	2463	2530	0.239	8.519	8.758	73.37	**16**
291.15	0.002063	4.183	1.873	1.001	65080	75.53	2459	2534	0.268	8.445	8.713	73.08	**18**
293.15	0.002337	4.182	1.874	1.002	57830	83.90	2454	2538	0.296	8.371	8.667	72.78	**20**
295.15	0.002642	4.181	1.876	1.002	51490	92.26	2449	2541	0.325	8.298	8.623	72.47	**22**
297.15	0.002982	4.180	1.878	1.003	45920	100.6	2444	2545	0.353	8.227	8.580	72.17	**24**
299.15	0.003360	4.180	1.879	1.003	41030	109.0	2440	2549	0.381	8.156	8.537	71.86	**26**
301.15	0.003779	4.179	1.881	1.004	36730	117.3	2435	2552	0.409	8.086	8.495	71.55	**28**
303.15	0.004242	4.179	1.883	1.004	32930	125.7	2430	2556	0.437	8.017	8.454	71.23	**30**
305.15	0.004754	4.179	1.885	1.005	29570	134.1	2425	2560	0.464	7.949	8.413	70.91	**32**
307.15	0.005318	4.179	1.887	1.006	26600	142.4	2421	2563	0.491	7.882	8.373	70.59	**34**
309.15	0.005940	4.179	1.889	1.006	23970	150.8	2416	2567	0.519	7.816	8.335	70.27	**36**
311.15	0.006624	4.179	1.891	1.007	21630	159.1	2411	2570	0.545	7.750	8.295	69.94	**38**
313.15	0.007375	4.179	1.894	1.008	19550	167.5	2406	2574	0.572	7.684	8.258	69.61	**40**
315.15	0.008198	4.179	1.896	1.009	17690	175.9	2402	2578	0.599	7.621	8.220	69.28	**42**
317.15	0.009100	4.179	1.899	1.010	16040	184.2	2397	2581	0.625	7.558	8.183	68.94	**44**
319.15	0.010085	4.180	1.901	1.010	14560	192.6	2392	2585	0.652	7.496	8.148	68.61	**46**
321.15	0.011161	4.180	1.904	1.011	13240	200.9	2387	2588	0.678	7.434	8.112	68.27	**48**
323.15	0.01234	4.181	1.907	1.012	12050	209.3	2383	2592	0.704	7.373	8.077	67.93	**50**
328.15	0.01574	4.182	1.915	1.015	9581	230.2	2370	2601	0.768	7.224	7.992	67.06	**55**
333.15	0.01992	4.185	1.924	1.017	7681	251.1	2358	2609	0.831	7.079	7.910	66.19	**60**
338.15	0.02501	4.187	1.934	1.020	6204	272.1	2346	2618	0.893	6.939	7.832	65.30	**65**
343.15	0.03116	4.191	1.944	1.023	5048	293.0	2334	2627	0.955	6.802	7.757	64.40	**70**
348.15	0.03855	4.194	1.956	1.026	4136	314.0	2321	2635	1.016	6.668	7.684	63.49	**75**
353.15	0.04736	4.198	1.969	1.029	3410	335.0	2309	2644	1.075	6.538	7.613	62.57	**80**
358.15	0.05780	4.203	1.983	1.033	2830	356.0	2296	2652	1.134	6.411	7.545	61.63	**85**
363.15	0.07011	4.207	1.999	1.036	2362	377.0	2283	2660	1.193	6.288	7.481	60.69	**90**
368.15	0.08453	4.212	2.016	1.040	1983	398.1	2270	2668	1.250	6.167	7.417	59.74	**95**
373.15	0.1013	4.218	2.034	1.044	1673.9	419.2	2257	2676	1.307	6.049	7.356	58.78	**100**
383.15	0.1433	4.230	2.075	1.052	1210.6	461.4	2230	2692	1.419	5.821	7.240	56.83	**110**
393.15	0.1985	4.244	2.124	1.060	892.0	503.8	2203	2707	1.528	5.603	7.131	54.85	**120**
403.15	0.2701	4.262	2.180	1.070	668.5	546.4	2174	2721	1.635	5.393	7.028	52.83	**130**
413.15	0.3614	4.282	2.245	1.080	508.8	589.2	2145	2734	1.739	5.191	6.930	50.79	**140**
423.15	0.4760	4.306	2.320	1.091	392.6	632.2	2114	2746	1.842	4.996	6.838	48.70	**150**
433.15	0.6180	4.334	2.406	1.102	306.8	675.5	2082	2758	1.943	4.807	6.750	46.59	**160**
443.15	0.7920	4.366	2.504	1.114	242.6	719.1	2049	2768	2.042	4.624	6.666	44.44	**170**
453.15	1.0027	4.403	2.615	1.127	193.8	763.0	2014	2777	2.139	4.445	6.584	42.26	**180**
463.15	1.2552	4.446	2.741	1.142	156.3	807.4	1978	2785	2.235	4.270	6.505	40.05	**190**
473.15	1.555	4.494	2.883	1.157	127.1	852.2	1939	2791	2.330	4.099	6.429	37.81	**200**
483.15	1.908	4.550	3.043	1.173	104.2	897.6	1899	2796	2.424	3.931	6.355	35.53	**210**
493.15	2.320	4.613	3.223	1.190	86.01	943.5	1856	2800	2.517	3.764	6.281	33.23	**220**
503.15	2.798	4.685	3.426	1.209	71.43	990.1	1812	2802	2.610	3.601	6.209	30.90	**230**
513.15	3.348	4.769	3.636	1.229	39.64	1037.0	1764	2801	2.702	3.438	6.140	28.56	**240**
523.15	3.978	4.866	3.918	1.251	50.02	1086	1714	2800	2.793	3.276	6.069	26.19	**250**
533.15	4.694	4.985	4.220	1.275	42.12	1135	1660	2795	2.885	3.114	5.999	23.82	**260**
543.15	5.505	5.133	4.574	1.302	35.57	1185	1603	2788	2.976	2.952	5.928	21.44	**270**
553.15	6.419	5.306	4.995	1.332	30.11	1237	1542	2779	3.068	2.788	5.856	19.07	**280**
563.15	7.445	5.521	5.508	1.365	25.51	1290	1475	2765	3.162	2.619	5.781	16.71	**290**
573.15	8.592	5.793	6.146	1.404	21.62	1346	1403	2749	3.256	2.448	5.704	14.39	**300**
583.15	9.870	6.142	6.967	1.448	18.30	1403	1324	2727	3.353	2.270	5.623	12.11	**310**
593.15	11.29	6.603	8.080	1.499	15.44	1464	1236	2700	3.452	2.084	5.536	9.89	**320**
603.15	12.86	7.240	9.585	1.562	12.96	1528	1137	2665	3.555	1.886	5.441	7.75	**330**
613.15	14.61	8.223	11.86	1.639	10.78	1597	1026	2623	3.664	1.673	5.337	5.71	**340**
623.15	16.54	10.06	15.80	1.741	8.838	1673	894	2567	3.782	1.435	5.217	3.79	**350**
633.15	18.67	14.98	27.71	1.893	6.967	1763	723	2486	3.919	1.142	5.061	2.03	**360**
643.15	21.05	55.23	130.1	2.232	4.917	1897	432	2329	4.121	0.672	4.793	0.47	**370**
647.29	22.12	∞	∞	3.155	3.155	2095	0	2095	4.424	0	4.424	0	374.14

1 MPa = 10^6 N/m² = 10 bar

Figure 1.3: Abridged steam tables.

1.7 Conservation of energy and energy balances

Energy is expensive. An energy balance accounts for all the energy entering and leaving a system. If we overlook one of the forms of energy entering or leaving the system, we will not calculate the energy balance correctly.

1.7.1 Energy balances

Conservation of energy is one of the fundamental laws of the universe, along with conservation of mass, except where nuclear reactions are involved, in which case mass can be converted to energy via the equation

$$E = mc^2$$

Because c^2 is such a large term, a very small amount of mass is converted into vast amounts of energy – this is why nuclear energy is such an attractive prospect.

1.7.2 Statement of the Law of Conservation

But in most process industries, except the nuclear industry, we don't get nuclear reactions occurring, so we can take it as fundamentally true that energy, and mass, are conserved. This means that all the energy that enters a system must leave, one way or another, or else must accumulate in the system. We write this as follows:

$$\text{IN} - \text{OUT} + \text{GENERATION} - \text{DISSIPATION} = \text{ACCUMULATION}$$

This applies for heat transfer, mass transfer and momentum transfer. This is a fundamental statement for chemical engineers, and it applies to all situations, from overall plant balances to individual unit operations and to small elements within equipment. Applying this law to conservation of energy within process equipment results in sets of equations, either algebraic or differential, which describe the variation of temperature or heat flow within the equipment. If the process or operation involves no generation or dissipation of heat (e.g. no reactions producing or removing energy), then the above equation simplifies to

$$\text{IN} - \text{OUT} = \text{ACCUMULATION}$$

If the system is also operating at steady state i.e. nothing is changing, then there cannot be any accumulation within the system, therefore

$$\text{IN} - \text{OUT} = 0$$

or

$$\text{TOTAL ENERGY IN} = \text{TOTAL ENERGY OUT}$$

We must remember to include all forms of energy involved, and recognise that a particular form of energy is not necessarily conserved, as energy can be transformed, e.g. from mechanical work into heat.

It helps enormously when performing an energy balance (or a mass balance) to draw a dashed line around the system of interest, whether the system is an entire process, some section of it, a single unit operation, or a differential element within an item of process equipment. This helps to define clearly what energy flows are entering and leaving the system, and helps you to avoid overlooking any.

Mass and energy balance examples can be deceptively easy, but more difficult examples require a systematic approach. Five helpful steps to performing mass and energy balances are as follows:

1. Draw a picture, with all streams (mass and energy) entering and leaving the system. Draw a dashed line to indicate the boundaries of the system. Label the streams (with numbers if necessary).
2. Decide a basis on which to perform the calculations.
3. Draw up two balance tables, one for the mass balance, the other for the energy balance.
4. Perform preliminary calculations – fill in the balance tables.
5. Set x = the unknown quantity to be calculated. Solve for x.

1.7.3 Temperature-Enthalpy Diagrams

You will remember the enthalpy-temperature diagram for water, where we showed how enthalpy changed with temperature. We showed it this way, because it makes intuitive sense to consider, when we change the temperature of water, how its enthalpy would change. But it might actually be more sensible to look at it as, by adding energy to the water, we are changing its enthalpy, and are seeing how the temperature changes. In other words, it might make more sense to make temperature the dependent axis, and enthalpy the independent axis. This would make our temperature-enthalpy diagram for a pure component look like Figure 1.4.

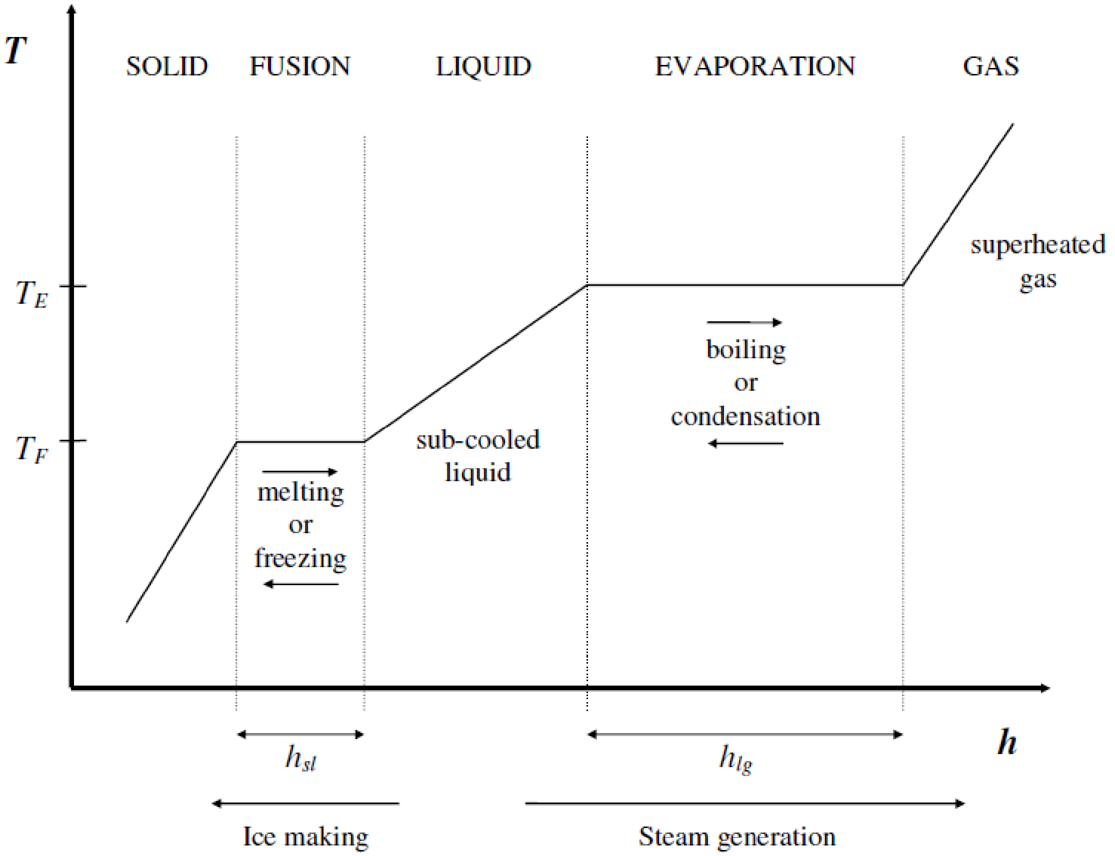


Figure 1.4: Temperature-enthalpy plot for a pure component at constant pressure.

T_E is the temperature of evaporation, or the boiling point. For a multicomponent stream, this would vary over the evaporation region from the bubble point to the dew point.

T_F is the temperature of fusion, or the freezing point. For a multicomponent stream, this would vary over the fusion region from the solidus point to the liquidus point.

h_{lg} is the enthalpy (or latent heat) of evaporation $= fn[T, P, \text{composition}]$ h_{sl} is the enthalpy (or latent heat) of fusion $= fn[T, P, \text{composition}]$ h_{sg} is the enthalpy (or latent heat) of sublimation $= fn[T, P, \text{composition}]$

Chapter 2

General Heat Transfer Equation

Contents

2.1 Introduction

We have talked a lot about energy balances, and that energy is conserved over a system, but may be transferred within the system. For example, in a heat exchanger the total energy entering and leaving the system is constant, but heat is transferred from the hot to the cold stream. And we established that the rate of change of enthalpy of either stream, H & D , is equal in magnitude to the heat transfer rate, $\dot{Q}$ (in $\mathrm{J\,s^{-1}}$ or Watts, W), through the heat exchanger walls.

So energy balances tell us how much heat is transferred, but they do not tell us how (i.e. by what mechanisms) this heat is transferred, or how we can design our heat exchanger (or whatever) to achieve this rate of transfer. To decide these questions, we need to move from a study of energy balances to Heat Transfer.

2.2 General Heat Transfer Equation

Let us start by considering what factors we might expect to affect heat transfer. If I have an external wall in my office, with a window, and I lose heat through the wall and the window, what factors might affect the rate at which I lose heat and therefore the size of the heater I need in my office?

- size of the wall – Area, A
- temperature difference between my office and the outside – $\Delta\,T$
- thickness of the wall – x
- what the wall is made of – its thermal conductivity – λ or k
- rate of transfer of heat from air inside to the wall, and from the wall to the outside
- Sunny, raining, windy
- Open or closed window
- etc.

In a general sense, the thing that is causing heat transfer to occur is the temperature difference, $\Delta\,T$

If $\Delta\,T$ is larger, the rate of heat transfer will be greater. The fact that there is a wall in the way means that there is a resistance to heat transferring. If we have a thicker wall, for example, the resistance to heat transfer will be greater, and heat will be transferred more slowly. So the rate of heat transfer could be described by an equation:

$$\dot{Q} = \frac{\Delta\,t}{R}$$

where R incorporates all the factors that contribute to the resistance to heat transfer, such as the area of the wall, its thickness, what it's made of, etc. The units of R are clearly $\mathrm{K\,W^{-1}}$, in other words, the amount of temperature driving force required to cause heat to be transferred at a rate of 1 W. N.B. This is only true for steady state conditions.

Alternatively, the rate of heat transfer could be described by:

$$\dot{Q} = UA\Delta\,T$$

where U represents some sort of overall heat transfer coefficient (units: $\mathrm{W\,m^{-1}\,K^{-1}}$), which would incorporate the thickness of the wall, what it's made of, rate of heat transfer to it from the air inside and from it to the air outside. Clearly, $R = 1/UA$.

This is the General Heat Transfer Equation, and ranks alongside $\dot{Q} = \dot{M}C_p\left(T_{\text{out}} - T_{\text{in}}\right)$ as one of the most important heat transfer equations you will learn – memorise it and digest it! The General Heat Transfer Equation can also be written as

$$\dot{q} = \frac{\dot{Q}}{A} = U\Delta T$$

where $\dot{q}$ is called the heat flux or heat transfer per unit area (units: $\mathrm{W\,m^{-2}}$). Often it is the heat flux that we are interested in.

2.3 Mechanisms of Heat Transfer

Let's start by identifying the major energy transfer mechanisms.

Heat transfer	Other energy transfer mechanisms
Conduction	Mechanical
Convection	Electrical
Radiation	Electromagnetic (e.g microwave)
(Phase change)	Chemical reaction
	Nuclear reaction

The difference between the heat transfer list and the other mechanisms identified above for energy transfer is that in the first list, the driving force for energy transfer is a temperature difference. For heat transfer, energy will only flow if there is a temperature difference. The other mechanisms can generate thermal energy within a material without the requirement for a temperature gradient.

To quote from Ozisik, page 1, "Since heat flow takes places whenever there is a temperature gradient in a system, a knowledge of the temperature distribution in a system is essential in heat transfer studies." We will therefore aim, as much as possible, to focus on what the temperature distribution is in the systems we are considering, i.e. the temperature profile.

2.3.1 Conduction

Conduction is the mechanism of heat transfer in which energy exchange takes place from a region of high temperature to one of lower temperature by the kinetic motion or direct impact of molecules and by the drift of electrons. The latter applies particularly to metals, which are both good electrical conductors and heat conductors. Essentially, conduction is heat transfer by molecular motion in solids or fluids at rest.

The empirical law of heat conduction, based on experimental observation, was proposed by Joseph Fourier, who stated that the rate of heat flow by conduction in a given direction is proportional to the area normal to the direction of heat flow and to the gradient of the temperature in that direction:

Fourier's law: $\dot{Q}_x = -\lambda A \dfrac{\mathrm{d}\,T}{\mathrm{d}\,x}$

In terms of heat flux: $\dot{q}_x = -\lambda \dfrac{\mathrm{d}\,T}{\mathrm{d}\,x}$

$\dot{Q}_x$ is the rate of heat flow through area A in the positive x-direction. Heat will only flow in the positive x-direction if the temperature is decreasing in that direction – hence the negative sign, as $\mathrm{d}\,T/\mathrm{d}\,x$ must be negative for $\dot{Q}_x$ to be positive – i.e. heat flows "downhill". The proportionality constant, λ (or k as is often used) is called the thermal conductivity of the material. Good conductors have high values of λ, good insulators low values. λ has SI units $\mathrm{W\,m^{-1}\,K^{-1}}$, and varies from around $0.1\,\mathrm{W\,m^{-1}\,K^{-1}}$ for gases and insulating materials to up to $1000\,\mathrm{W\,m^{-1}\,K^{-1}}$ for highly conducting metals such as copper or silver. If the temperature gradient through the material is uniform (as it would be in a slab of isotropic material) and at steady state, and the thermal conductivity does not change significantly with temperature, then Fourier's equation can be written in its steady state form:

$$\begin{aligned}\dot{Q}_x &= \frac{\lambda}{x} A \Delta\,T \\ &= U A \Delta\,T \\ &= \frac{\Delta\,T}{R}\end{aligned}$$

Therefore, $U = \frac{\lambda}{x}$ and $R = \frac{x}{\lambda A}$ for pure conduction.

Comparing with the General Heat Transfer Equation, we see that for pure conduction through a slab, U, the overall heat transfer coefficient, under steady state conditions is given by $\frac{\lambda}{x}$. If we think in terms of the resistance to heat transfer, then $R = \frac{x}{\lambda A}$. This makes sense: as the thickness of the wall increases, so must its resistance to heat transfer. But if the thermal conductivity is very large, then heat is transferred easily, and the resistance to heat transfer is small. Similarly, if the area is very large, then a lot of heat will be lost through it.

Figure 2.1 overleaf shows typical ranges of thermal conductivities for various materials. Note that it is a logarithmic scale, and that metals have thermal conductivities typically $1\,000 - 10\,000$ times greater than insulators and gases. Figure 2.2 shows the effect of temperature on thermal conductivities of some representative materials. Table 2.1 below gives thermal conductivities of various materials at 0 °C.

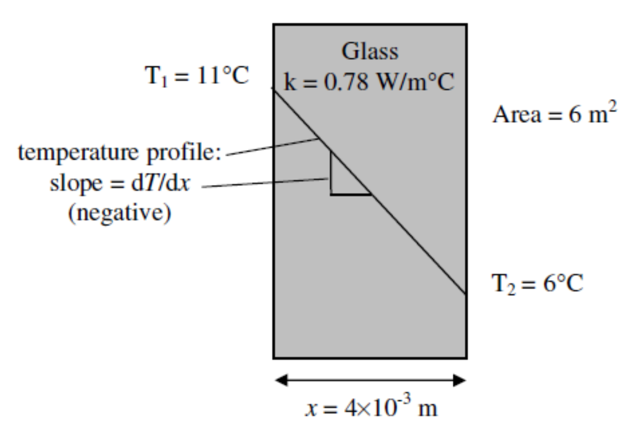
? Question

Example: Heat loss through a window.

The window in my office is 2 metres by 3 metres. Assuming all other walls are well insulated and that heat loss only occurs through the window, calculate what size of heater I need in my office to maintain the temperature, if the inside surface of the glass is at 11°C and the outside surface at 6°C. Sketch the temperature profile through the glass.

Data: Thickness of glass = 4 mm Thermal conductivity of glass = 0.78 W m^{-1} K^{-1}.

Draw a picture:



$$\begin{aligned}\dot{Q}_x &= \frac{\lambda}{x} A \Delta T \\ &= \frac{0.78 \times 6 \times (11-6)}{4 \times 10^{-3}} \\ &= 5850\,\mathrm{W}\end{aligned}$$

So I need nearly a 6 kW heater in my room to maintain it at a comfortable temperature (which is quite a lot).

2.3.2 Convection

Convection is the mechanism of heat transfer between a flowing fluid and a solid body (or between a gas and a liquid at rest). Imagine a fluid flowing past a solid wall, Figure 2.3.

Clearly there will be a temperature profile between the bulk fluid and the wall – in the bulk the fluid will be fairly uniformly at its bulk average temperature, T_f, and near the wall it will approach the wall temperature, T_w.

Imagine now a layer near the wall in which the temperature varies between T_w and T_f. If we were to consider heat conduction in this layer, we could describe it by

$$\dot{Q} = \frac{\lambda_l}{\delta} A \Delta T$$

where λ_l is the thermal conductivity of the liquid layer near the wall, and δ the thickness of our imaginary layer. The trouble is, we don't know the value of either λ_l or δ, and both will change depending on the fluid and the flow conditions. So as we don't know either,

Table 2.1: Thermal conductivities of various materials at 0 °C.

Material	Thermal Conductivity	
	$\mathrm{W\,m^{-1}\,K^{-1}}$	$\mathrm{Btu^{-1}hr^{-1}ft\,^{\circ}F^{-1}}$
Metals		
Silver (pure)	400	237
Copper (pure)	385	223
Aluminium (pure)	202	117
Nickel (pure)	93	54
Iron (pure)	73	42
Carbon steel, 1%	43	25
Lead (pure)	35	20.3
Stainless steel (15% Cr, 10% Ni)	19	11.3
Chrome-nickel steel (18% Cr, 8% Ni)	16.3	9.4
Non-metallic solids		
Quartz, parallel to axis	41.6	24
Magnesite	4.15	2.4
Marble	$2.08 - 2.94$	$1.2 - 1.7$
Ice	2.0	1.19
Sandstone	1.83	1.06
Mortar	1.16	0.69
Glass, window	0.78	0.45
Maple or oak	0.17	0.096
White pine	0.112	0.066
Corrugated cardboard	0.064	0.038
Sawdust	0.059	0.034
Glass wool	0.038	0.022
Liquids		
Mercury	8.21	4.74
Water	0.556	0.327
Ammonia	0.540	0.312
Lubricating oil, SAE50	0.147	0.085
Freon 12, $\mathrm{CCl_2F_2}$	0.073	0.042
Gases		
Hydrogen	0.175	0.101
Helium	0.141	0.081
Air	0.024	0.0139
Water vapour (saturated)	0.0206	0.0119
Carbon dioxide	0.0146	0.00844

we may as well replace them both by a single term, which we will call the convective heat transfer coefficient, h:

$$\dot{Q} = hA\Delta T$$

To be dimensionally correct, h must have SI units of $\mathrm{W\,m^{-2}\,K^{-1}}$. These are the same units as U; clearly, for pure convection, U is equal to h. The resistance to heat transfer by convection is given by $R = 1/hA$.

The convective heat transfer coefficient varies with the type of flow (turbulent or laminar), the geometry of the system, the physical properties of the fluid, the average temperature, the position along the surface of the body, and time. So calculating h is quite complicated.

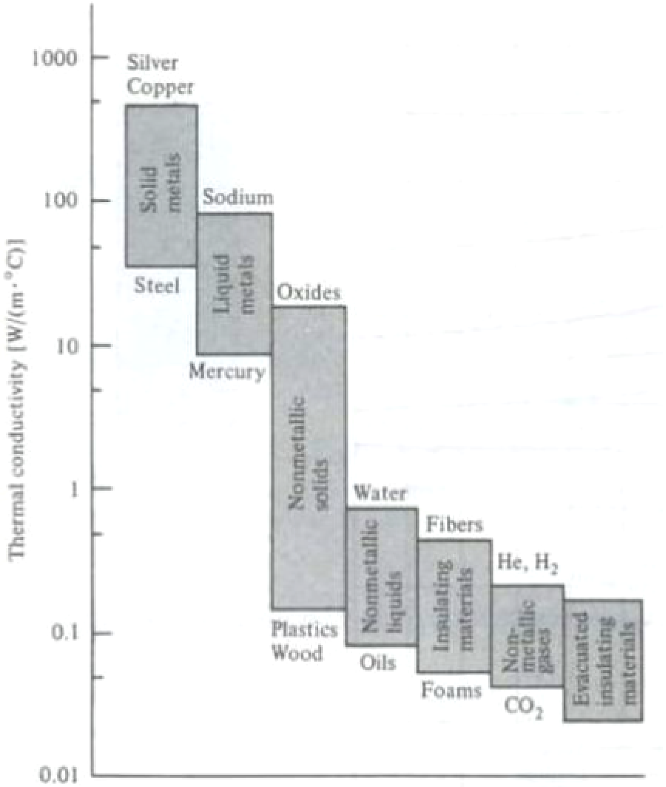


Figure 2.1: Typical range of thermal conductivity of various materials.

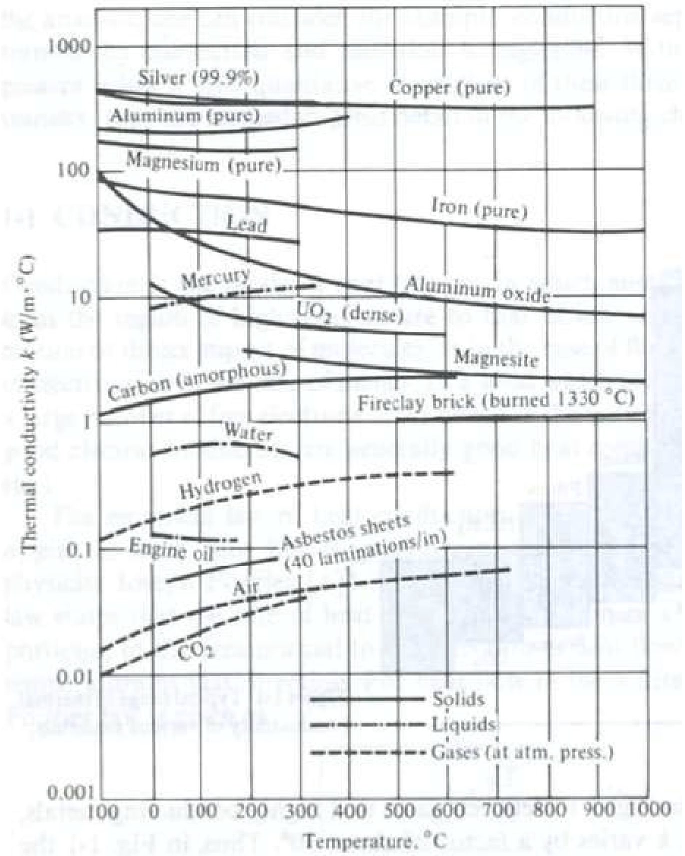


Figure 2.2: Effect of temperature on thermal conductivity of materials.

We will consider ways to calculate it later in the course.

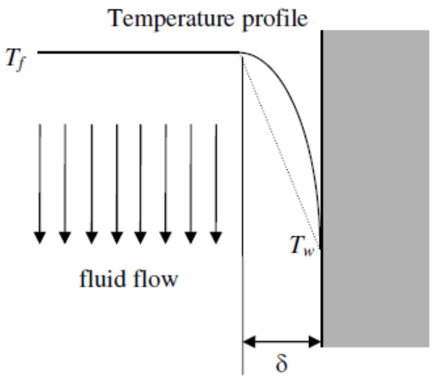


Figure 2.3: Heat transfer by convection between a hot fluid and a wall.

h also depends on whether the heat transfer is by forced convection or free convection.

Free (or natural) convection arises when the fluid motion is caused by buoyancy effects caused by density differences caused by temperature differences in the fluid. So a hot plate suspended in air causes the immediately adjacent air to heat up; this air rises, giving fluid motion relative to the solid plate.

Forced convection, by contrast, arises when the fluid motion is artificially induced, e.g. by a pump or a fan which forces the fluid to flow over the surface of the solid.

To summarise, the convective heat transfer coefficient, h, is affected by the fluid mechanics of the system, so is somewhat difficult to determine – once known, however, it is easy to use. Table 2.2 shows typical values of h. Note that these values differ by $4 - 5$ orders of magnitude. In practice, this means that if you have a wall with convective heat transfer on both sides, often the coefficient on one side will be the controlling one.

Note too that condensing steam uses a heat transfer coefficient in the same way and with the same units, although the mechanism of heat transfer is not quite the same (involving phase change rather than conduction through an imaginary thin layer of fluid near the wall).

The very high value of heat transfer coefficient for condensing steam is one reason that steam is a popular heat transfer fluid in the process industries. The two other major reasons are:

- condensing steam has a very high latent heat of condensation, so delivers a lot of heat

Table 2.2: Representative values of convective heat transfer coefficient.

Condition	h (W m^{-2} K^{-1})
Free convection, air	$6 - 35$
Forced convection, air	$28 - 851$
Free convection, water	$170 - 1140$
Forced convection, water	$570 - 22\,700$
Boiling water	$5\,700 - 85\,000$
Condensing steam	$57\,000 - 170\,000$
Forced convection, sodium	$113\,000 - 227\,000$

- the temperature at which the steam condenses can be easily controlled by changing the pressure

Additional, more minor, reasons include: steam is cheap, non-toxic, environmentally friendly

Example: Steam condenses on the inside of an insulated pipe 10 m in length and with outer diameter 5 cm. Heat is lost at a rate of 900 W to the external air at 15 °C with a convective heat transfer coefficient of $20\,\mathrm{W\,m^{-2}\,K^{-1}}$. Calculate the temperature of the outer surface of the pipe.

$$\begin{aligned} \dot{Q} &= hA\delta T \\ 900 &= 20 \times (\pi \times 0.05 \times 10) \times (T_s - 15) \\ T_s &= 43.6\,^{\circ}\mathrm{C} \end{aligned}$$

Answer: 43.6 °C

2.3.3 Radiation

All bodies emit electromagnetic energy as a result of their temperature; this energy is called thermal radiation. The internal energy of the body is converted into electromagnetic waves which travel through space. The emitted radiation depends on the temperature of the body, the wave length (or range of wave lengths) and the condition of the surface.

Similarly, radiation falling on a body may be absorbed, transmitted or reflected (or a combination of all three). Radiation incident on an absorbing body is attenuated as it passes through the body. If it is attenuated over a very short distance (a few angstroms), then the body is considered opaque to thermal radiation. Water and glass partially reflect, partially absorb and partially transmit, and are therefore considered semi-transparent. Only in a vacuum does thermal radiation propagate with no attenuation. However, air and most gases are transparent to thermal radiation for most practical purposes, although some gases such as CO_2, carbon monoxide, water vapour and ammonia can absorb significant thermal radiation over certain wavelength bands.

Bodies which absorb all radiation falling on them (without transmitting or reflecting any) are called black bodies. Black bodies are also perfect emitters or ideal radiators – they emit the maximum possible amount of thermal radiation. Bodies which reflect or transmit some of the radiation incident on them, and emit less than the maximum possible, are called grey bodies.

So, thermal radiation emitted from one body and absorbed by another represents a mechanism of heat transfer. The difference between radiation and the other two forms of heat transfer discussed above, conduction and convection, is that radiation does not require a medium through which to travel. Radiation can transfer heat in a vacuum. This is how heat energy from the sun can travel to Earth, despite there being nothing (substantial) in between.

The maximum emitted radiation from a black body is given by the Stefan-Boltzmann law:

$$E_b = \sigma T^4$$

where E_b is the emitted black body radiation (W m^{-2}), T is the absolute temperature in Kelvins (K), and σ is the Stefan-Boltzmann constant.

Clearly the amount of energy radiated by a body increases very rapidly with temperature, as this is raised to the fourth power.

Real (i.e. grey) bodies do not emit the maximum radiation given by the above equation, but some smaller amount given by

$$E = \varepsilon E_b = \varepsilon \sigma T^4$$

where ε is the emissivity of the surface.

The emissivity, ε, depends on the surface conditions. It is unity for a true black body, and between 0 and 1 for all real bodies.

When radiation falls onto a black body, it is completely absorbed. But for real (grey) bodies, the energy absorbed is less than all of the incident radiation, and is given by

$$\dot{q}_{\text{abs}} = \alpha \dot{q}_{\text{inc}}$$

where $\dot{q}_{\text{abs}}$ is the energy absorbed (W m^{-2}), $\dot{q}_{\text{inc}}$ is the energy incident on the surface (W m^{-2}), and α is the absorptivity of the surface.

Like ε, α is between zero and unity, and is only unity for black bodies. The absorptivity of a body is generally different from its emissivity, but often, to simplify the analysis, α is assumed to equal ε.

The net heat transfer flux (heat transfer per unit area) for two very large black bodies separated by a vacuum is given by

$$\begin{aligned}\dot{q} &= E_{b1} - E_{b2} \\ &= \sigma \left(T_1^4 - T_2^4 \right)\end{aligned}$$

However, in real situations we are dealing with surfaces of finite areas which may be separated such that not all of the radiation leaving one strikes the other. We are interested in the net heat transfer in these situations, which we can describe by

$$\dot{Q} = \mathfrak{F}_{1,2} \sigma A_1 \left(T_1^4 - T_2^4 \right)$$

where $\mathfrak{F}_{1,2}$ is the view factor between the two surfaces which depends on the geometry of the system, and is defined as “the fraction of the radiation leaving surface 1 that is intercepted by surface 2”. This equation describes the heat transfer rate from a surface at temperature T_1 of area A_1 falling on a surface at temperature T_2 and of area A_2 (which is included in $\mathfrak{F}_{1,2}$).

If one of the bodies is a black body or has a very large area (e.g. the sky), then (as we’ll see later) the equation simplifies to the following:

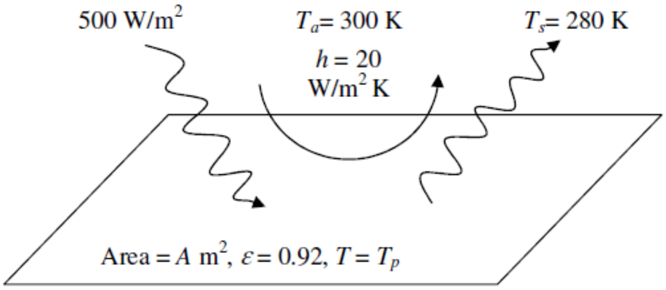
$$\dot{Q} = \sigma A_1 \varepsilon_1 \left(T_1^4 - T_2^4 \right)$$

This is the equation that we’ll use at this stage, as we’ll confine ourselves mostly to geometries where the area and emissivity of only one of the two bodies exchanging heat via thermal radiation matters.

Heat transfer by thermal radiation is non-linear with respect to temperature. This can give rise to iterative calculations when combined with convection.

Example. Combined convection and radiation

A small, thin metal plate of area $A\,\mathrm{m}^2$ is kept insulated on one side and exposed to the sun on the other. The plate absorbs solar energy at a rate of $500\,\mathrm{W\,m^{-2}}$. It dissipates heat by convection to the ambient air at $T_a = 300\,\mathrm{K}$ with a convection heat transfer coefficient of $20\,\mathrm{W\,m^{-2}\,K^{-1}}$, and by radiation into the sky, which may be assumed to be a black body with a temperature of 280 K. The emissivity of the plate is $\varepsilon = 0.92$. Determine the equilibrium temperature of the plate, and the proportions of energy dissipated by convection and by radiation. (Hint: you will need to find the temperature of the plate by iteration.)



$$
\begin{aligned}
\dot{q}_{\text{inc}} &= 500\,\mathrm{W\,m^{-2}} \\
\dot{q}_{\text{conv}} &= h\left(T_p - T_a\right) = 20 \times \left(T_p - 300\right)\ \mathrm{W\,m^{-2}} \\
\dot{q}_{\text{rad}} &= \varepsilon\sigma\left(T_p^4 - T_s^4\right) = 0.92 \times 5.67 \times 10^{-8}\left(T_p^4 - 280^4\right)\ \mathrm{W\,m^{-2}} \\
\dot{q}_{\text{inc}} &= \dot{q}_{\text{conv}} + \dot{q}_{\text{rad}} \\
500 &= 20 \times \left(T_p - 300\right) + 0.92 \times 5.67 \times 10^{-8}\left(T_p^4 - 280^4\right)
\end{aligned}
$$

The easiest method to solve the quartic equation, is to rearrange to separate T_p out on its own on the LHS and make it a function of T_p on the RHS. We can choose either T_p to separate out – we'll choose the first one:

$$
T_p = \frac{500 - 5.2164 \times 10^{-8}\left(T_p^4 - 280^4\right)}{20} + 300
$$

Now we guess a value for T_p, put it in on the RHS, calculate the resulting value of T_p on the LHS, then put that back into the RHS and keep going until it converges. A suitable initial guess (given that we might expect a car bonnet in the sun, which this is similar to, to feel quite warm, i.e. around $40 - 50\,^\circ\mathrm{C}$ or $313 - 323\,\mathrm{K}$) – any sensible initial guess should result in iteration to 315.26 K, giving proportions of convection and radiation heat transfer of 61% and 39%, respectively.

N.B. If this didn't converge, we would try writing the equation to separate out the other T_p.

Enclosed Systems

A two surface enclosure is a system involving two surfaces which only exchange radiation with each other. The two surfaces may not be black bodies, and may have differing emissivities. In this situation, the net heat transfer may be described by

$$\dot{Q} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 \mathfrak{F}_{1,2}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

For large (infinite) parallel planes, where $\mathfrak{F}_{1,2} = 1$ and $A_1 = A_2 = A$, this simplifies to

$$\dot{Q} = \frac{\sigma A\left(T_1^4 - T_2^4\right)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

while for long (infinite) concentric cylinders, where $\mathfrak{F}_{1,2} = 1$ and $A_1/A_2 = r_1/r_2$, this simplifies to

$$\dot{Q} = \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\dfrac{1}{\varepsilon_1} + \dfrac{1-\varepsilon_2}{\varepsilon_2}\left(\dfrac{r_1}{r_2}\right)}$$

If one of the bodies is a black body or has a very large area (e.g. the sky), then the equation simplifies to

$$\dot{Q} = \sigma A_1 \varepsilon_1 \left(T_1^4 - T_2^4\right)$$

If a radiation shield of emissivities ε_{s1} and ε_{s2} on its two sides is inserted between two large parallel planes to reduce the heat transfer, it can be shown that the rate of heat transfer is then given by

$$\dot{Q} = \frac{\sigma A\left(T_1^4 - T_2^4\right)}{\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1\right) + \left(\dfrac{1}{\varepsilon_{s1}} + \dfrac{1}{\varepsilon_{s2}} - 1\right)}$$

If the emissivity of the shield is the same on both sides, this simplifies to

$$\dot{Q} = \frac{\sigma A\left(T_1^4 - T_2^4\right)}{\dfrac{1}{\varepsilon_1} + \dfrac{2}{\varepsilon_s} + \dfrac{1}{\varepsilon_2} - 2}$$

and so on for additional shields.

2.4 The Standard Engineering Equation, and Analogy with Electrical Circuits

The General Heat Transfer Equation can be expressed in the form of the standard engineering equation, which is:

$$\begin{aligned} \text{Rate} &= \frac{\text{Driving Force}}{\text{Resistance}} \\ \dot{Q} &= UA\Delta T = \frac{\Delta T}{R} \qquad (2.4.1) \end{aligned}$$

where clearly, the resistance to heat transfer, R, is equal to $1/UA$. Remember, this is only true for steady state conditions; under unsteady state (i.e. changing) conditions, the differential form of the equation would need to be applied.

This form of equation is most familiar to us as

$$V = IR$$

or

$$I\,(\text{rate of flow of electrons}) = \frac{V\,(\text{driving force})}{R\,(\text{resistance})}$$

In fact, a circuit-type diagram can be drawn to represent heat transfer, as shown in Figure 2.4.

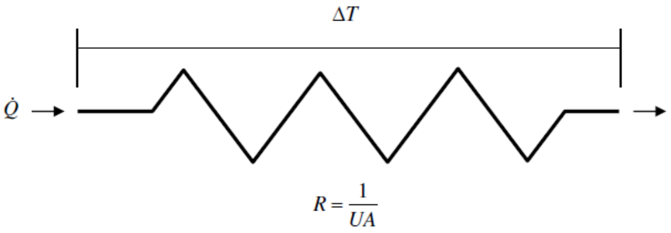


Figure 2.4: Resistance diagram for heat transfer.

The concept of using circuit diagrams to represent heat transfer helps when we come to consider several heat transfer operations in series, such as convection on one side of a wall, conduction through the wall, and convection heat transfer from the other side. It becomes very clear that we determine the overall resistance by summing the individual resistances, as shown in Figure 2.5:

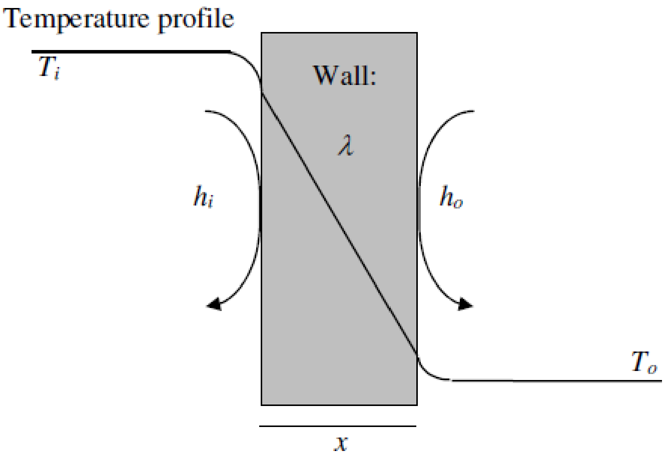


Figure 2.5: Summing heat transfer resistances in series, for convection and conduction through a wall.

From above

$$\dot{Q} = UA\Delta T = \frac{(T_i - T_o)}{R}$$

where

$$\begin{aligned}R &= \frac{1}{UA}\\ &= \frac{1}{A}\left(\frac{1}{h_i}+\frac{x}{\lambda}+\frac{1}{h_o}\right)\end{aligned}$$

As the areas for heat transfer are equal, we can simplify things by writing

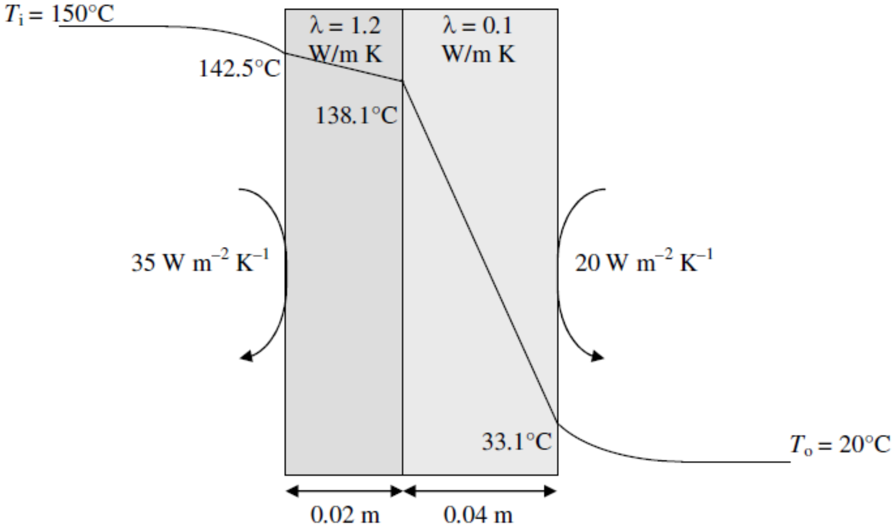
$$\dot{Q} = UA\Delta T$$

and calculating U directly from

$$\frac{1}{U} = \left(\frac{1}{h_i}+\frac{x}{\lambda}+\frac{1}{h_o}\right)$$

Example: Conduction through a wall of two layers

A wall consists of two layers, of thicknesses and thermal conductivities as shown in the diagram below. The temperature on the inside is 150 °C, and on the outside is 20 °C, and the convection heat transfer coefficient on the inside wall is 35 W m^{-2} K^{-1}, and on the outside is 20 W m^{-2} K^{-1}. Assuming close bonding between the two layers, calculate the temperature at the interface and at the outer walls. Draw a scale diagram showing the temperature profile. Why, at steady state, is the heat flux across each layer the same, even though the thicknesses and thermal conductivities are different? How does this happen?



Calculate the overall heat transfer coefficient, assuming close bonding between the two layers.

$$\begin{aligned}\frac{1}{U} &= \frac{1}{h_i}+\frac{x_1}{\lambda_1}+\frac{x_2}{\lambda_2}+\frac{1}{h_o}\\ &= \frac{1}{35}+\frac{0.02}{1.2}+\frac{0.04}{0.1}+\frac{1}{20}\\ &= 0.0286+0.0167+0.4+0.05\\ &= 0.4953\end{aligned}$$

$$N.B. \qquad 6\,\% \qquad 3\,\% \qquad 81\,\% \qquad 10\,\%$$

– most of the resistance is in the second layer.

$$U = 2.02\,\mathrm{W\,m^{-2}\,K^{-1}}$$

Calculate the heat flux.

$$\begin{aligned}\dot{q} &= U\Delta T\\ &= 2.02 \times (150 - 20)\\ &= 262.6\,\mathrm{W\,m^{-2}}\end{aligned}$$

Note – this same rate of heat transfer must pass from the inside to the wall, through each layer, and from the wall to the outside

Calculate the temperature at the left hand wall and at the interface between the two layers.

$$\begin{aligned}\dot{q} &= h_i\Delta T\\ 262.6 &= 35 \times (150 - T_1)\\ T_1 &= 142.5\,^{\circ}\mathrm{C}\end{aligned}$$

$$\begin{aligned}\dot{q} &= \frac{\lambda_1}{x_1}\Delta T\\ 262.6 &= \frac{1.2}{0.02} \times (142.5 - T_2)\\ T_2 &= 138.1\,^{\circ}\mathrm{C}\end{aligned}$$

Calculate the temperature at the right hand wall, and check that the final calculation yields a temperature of 20 °C.

$$\begin{aligned}\dot{q} &= \frac{\lambda_2}{x_2}\Delta T\\ 262.6 &= \frac{0.1}{0.04} \times (138.1 - T_3)\\ T_2 &= 33.1\,^{\circ}\mathrm{C}\end{aligned}$$

$$\begin{aligned}\dot{q} &= h_o\Delta T\\ 262.6 &= 20 \times (33.1 - T_1)\\ T_1 &= 20\,^{\circ}\mathrm{C}\end{aligned}$$

Sketch the temperature profile on the above picture.

Why, at steady state, is the heat flux across each layer the same, even though the thicknesses and thermal conductivities are different? How does this happen?

The heat flux must be the same across each layer, otherwise there would be accumulation of heat, the temperature would rise, and we would not be at steady state. How this is achieved is by matching the temperature driving force across each layer to the resistance – a layer with more resistance requires a larger temperature driving force to get the same amount of heat through it. Remember the units of resistance, $\mathrm{K\,W^{-1}}$ i.e. the temperature driving force required to achieve 1 W of heat transfer. A larger resistance means a larger temperature driving force is required.

2.5 Problems

1. A tank of water is kept boiling with an internal 10 kW heater, and loses energy through evaporation and through the walls of the tank. The surrounding environment has a temperature of 20 °C.

    (a) Calculate the heat loss through the walls, which are made of 5 mm thick stainless steel with a total surface area of 2.8 m^2. Calculate the percentage of resistance to heat transfer contributed by the boiling water, the walls and the air. Assume natural convection of the air. Which is the controlling resistance?

    (b) Calculate the rate of evaporation of water.

    Data:

    Thermal conductivity of stainless steel = 19 $\mathrm{W\,m^{-1}\,K^{-1}}$

    Latent heat of vaporisation of water = 2257 $\mathrm{kJ\,kg^{-1}}$

    Take the convective heat transfer coefficient for boiling water to be about 45000 $\mathrm{W\,m^{-2}\,K^{-1}}$, and for air under free convection to be 35 $\mathrm{W\,m^{-2}\,K^{-1}}$.

2. A thermocouple is located inside a ceramics oven for temperature control. The walls of the oven are at 600 °C, and the air temperature is 527 °C. The thermocouple can be assumed to be spherical with a diameter of 2 mm, and to have a uniform temperature throughout.

    Assuming that both the oven walls and the thermocouple are black bodies with respect to radiation, calculate the steady state temperature that would be recorded by the thermocouple for an air velocity of 4 $\mathrm{m\,s^{-1}}$ if the convection heat transfer coefficient can be found from the following correlation:

    $$\frac{hD}{l} = 0.37\mathrm{Re}^{0.6}$$

    Comment on the recorded measurements. Suggest how a more accurate air temperature reading could be obtained. [Hint: you will need to determine the steady state temperature iteratively.]

    Data: Physical properties of air at 527 °C

    $\rho = 0.435\,\mathrm{kg\,m^{-3}}$

    $\mu = 0.370 \times 10^{-4}\,\mathrm{Pa\,s}$

    $\lambda = 0.0577\,\mathrm{W\,m^{-1}\,K^{-1}}$

Chapter 3

Heat Transfer by Conduction

Contents

3.1 Introduction

Conduction is the mechanism of heat transfer in solids. Solids tend to have temperature distributions or profiles across them, i.e. they are hotter on the outside or inside, or on one side etc. Fluids tend to be well-mixed, so that we are not really concerned about their temperature distribution – it tends to even out to a uniform temperature just through mixing. But solids don't – heat is transferred by conduction, and the nature of the heat transfer depends on the temperature distribution. We noted earlier that "Since heat flow takes places whenever there is a temperature gradient in a system, a knowledge of the temperature distribution in a system is essential in heat transfer studies." So we now want to consider how to calculate firstly the temperature distribution in a system, and from it the heat transfer rate.

3.2 Derivation of one-dimensional conduction heat transfer equation

To do this we will develop the basic mathematical equation which describes the temperature distribution through a system. We will do this based on the energy conservation equation, which we will apply to a very small element. As we noted earlier when we were considering energy balances, conservation of energy applies whether we are considering a whole plant, a single item of equipment, or a small differential element within some equipment. Here we are going to consider conservation of energy over a small differential element.

To simplify the mathematical notation, and to illustrate the meaning of the terms in the equation, we will start by considering just one-dimensional conduction. This would apply when considering, for example, conduction through a wall where the height and width of the wall are very large compared with its thickness, so that essentially there is no difference in temperature along the height or width, and the only difference to be considered is that across the thickness of the wall.

So, considering one-dimensional heat conduction in the x-dimension, consider a volume element of thickness $\Delta\, x$ and area normal to the x-direction, as shown in Figure 3.1 below.

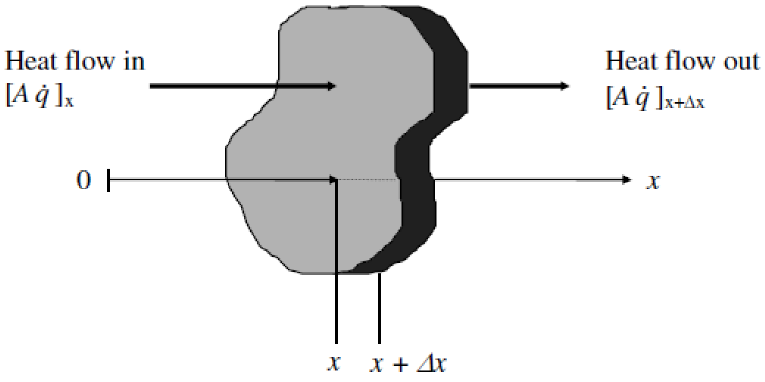


Figure 3.1: Nomenclature for derivation of the one-dimensional heat conduction equation.

An energy balance for this volume element is stated as

$$\begin{pmatrix}\text{Net rate of heat gain}\\ \text{by conduction}\end{pmatrix} + \begin{pmatrix}\text{Rate of}\\ \text{energy generation}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase}\\ \text{of internal energy}\end{pmatrix}$$

$$\text{I} \qquad\qquad\qquad \text{II} \qquad = \qquad \text{III}$$

Note that this energy balance does not assume steady state, and it includes the possibility of energy generation. This energy generation could arise from, for example, electrical resistance, nuclear reaction, or chemical reaction. Each of the terms I, II and III will be considered in turn.

I Net rate of heat gain by conduction

Heat enters the volume element by conduction through the area A normal to the x-coordinate at the point x, and leaves by conduction through the area A at the point $x + \Delta\, x$. Let $\dot{q}$ be the conduction heat flux (heat flow per unit area, $\mathrm{W\,m^2}$) at point x in the positive x-direction at the surface A of the element. Then the rate of heat flow into the volume element through the surface A at location x is written as

$$[A\dot{q}]_x$$

Similarly, the rate of heat flow out of the element by conduction at the location $x + \Delta\, x$ is given by

$$[A\dot{q}]_{x+\Delta\, x}$$

Then the net rate of heat flow by conduction into the element is the difference between the flow in and flow out:

$$\text{I} \equiv [A\dot{q}]_x - [A\dot{q}]_{x+\Delta\, x}$$

II Rate of energy generation

The rate of energy generation in an element of volume ADx is given by

$$\text{II} \equiv A\Delta\, xg$$

where $g = g(x, t)$ is the energy generation per unit volume ($\mathrm{W\,m^3}$) at the point x and at time t.

III Rate of increase of internal energy

The relationship between the rate of energy input into a mass of material and the rate of energy change is given by

$$\dot{Q} = mC_p \frac{\mathrm{d}\,T}{\mathrm{d}\,t} \qquad \mathrm{kg} \times \frac{\mathrm{J}}{\mathrm{kg\,K}} \times \frac{\mathrm{K}}{\mathrm{s}} = \mathrm{W}$$

Mass is given by density × volume, therefore

$$\text{III} \equiv A\Delta\, x\rho C_p \frac{\partial\, T\,(x, t)}{\partial\, t}$$

Note that we use the partial derivative, ∂, as we are considering in the term the change in temperature with respect to time, but not distance.

Putting it all together, rearranging slightly and dividing through by the volume, $A\Delta\, x$, we get:

$$\begin{aligned}[A\dot{q}]_x - [A\dot{q}]_{x+\Delta\, x} + gA\Delta\, x &= A\Delta\, x\rho C_p \frac{\partial\, T\,(x,t)}{\partial\, t} \\ -\frac{1}{A}\frac{[A\dot{q}]_x - [A\dot{q}]_{x+\Delta\, x}}{\Delta\, x} + g &= \rho C_p \frac{\partial\, T\,(x,t)}{\partial\, t}\end{aligned}$$

Now, as $\Delta\, x$ tends to 0, the first term on the left hand side becomes, by definition, the derivative of $[A\dot{q}]$ with respect to x (that's why we've written it this way). So the equation becomes

$$-\frac{1}{x}\frac{\partial\,(A\dot{q})}{\partial\, x} + g = \rho C_p \frac{\partial\, T\,(x,t)}{\partial\, t}$$

Here, the first term describes the change of heat flow with distance, x. The negative sign arises because heat flow is defined as positive into the system and negative out of it, but the definition of the derivative with respect to x reverses this order.

Now, from Fourier's law, the heat flux, $\dot{q}$, is given by the temperature gradient times the thermal conductivity (with an appropriate negative sign):

$$\dot{q} = -\lambda \frac{\partial\, T\,(x,t)}{\partial\, x}$$

Substituting into the first term of the above equation gives

General Heat Conduction Equation

$$\frac{1}{A}\frac{\partial}{\partial\, x}\left(A\lambda\frac{\partial\, T}{\partial\, x}\right) + g = \rho C_p \frac{\partial\, T\,(x,t)}{\partial\, t}$$

This is the general heat conduction equation. The first term represents the net rate at which heat flows by conduction into the system. Heat flows by conduction only when there is a temperature gradient, so this term includes, inside the brackets, the temperature gradient, without which there would be no flow of heat through the system at all. But if there were no difference in the rate of flow of energy into and out of the system by conduction, there would be no accumulation, so the temperature would not change. This is why there is an additional differentiation with respect to x. If there is a difference (i.e. this differential is not equal to 0), it will show up as an accumulation within the system, so the temperature would change with time. This is shown by the term on the right hand side.

If there were internal energy generation somehow, this would also cause the temperature to change with time. So the two terms on the left are potential sources by which the temperature might change with time, and the term on the right describes the rate at which the temperature would change with time due to these sources.

Now, you might say that the area in the first term doesn't change with distance in the xdirection, so we could take it out of the differential and cancel it with the $1/A$. And you would be right, for rectangular co-ordinates. But what about cylindrical or spherical coordinates? In these cases, area does change with distance. So let's take our general equation and write it out for all three co-ordinate systems.

3.2.1 Rectangular Coordinates

These are our familiar, Cartesian, x, y, z coordinates, and in this system of coordinates, area normal to the x-direction does not change as x increases. So A in the general equation is constant, and the equation simplifies to

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+g=\rho C_p\frac{\partial T\,(x,t)}{\partial t}$$

This is the one-dimensional, time-dependent heat conduction equation in the rectangular coordinate system.

3.2.2 Cylindrical Coordinates

In cylindrical co-ordinates our three axes are given by r, ϕ, and z, and for one-dimensional heat conduction we are concerned with conduction in the radial direction, r. Area increases proportionally with r, so the general equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)+g=\rho C_p\frac{\partial T\,(r,t)}{\partial t}$$

This is the one-dimensional, time-dependent heat conduction equation in the cylindrical coordinate system.

3.2.3 Spherical Coordinates

In spherical co-ordinates, once again we denote the radial coordinate by r instead of x. In this case, area A is proportional to r^2, so the general equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda\frac{\partial T}{\partial r}\right)+g=\rho C_p\frac{\partial T\,(r,t)}{\partial t}$$

This is the one-dimensional, time-dependent heat conduction equation in spherical coordinates.

3.2.4 Special Cases

Let's consider some special cases, which will help us understand what each of the terms means in a physical sense. And, to avoid excessive brain strain, let's look just at the familiar rectangular co-ordinate system, for which the equation is

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+g=\rho C_p\frac{\partial T\,(x,t)}{\partial t}$$

Constant Thermal Conductivity

If the thermal conductivity, λ, were constant, then we could take it out of the first term. We could also divide through by λ, to give

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)+\frac{g}{\lambda}=\frac{\rho C_p}{\lambda}\frac{\partial T\left(x,t\right)}{\partial t}$$
$$\frac{\partial^2 T}{\partial x^2}+\frac{g}{\lambda}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where α is the thermal diffusivity ($\lambda/\rho C_p$). The thermal diffusivity is a measure of the rate at which heat is propagated through a medium; the larger the thermal diffusivity, the faster heat is propagated.

No Internal Energy Generation

If, in addition, there were no internal energy generation, and we were just considering pure conduction heat transfer, then we get

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{Fourier Equation}$$

This means quite simply that the rate of change of temperature, which indicates a rate of change of internal energy, must be balanced by the net rate at which energy is flowing by conduction into the system.

No Heat Conduction

If both surfaces of the system were well insulated so that there was no conduction into or out of the element, then the equation becomes

$$\frac{g}{\lambda}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which is equivalent to

$$g=\rho C_p\frac{\partial T}{\partial t}$$

then multiply both sides by m^3 to get

$$\text{Power (W)}=mC_p\frac{\partial T}{\partial t}$$

This quite clearly means that the rate of energy generation is balanced by the rate of change of temperature of the material.

Steady State Heat Conduction

Under steady state conditions, there is no change with respect to time. Therefore the right hand side of the above equation must be zero:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\, x^2} + \frac{g}{\lambda} = 0 \qquad \text{Poisson Equation}$$

Note that for steady state conditions the derivatives are no longer partial, so d is used instead of ∂.

If there is also no internal energy generation, then

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\, x^2} = 0 \qquad \text{Laplace Equation}$$

This is the Laplace equation, and it simply means that, while there is a temperature gradient causing conduction heat flow (i.e. the first derivative is not zero), there is no difference in the heat flow that would cause accumulation (i.e. the second differential does equal zero), so there is no accumulation and therefore no temperature change.

If there is internal energy generation, then the Poisson equation simply means that the rate of internal energy generation is balanced by the rate of conduction out of the system, and there is no accumulation and no temperature change.

Equivalent equations exist for cylindrical and spherical co-ordinates (see below), and for three-dimensional heat conduction. For three dimensional steady state heat conduction with no internal energy generation, Laplace's equation becomes

$$\frac{\partial^2 T}{\partial\, x^2} + \frac{\partial^2 T}{\partial\, y^2} + \frac{\partial^2 T}{\partial\, z^2} = \nabla^2 T = 0$$

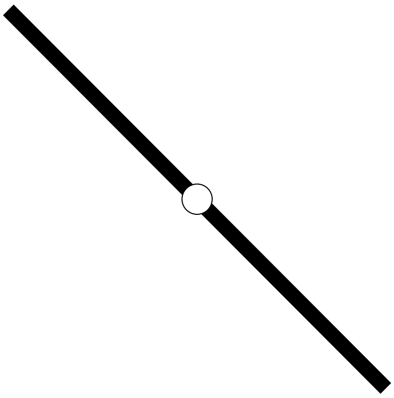
which is the three-dimensional form of Laplace's equation.

Table 3.1: Summary of the Poisson and Laplace equations in different coordinate systems.

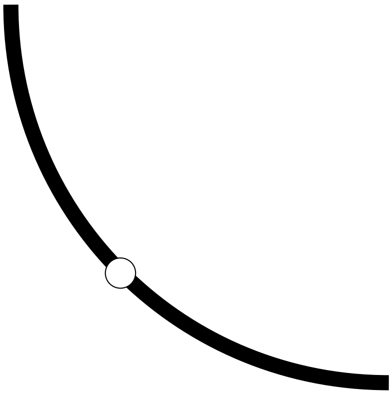
Coordinate System	Poisson equation	Laplace equation
Rectangular	$\frac{\mathrm{d}^2 T}{\mathrm{d}\, x^2} + \frac{g}{\lambda} = 0$	$\frac{\mathrm{d}^2 T}{\mathrm{d}\, x^2} = 0$
Cylindrical	$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\, r}\left(r\frac{\mathrm{d}T}{\mathrm{d}\, r}\right) + \frac{g_0}{\lambda} = 0$	$\frac{\mathrm{d}}{\mathrm{d}\, r}\left(r\frac{\mathrm{d}T}{\mathrm{d}\, r}\right) = 0$
Spherical	$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}\, r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}\, r}\right) + \frac{g_0}{\lambda} = 0$	$\frac{\mathrm{d}}{\mathrm{d}\, r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}\, r}\right) = 0$

3.2.5 Physical Meaning of the Laplace Equation

Heat flow by conduction is proportional to the temperature gradient...



There will only be accumulation of heat if the temperature gradient changes...

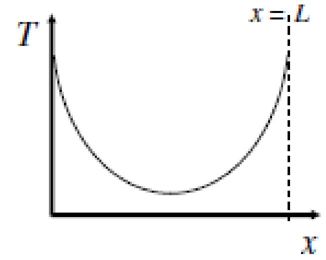


Let's think about the relationship between the physical situation and the mathematics describing that physical situation a bit further. The figure below shows an instantaneous temperature profile in a solid slab of thickness L. The temperature profile might arise, for example, because the slab has been plunged into hot water such that the sides have begun to heat, while the centre is still cool. Eventually the whole slab will heat to the same uniform temperature throughout, but at a given moment in time, the temperature profile is as shown in the figure. For the sake of illustration, it can be described by a quadratic function, as shown to the right of the figure.

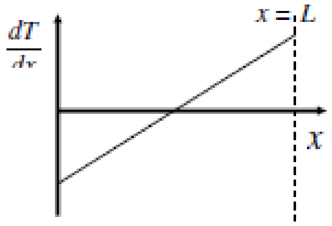
Sketch the profiles of $\mathrm{d}\,T/\mathrm{d}\,x$, $\dot{q}$ and $\mathrm{d}^2T/\mathrm{d}\,x^2$ across the slab, based on thinking about what is going on physically as indicated by the figure, and based on differentiating the mathematical expression describing the physical situation. To simplify the maths, assume $\lambda = 1\,\mathrm{W\,m^{-}1\,K^{-1}}$.

Graphical representation

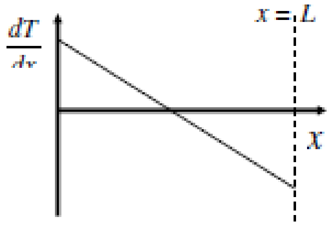
Mathematical representation
(and physical interpretation)



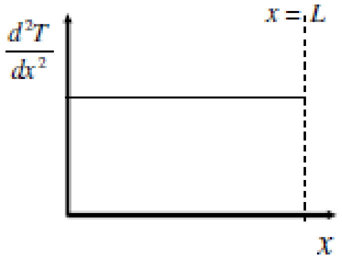
$T(x) = 3x^2 - 20x + 100$
– *i.e.* hotter on the sides, colder in the middle



$\frac{dT}{dx} = 6x - 20$
– *i.e.* the ***slope*** or ***gradient*** changes from negative to zero to positive



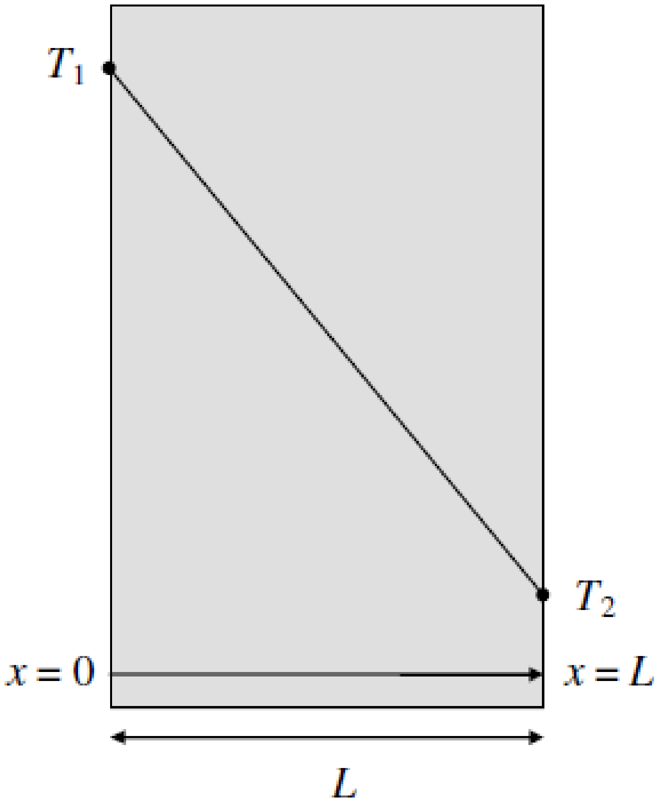
$\dot{q} = -6x + 20$
– *i.e.* the ***heat flux*** is in the positive x-direction on the left hand side and in negative x-direction on the right hand side, so heat is flowing towards the middle, where it will accumulate and cause the temperature to rise, hence not steady state



$\frac{d^2T}{dx^2} = 6$

3.3 Steady State Heat Conduction across a Slab

Consider a slab of thickness L, with the temperature of the walls maintained at T_1 and T_2, respectively. There is no energy generation in the solid, and thermal conductivity is constant. Develop expressions for the temperature distribution, $T(x)$, and heat flux through the slab.



The mathematical formulation of this steady state heat conduction problem with no internal energy generation and constant thermal conductivity is given by Laplace's equation:

$$\frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} = 0$$

where $T(x) = T_1$ at $x = 0$ and $T(x) = T_2$ at $x = L$

Integrating once gives

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = C_1$$

and again gives

$$T(x) = C_1 x + C_2$$

We have two equations, and two boundary conditions. From the boundary condition at $x = 0$, we get

$$C_2 = T_1$$

Then, from the boundary condition at $x = L$, we get

$$T_2 = C_1 L + T_1$$
$$C_1 = \frac{(T_2 - T_1)}{L}$$

Therefore the equation describing the variation of T with x is

$$T(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

which is a straight line starting at T_1, with slope $(T_2 - T_1)/L$. If $T_2 < T_1$, then this slope is negative, and heat flows from T_1 to T_2.

So we know the temperature distribution. But we want to know the heat flowrate or flux. This we get from Fourier's law:

$$\dot{q} = -\lambda \frac{\mathrm{d}\,T}{\mathrm{d}\,x}$$

Differentiating our temperature distribution with respect to x gives

$$\frac{\mathrm{d}\,T}{\mathrm{d}\,x} = \frac{(T_2 - T_1)}{L}$$

Therefore

$$\dot{q}\,(x) = -\frac{\lambda}{L}\,(T_2 - T_1)$$

or

$$\dot{Q} = \frac{\lambda}{L} A \Delta\,T$$

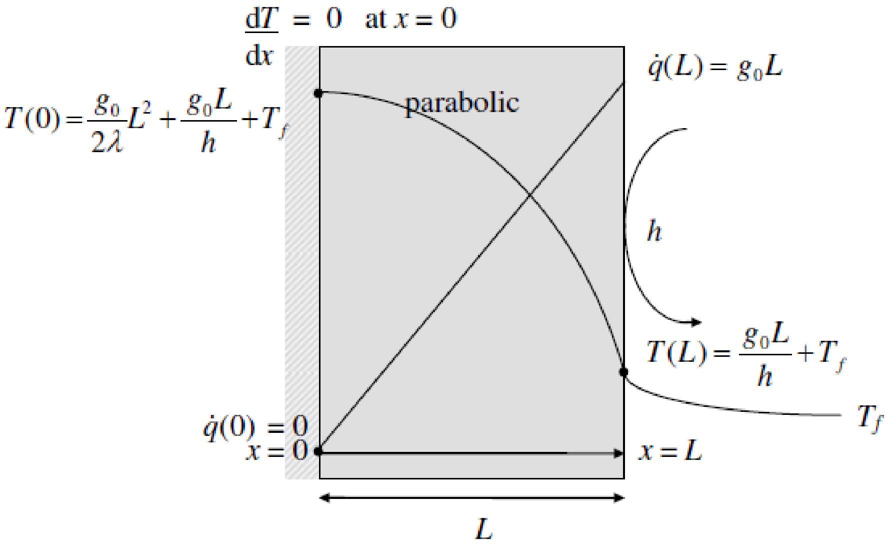
which is the same (essentially) as the conduction heat transfer equation which we developed earlier. Note that $\dot{q}(x)$ is independent of x, i.e. is the same throughout the slab.

So we have learned from this exercise that we solve the general heat conduction equation by integrating, and we find the constants of integration from the boundary conditions. We therefore need as many boundary conditions as constants of integration (two in this case). We have also learned that, having found the temperature distribution in this way, we differentiate it to find the heat flux, and that the equation obtained in this way is the same as that which we encountered earlier for describing heat conduction through a slab – in other words, what we are learning now is no different to what we learned about conduction earlier, it's just that we've made the fundamental basis for it more explicit.

3.4 Steady State Heat Generation in a Slab Insulated on One Side

Consider a slab of thickness L and constant thermal conductivity λ in which energy is generated at a constant rate of $g_0\,\mathrm{W\,m^3}$. The boundary surface at $x = 0$ is insulated so that there is no heat flow across this boundary, and that at $x = L$ dissipates heat by convection into a fluid at temperature T_f with a heat transfer coefficient h.

In this example, there is energy generation in the medium. This could apply to an electrical heating element or a nuclear fuel rod, for example. This means that we are unable to calculate the heat flow by considering the driving force and resistance to flow – we cannot use the resistance approach when there is energy generation. To solve this one, we need to solve the heat conduction equation.



The mathematical formulation of the steady state problem is given by the Poisson equation:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\, x^2} = -\frac{g_0}{\lambda}$$

with boundary conditions,

$$\frac{\mathrm{d}\, T\,(x)}{\mathrm{d}\, x} = 0 \qquad \text{at } x = 0 \text{ (no heat flux must mean there is no temperature gradient)}$$

$$\dot{q} = -\lambda\frac{\mathrm{d}\, T\,(x)}{\mathrm{d}\, x} = h\,(T\,(x) - T_f) \qquad \text{at } x = L$$

i.e. the rate of heat loss by conduction at the surface must equal the rate at which heat is removed by convection. Therefore

$$T\,(L) = T_f - \frac{\lambda \left.\frac{\mathrm{d}\, T}{\mathrm{d}\, x}\right|_{x=L}}{h}$$

Integrating the Poisson equation gives

$$\frac{\mathrm{d}\, T\,(x)}{\mathrm{d}\, x} = -\frac{g_o}{\lambda}x + C_1$$

Putting in $x = 0$ shows that $C_1 = 0$.

The expression for the heat flux is given by substituting into Fourier's law:

$$\dot{g} = -\lambda\frac{\mathrm{d}\, T}{\mathrm{d}\, x} = g_0 x$$

note, the heat flux varies with position across the slab.

This also allows us to simplify our boundary condition expression for $T(L)$:

$$T\,(L) = T_f + \frac{g_0 L}{h}$$

Integrating again gives

$$T\left(x\right)=-\frac{g_o}{2\lambda}x^2+C_2$$

Applying the boundary condition at $x=L$ gives

$$T\left(L\right)=-\frac{g_o}{2\lambda}L^2+C_2=T_f+\frac{g_0L}{h}$$

Therefore

$$C_2=\frac{g_o}{2\lambda}L^2+\frac{g_0L}{h}+T_f$$

Therefore the temperature distribution becomes

$$\begin{aligned}T\left(x\right)=&-\frac{g_o}{2\lambda}x^2+\frac{g_o}{2\lambda}L^2+\frac{g_0L}{h}+T_f\\=&\frac{g_o}{2\lambda}L^2\left(1-\left(\frac{x}{L}\right)^2\right)+\frac{g_0L}{h}+T_f\end{aligned}$$

The physical significance of these terms is that the first is due to the energy generation, and the second to the presence of the finite convection heat transfer coefficient at the surface. If this were infinite, this term would vanish.

Note that the temperature distribution is not uniform across the slab (it is parabolic), and neither is the heat flux (which is linear with respect to x, starting at 0).

3.5 Temperature Profiles across Cylindrical Walls

Cylindrical systems are very frequently encountered in the process industries, in particular in pipework. Often we have pipes conveying hot fluids, and it is important that we know the temperature distribution and therefore heat losses across the pipe walls. Other examples include heat generation in cylindrical fuel elements in nuclear reactors or in electrical wires carrying currents.

The steady state heat conduction equation with constant, uniform energy generation for the cylindrical co-ordinate system is given by

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)+\frac{g_0}{\lambda}=0$$

or,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)=-\frac{g_0}{\lambda}r$$

Integrating twice gives

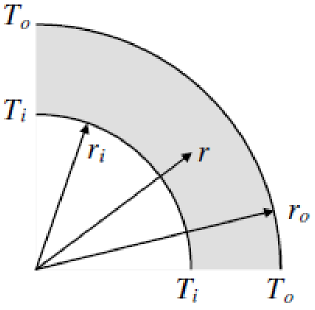
$$\begin{aligned}\frac{\mathrm{d}T\left(r\right)}{\mathrm{d}r}=&-\frac{g_0}{2\lambda}r+\frac{C_1}{r}\\T\left(r\right)=&-\frac{g_0}{4\lambda}r^2+C_1\ln r+C_2\end{aligned}$$

Clearly, we need two boundary conditions to determine the two integration constants.

To solve this for a solid cylinder with energy generation (e.g. a wire carrying current), we would specify the surface temperature and that the heat flux at the centre of the wire = 0, due to the temperature gradient at the centre being 0 (as the cylinder is symmetric).

Let us consider the more common (for process engineers) situation of a hollow pipe with no internal energy generation containing flowing fluid, where the inner surface at $r = r_i$ and the outer surface at $r = r_o$ are maintained at temperatures T_i and T_o, respectively.

r_i = inner radius of pipe (= constant)
r_o = outer radius of pipe (= constant)
r = radius at some point within the walls (= variable, varying from r_i to r_o)
T_i = temperature of inner wall surface
T_o = temperature of outer wall surface



Therefore the above equations become

$$\begin{aligned} \frac{\mathrm{d}\,T(r)}{\mathrm{d}\,r} &= \frac{C_1}{r} \\ T(r) &= C_1 \ln r + C_2 \end{aligned}$$

The boundary conditions give us

$$\begin{aligned} T_i &= C_1 \ln r_i + C_2 \\ T_o &= C_1 \ln r_o + C_2 \end{aligned}$$

Simultaneous solution of these equations gives

$$\begin{aligned} C_1 &= \frac{T_o - T_i}{\ln\left(\dfrac{r_o}{r_i}\right)} \\ C_2 &= T_i - (T_o - T_i)\,\frac{\ln r_i}{\ln\left(\dfrac{r_o}{r_i}\right)} \end{aligned}$$

Introducing these into the above equation for $T(r)$ gives

$$\begin{aligned} T(r) &= \frac{T_o - T_i}{\ln\left(\dfrac{r_o}{r_i}\right)} \ln r + T_i - (T_o - T_i)\,\frac{\ln r_i}{\ln\left(\dfrac{r_o}{r_i}\right)} \\ &= T_i + (T_o - T_i)\,\frac{\ln r - \ln r_i}{\ln\left(\dfrac{r_o}{r_i}\right)} \end{aligned}$$

from which

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\ln\left(\dfrac{r}{r_i}\right)}{\ln\left(\dfrac{r_o}{r_i}\right)}$$

The heat flux is given, as usual, by Fourier's law:

$$\dot{q} = -\lambda \frac{\mathrm{d}\,T(r)}{\mathrm{d}\,r}$$

From earlier,

$$\frac{\mathrm{d}\,T(r)}{\mathrm{d}\,r} = \frac{C_1}{r} = \frac{1}{r}\frac{T_o - T_i}{\ln\left(\dfrac{r_o}{r_i}\right)}$$

Hence

$$\dot{q} = \frac{\lambda}{r}\frac{T_i - T_o}{\ln\left(\dfrac{r_o}{r_i}\right)}$$

In other words, $\dot{q}$ is inversely proportional to r. The equation above tells us the temperature profile across the wall of the tube. If T_i is arbitrarily assigned a value of 1, and T_o a value of 0, then the temperature profile would be as follows, for different ratios of r_o/r_i, Figure 3.2.

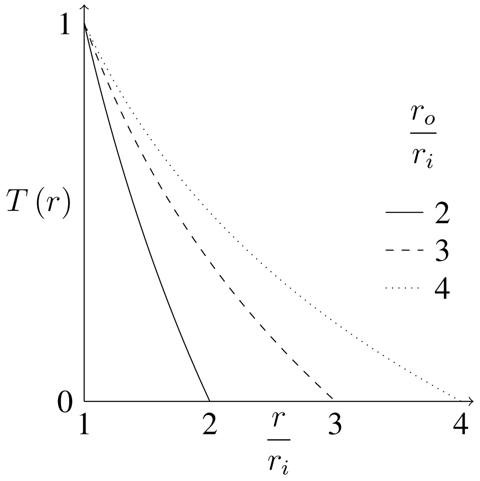


Figure 3.2: Example temperature profile through a cylinder.

$\dot{Q}$, the product of the heat flux × the area, must be constant (heat must enter one side of the pipe wall at the same rate that it exits the other side). The heat flux therefore decreases as the radius increases, because the area increases proportionally to the radius. The decrease in $\dot{q}$ is reflected by the change in the slope of $\mathrm{d}\,T/\mathrm{d}\,r$ as r increases.

Now we want to know the heat flowrate out of the pipe:

$$\begin{aligned}
\dot{Q} = \dot{q} \times A = & -\lambda \frac{\mathrm{d}\,T(r)}{\mathrm{d}\,r} 2\pi r H \\
= & -\lambda \frac{1}{r} \frac{1}{\ln\left(\dfrac{r_o}{r_i}\right)} (T_o - T_i)\, 2\pi r H \\
= & \frac{2\pi H \lambda (T_i - T_o)}{\ln\left(\dfrac{r_o}{r_i}\right)}
\end{aligned}$$

Comparing with

$$\dot{Q} = \frac{(T_i - T_o)}{R}$$

clearly, the resistance to heat transfer is given by

$$R = \frac{1}{UA} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi H\lambda}$$

This analysis is appropriate when the thickness of the pipe walls is large compared with the pipe radius. For thin-walled pipes (where the pipe thickness is small compared with the radius), a rectangular approximation can be used.

Alternatively, the log-mean area, A_m, can be used, defined by

$$R = \frac{x}{\lambda A_m} = \frac{(r_o - r_i)}{\lambda A_m} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi H\lambda}$$

$$A_m = \frac{(r_o - r_i)\,2\pi H}{\ln\left(\frac{r_o}{r_i}\right)} = \frac{(A_o - A_i)}{\ln\left(\frac{A_o}{A_i}\right)}$$

The log-mean area will be useful for heat exchanger calculations.

3.6 Temperature Profiles Across Spherical Walls

Spherical or hemi-spherical shaped containers are often used in the process industries, because the have the minimum surface area per unit volume (so minimise heat losses) and have uniform distribution of stresses within the container walls. It is left as a tutorial exercise for students to show that the temperature distribution across the wall of a hollow spherical container of constant thermal conductivity, with inner and outer surfaces maintained at temperatures T_i and T_o respectively, for steady state heat conduction with no internal energy generation, is given by

$$T(r) = \frac{r_i}{r}\frac{r_o - r}{r_o - r_i}T_i + \frac{r_o}{r}\frac{r - r_i}{r_o - r_i}T_o$$

and the heat flow by

$$\dot{Q} = 4\pi\lambda\frac{r_i r_o}{r_o - r_i}(T_i - T_o)$$

The resistance can then given by

$$R = \frac{1}{4\pi\lambda}\frac{r_o - r_i}{r_i r_o}$$

3.7 Boundary Conditions

We have been using boundary conditions throughout the last two lectures to determine the constants of integration that arise when we solve the heat conduction equation. It is not always immediately obvious what the boundary conditions are or how to use them. It is therefore helpful to point out that there are generally three types of boundary condition âĂŞ if you know this, it will help you spot them. The three types are the Prescribed Temperature (First Kind), Prescribed Heat Flux (Second Kind) and Convection (Third Kind) Boundary Conditions.

3.7.1 Prescribed Temperature Boundary Condition (First Kind)

This situation occurs simply when the temperature at a surface is considered known. This might occur when, for example, a boundary surface is in contact with melting ice or condensing steam at atmospheric pressure, so that the surface is known to be at a constant temperature of 0 °C or 100 °C, respectively.

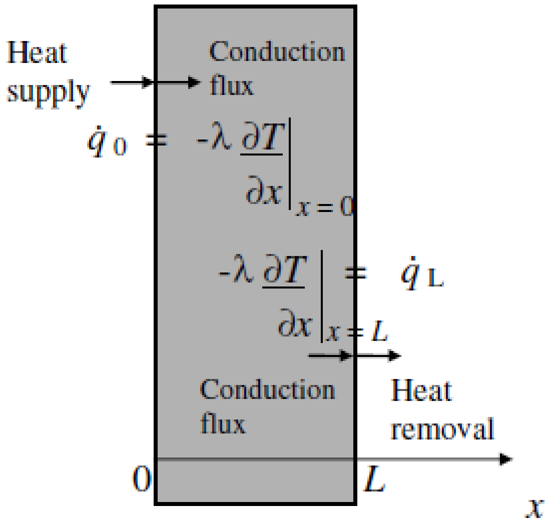
If a slab of thickness L has its surface at $x = 0$ maintained at temperature T_1, and its surface at $x = L$ maintained at temperature T_2, then the slab has prescribed temperature boundary conditions at both surfaces, and these are written as

$$\begin{aligned} T(x,t)|_{x=0} &\equiv T(0,t) = T_1 \\ T(x,t)|_{x=L} &\equiv T(L,t) = T_2 \end{aligned}$$

More generally, the distribution of temperature over the boundary surface may be specified as a function of position and/or time. Either way, when the value of temperature at a surface is specified, this is known as a boundary condition of the first kind.

3.7.2 Prescribed Heat Flux Boundary Condition (Second Kind)

In some situations, the rate at which heat is supplied to (or removed from) a surface is known, for example, if the surface is being electrically heated or heated by thermal (e.g. solar) radiation.



If a slab of thickness L has a heat supply of $\dot{q}_0\,\mathrm{W\,m^2}$ to its surface at $x = 0$, and a heat removal of $\dot{q}_L\,\mathrm{W\,m^2}$ to its surface at $x = L$, then the slab has prescribed heat flux boundary conditions at both surfaces. These external heat supplies are equated with the conduction heat flux into the solid, and are written as:

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \dot{q}_0$$

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=L} = \dot{q}_L$$

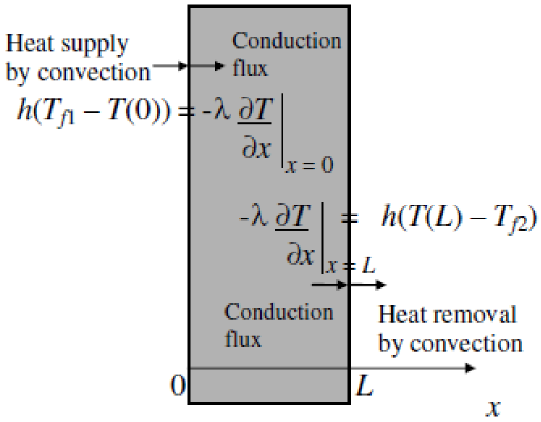
When the heat flux is prescribed at a boundary surface, this is known as a boundary condition of the second kind. Similar equations apply to the boundary surfaces of spheres and cylinders, replacing x with r in the above equations.

Sometimes the prescribed heat flux boundary condition is invoked from the symmetry of a system. For example, sometimes we have a solid slab, cylinder or sphere with internal heat generation, and we are only given a single surface temperature, but we need two boundary conditions to determine our two constants of integration. We can determine the other boundary condition by considering the symmetry of the system, which says that at the centre, the temperature profile must be flat, therefore $\mathrm{d}T/\mathrm{d}x = 0$, therefore $\dot{q} = 0$.

3.7.3 Convection Boundary Condition (Third Kind)

In most practical situations, heat transfer at the boundary of a solid is by convection into an ambient fluid with a prescribed temperature, and with a known heat transfer coefficient h.

Consider again a slab of thickness L with a fluid at temperature T_{f1} flowing over its surface at $x = 0$ with a heat transfer coefficient of h_1. The mathematical formulation of the convection boundary condition is obtained by considering an energy balance at the surface:



$$\left(\begin{array}{c}\text{Convection heat flux}\\ \text{from the fluid at } T_{f1} \text{ to}\\ \text{the surface at } x=0\end{array}\right) = \left(\begin{array}{c}\text{Conduction heat flux}\\ \text{from the surface at}\\ x=0 \text{ into the plate}\end{array}\right)$$

$$h_1 \left[T_{f1} - T\left(x,t\right)\big|_{x=0}\right] = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0}$$

If a fluid at temperature T_{f2} with heat transfer coefficient h_2 flows over the surface at $x = L$, the energy balance is staged as

$$h_2 \left[T\left(x,t\right)\big|_{x=L} - T_{f2}\right] = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=L}$$

A convection boundary condition is known as a boundary condition of the third kind. The convection boundary condition is the most general of the three – in fact, the other two are just special cases (see Ozisik for the reasoning why). The solution of conduction heat transfer problems hinges on the boundary conditions, and these are usually convection boundary conditions. This means that the next key to solving heat transfer problems is knowing how to calculate the convection heat transfer coefficient.

3.8 Problems

1. A commercial freezer is to be maintained at $-18\,^\circ\mathrm{C}$. The freezer is insulated with glass wool insulation sandwiched between two sheets of aluminum. Calculate the thickness of glass wool insulation required ($\lambda = 0.04\,\mathrm{W\,m^{-1}\,K^{-1}}$) if the external wall temperature is $25\,^\circ\mathrm{C}$, and the heat flux into the freezer is $170\,\mathrm{W\,m^{-2}}$. Ignore the thermal resistance of the aluminum sheets. How will a variation of $1\,^\circ\mathrm{C}$ in the external wall temperature impact on the thickness of insulation required?

   [1 cm]

2. A furnace wall is made up of three layers, the inner one of firebrick, the middle one of insulating brick and the outer one of red brick, of thermal conductivities 1.21, 0.13, and $0.85\,\mathrm{W\,m^{-1}\,K^{-1}}$, respectively, and thicknesses 200 mm, 70 mm, and 95 mm, respectively. The inner wall is at $900\,^\circ\mathrm{C}$, and the heat flux is $1000\,\mathrm{W\,m^{-2}}$. Assuming close bonding between each layer at their interfaces, calculate the temperature at each interface and at the outer wall. Is the outer wall at a safe temperature for nearby personnel? Which is the controlling resistance? Draw a scale diagram showing the temperature profile.

   [$900\,^\circ\mathrm{C}$, $735\,^\circ\mathrm{C}$, $196\,^\circ\mathrm{C}$, $84\,^\circ\mathrm{C}$]

3. A stainless steel fermentation vessel is to be maintained at $37\,^\circ\mathrm{C}$. The vessel has a cylindrical central section 2 m in diameter and 3 m in height, and the top and bottom of the vessel are hemispherical in shape with a 2 m diameter. The liquid in the vessel comes to the top of the central cylindrical section. Estimate the rate at which heat must be supplied to maintain the vessel temperature, if the surroundings are at $10\,^\circ\mathrm{C}$. Take the liquid-side convective heat transfer coefficient to be $200\,\mathrm{W\,m^{-2}\,K^{-1}}$, the internal air convective heat transfer coefficient to be $30\,\mathrm{W\,m^{-2}\,K^{-1}}$, and the external air convective heat transfer coefficient to be $7\,\mathrm{W\,m^{-2}\,K^{-1}}$. Ignore the thickness of the walls and the thermal resistance of the stainless steel, and ignore radiative heat losses.

   [5.55 kW]

4. The temperature of the water in a pipe is measured using a thermocouple, which may be considered a sphere of diameter 2 mm, thermal conductivity $25\,\mathrm{W\,m^{-2}\,K^{-1}}$, density $8400\,\mathrm{kg\,m^{3}}$, and specific heat capacity $400\,\mathrm{J\,kg^{-1}\,^\circ C^{-1}}$. The heat transfer coefficient between the water and the thermocouple is $700\,\mathrm{W\,m^{-2}\,K^{-1}}$. Is a lumped system analysis applicable? If the temperature of the water flowing past the thermocouple suddenly changes from $10\,^\circ\mathrm{C}$ to $40\,^\circ\mathrm{C}$, how long will it take for the thermocouple to reach a reading of $39\,^\circ\mathrm{C}$? What is the time constant for the thermocouple?

   [Yes ($\mathrm{Bi} = 9.33 \times 10^{-3} < 0.1$), 5.44 seconds, 1.6 seconds]

5. A wall is made up of three layers, the inner one of wood, the middle one of insulating glass wool and the outer one of cork, of thermal conductivities 0.112, 0.038 and, $0.04\,\mathrm{W\,m^{-1}\,K^{-1}}$, respectively, and thicknesses 12 mm, 20 mm and 16 mm, respectively. Air on the inner face of the wall is at $90\,^\circ\mathrm{C}$, and transfers heat with a convective heat transfer coefficient of $27\,\mathrm{W\,m^{-2}\,K^{-1}}$, while air on the outer face of the wall is at 5ÂřC and has a convective heat transfer coefficient of $8\,\mathrm{W\,m^{-2}\,K^{-1}}$. Assuming close bonding between each layer at their interfaces, calculate the overall heat transfer rate and the temperature at the wall surfaces and at the interfaces within the wall. Sketch the temperature profile.

[71.1 $\mathrm{W\,m^{-2}}$, 87.37 °C, 79.75 °C, 42.33 °C, 13.89 °C]

6. In many practical situations, heat transfer takes place simultaneously by convection to (or from) the ambient air and by radiation to (or from) the surroundings. Combined convection and radiation problems often require an iterative solution. Note that the temperature driving force for convective and radiative heat transfer may be different. A small, thin metal plate of area $A\,\mathrm{m}^2$ is kept insulated on one side and exposed to the sun on the other. The plate absorbs solar energy at a rate of 500 $\mathrm{W\,m^{-2}}$, and dissipates heat by convection to the ambient air at $T_a = 300\,\mathrm{K}$ with a convection heat transfer coefficient of 20 $\mathrm{W\,m^{-2}\,K^{-1}}$, and by radiation into the sky, which may be assumed to be a blackbody with a temperature of 280 K. The emissivity of the plate is $\varepsilon = 0.92$. Determine the equilibrium temperature of the plate, and the proportions of energy dissipated by convection and radiation. (Hint: you will need to find the temperature of the plate by trial and error.)

   [315.26 K, 61%, 39%]

7. Heat is lost by both convection and radiation from a steam pipe of 4 cm outer diameter with a wall surface temperature of 125 °C, into an external environment at 0 °C. The convection heat transfer coefficient is 20 $\mathrm{W\,m^{-2}\,K^{-1}}$, and the emissivity of the pipe surface is 0.92. Calculate the rate of heat loss per metre of pipe, and the proportions of heat transfer contributed by convection and radiation.

   [442.5 W/metre of pipe, 71%, 29%]

8. A room in Scotland has a glass window of width 1.5 m and height 2 m, 5 mm thick and with a thermal conductivity $\lambda = 1.4\,\mathrm{W\,m^{-1}\,K^{-1}}$. In the middle of winter, the air on the inside and outside of the glass is at 20 °C and -10 °C, respectively, with convection heat transfer coefficients of 25 $\mathrm{W\,m^{-2}\,K^{-1}}$ and 15 $\mathrm{W\,m^{-2}\,K^{-1}}$, respectively. The window is the only source of heat loss from the room, which is kept warm via an electric heater.

   (a) Calculate the rate of heat loss through the window and hence the power requirement of the heater. Assuming the heater is on for 24 hours per day, estimate the daily cost of heating the room, basing your answer on a typical domestic cost for electricity in the UK (i.e. check a recent electricity bill, quoting the supplier).

   (b) The window is replaced by a double glazed window in which two 5 mm thick panes of glass are separated by a 10 mm air gap (thermal conductivity of still air is 0.024 $\mathrm{W\,m^{-1}\,K^{-1}}$. Calculate the rate of heat loss through the new window, and the percentage contribution of each resistance.

Chapter 4

Heat Transfer by Convection

Contents

4.1 Introduction

We noted when we first came across convection heat transfer that it could be easily described using the convection heat transfer coefficient, h. However, we also noted that h depends on the fluid mechanics of the system, such as the type of flow (turbulent or laminar), the geometry of the system, the physical properties of the fluid, the average temperature, the position along the surface of the body, and whether the heat transfer is by forced convection or free convection. We said then that we would look at how to calculate h later in the course. To do so, we need to develop a picture of how convection heat transfer occurs, then find ways of relating the fluid mechanics to the heat transfer. Complete description of convection heat transfer would require solution of the equations of continuity, momentum and energy, which is beyond the scope of this course.

4.2 Thermal Boundary Layer

In Fluid Mechanics, you learned about the concept of a velocity boundary layer, which was the layer of fluid flowing near to the wall, in which the velocity of the fluid increased from zero at the wall up to close to the free stream velocity of the fluid far from the wall. Analogous to this idea, we can imagine the development of a thermal boundary layer, in which the temperature of a fluid flowing past a wall varies from the wall temperature to the bulk fluid temperature, as shown in Figure 4.1. Note that x represents direction along the wall surface, and y direction normal to the wall.

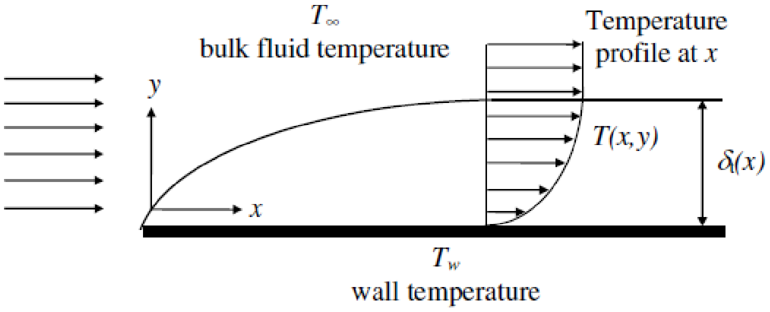


Figure 4.1: Development of the thermal boundary layer for flow over a wall.

The thermal boundary layer is the locus of points at which the temperature is equal to 99the bulk fluid temperature, i.e. $T(x, y) = 0.99T_\infty$ at $\delta_t(x)$.

The thicknesses of the thermal boundary layer and the velocity boundary layer are not necessarily the same! The relative thickness depends on the Prandtl number, which is a dimensionless number

$$\text{Pr} = \frac{C_p \mu}{\lambda}$$

$$\text{Pr} = \frac{\dfrac{\mu}{\rho}}{\dfrac{\lambda}{\rho C_p}} = \frac{v}{\alpha} = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\delta(x)}{\delta_t(x)}$$

The Prandtl number is a measure of the ratio of momentum diffusivity to heat diffusivity. If this ratio is equal to unity, as it is for gases, then the thermal and velocity boundary layers have the same thickness, i.e.

$$\delta(x) = \delta_t(x) \qquad \text{for Pr} = 1 \text{ (gases)}$$

For fluids for which Pr $\ll 1$, such as liquid metals, the thermal boundary layer is much thicker than the velocity boundary layer i.e. heat is propagated through the fluid to a greater extent than momentum. For fluids for which Pr $\gg 1$, such as ordinary liquids and oils, the thermal boundary layer is much thinner than the velocity boundary layer.

4.3 Relating Convection Heat Transfer to Fluid Mechanics using Dimensionless Numbers

Suppose the temperature distribution, $T(x, y)$, in the thermal boundary layer were known. Then the heat flux from the fluid to the wall at the point x along the wall would be given by

$$\dot{q}\,(x) = \lambda_f \left. \frac{\partial\, T\,(x, y)}{\partial\, y} \right|_{y=0}$$

where λ_f is the thermal conductivity of the fluid. (Note we are omitting the negative sign for convenience.) But we don't generally know the temperature distribution, so instead we define a local heat transfer coefficient $h(x)$ such that the heat flux between the fluid and the wall at point x along the surface of the wall is given by

$$\dot{q}\,(x) = h\,(x)\,(T_\infty - T_w)$$

Note that the local heat transfer coefficient can vary along the wall i.e. $h(x)$ is a function of x. Equating the above two equations gives

$$h\,(x) = \lambda_f \frac{\left. \dfrac{\partial\, T}{\partial\, y} \right|_{y=0}}{T_\infty - T_w}$$

Let us now define a dimensionless temperature $\theta(x, y)$ as

$$\theta\,(x, y) = \frac{T\,(x, y) - T_w}{T_\infty - T_w}$$

Clearly, $\theta(x, y)$ will vary from 0 at the wall to 1 in the bulk fluid. We can then write the above equation in terms of this dimensionless temperature as

$$h\,(x) = \lambda_f \left. \frac{\partial\, \theta\,(x, y)}{\partial\, y} \right|_{y=0}$$

4.4 Laminar Flow along a Flat Plate

An approximate expression for the temperature profile in the thermal layer for laminar flow along a flat plate is given by

$$\theta\left(x,y\right)=\frac{3}{2}\frac{y}{\delta_t\left(x\right)}-\frac{1}{2}\left[\frac{y}{\delta_t\left(x\right)}\right]^3$$

Similarly, an expression for the thickness of the thermal boundary layer is given by

$$\delta_t\left(x\right)=4.53\frac{x}{\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}}$$

where the local Reynolds number, Re_x is defined as

$$\mathrm{Re}_x=\frac{u_\infty x}{v}$$

where u_∞ is the bulk fluid velocity, v is the kinematic viscosity (μ/ρ), and x is the distance along the plate.

The heat transfer coefficient is related to the temperature profile by

$$h\left(x\right)=\lambda_f\left.\frac{\partial\,\theta\left(x,y\right)}{\partial\,y}\right|_{y=0}$$

$$\left.\frac{\partial\,\theta\left(x,y\right)}{\partial\,y}\right|_{y=0}=\frac{3}{2\delta_t\left(x\right)}$$

Therefore

$$h\left(x\right)=\frac{3\lambda_f}{2\delta_t\left(x\right)}$$

Substituting in our expression for $\delta_t(x)$ gives

$$\begin{aligned}h\left(x\right)&=\frac{3\lambda_f\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}}{2\times 4.53\times x}\\&=0.331\frac{\lambda_f}{x}\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}\end{aligned}$$

Rearranging to give three dimensionless groups gives

$$\mathrm{Nu}_x=\frac{h\left(x\right)x}{\lambda_f}=0.331\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}$$

where Nu_x is the local Nusselt number, and is a function of the Prandtl and Reynolds numbers. Note that $h(x)$ varies with $x^{-0.5}$, i.e. the convection heat transfer coefficient along the length of the wall decreases, as we noted earlier.

In fact, the exact expression for laminar flow along a flat plate is given by

$$\mathrm{Nu}_x=\frac{h\left(x\right)x}{\lambda_f}=0.332\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}$$

so our analysis is pretty close. The range of application for this correlation is $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 10$, and it applies when the wall temperature of the plate is constant along the plate.

The above example introduces the Nusselt number, which is an important dimensionless group for describing convection heat transfer. Looking closely at the general form of the Nusselt number, using a characteristic dimension L, it is clearly a ratio of

$$\text{Nu} = \frac{hL}{\lambda_f} = \frac{h\Delta T}{\dfrac{\lambda_f}{L}\Delta T}$$

which can be interpreted as the ratio of convection heat transfer to conduction heat transfer across the thermal boundary layer.

The example also demonstrates that the Nusselt number is a function of the Reynolds and Prandtl numbers, which describe the fluid mechanics of the system. This sort of expression therefore gives us our goal, which was to relate the convection heat transfer coefficient to the fluid mechanics. Knowing the fluid mechanics, we can calculate Re and Pr, and from these predict the convection heat transfer coefficient. Similar expressions have been developed for other systems e.g. flow in tubes, along ducts, turbulent flow etc. Clearly in these cases the temperature profile and thermal boundary layer profile will be different, so the equations will be different, but they often have this general form. In the next lecture we will look at some of these.

4.5 Correlations for Predicting the Convection Heat Transfer Coefficient

4.5.1 Forced Convection, Fully Developed Turbulent Flow in Smooth, Circular Pipes

One correlation (among many!) for forced convection fully developed turbulent flow inside smooth tubes is given by the Sieder and Tate equation:

$$\text{Nu} = 0.027\text{Re}^{0.8}\text{Pr}^{0.333}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

This equation is applicable for the following range of conditions:

- $0.7 < \text{Pr} < 16,700$
- $\text{Re} > 10,000$ ($\text{Re} = u_m D/v$)
- $L/D > 60$
- smooth pipes

The term $(\mu_b/\mu_w)^{0.14}$ is a correction factor which accounts for the fact that the fluid viscosity changes with temperature near the wall.

Calculation of convection heat transfer coefficient and heat flux for turbulent flow inside a circular tube

Calculate the convection heat transfer coefficient for water flowing with a mean velocity of $2\,\mathrm{m\,s^{-1}}$ inside a smooth-walled circular pipe of inside diameter 5 cm. The inside wall of the (thin, conducting) pipe is maintained at a constant temperature of 100 °C by steam condensing on the outside. At a location where the fluid flow is hydrodynamically and thermally developed, the bulk mean temperature of the water is 60 °C. Calculate the convection heat transfer coefficient and the heat flow per unit length of pipe.

Data: Physical properties of water at 60 °C

$$\begin{aligned} \rho &= 983\,\mathrm{kg\,m^{3}} \\ \mu_b &= 4.67 \times 10^{-4}\,\mathrm{kg\,m^{-1}\,s^{-1}} \\ \lambda &= 0.653\,\mathrm{W\,m^{-1}\,K^{-1}} \\ C_p &= 4185\,\mathrm{J\,kg^{-1}\,K^{-1}} \end{aligned}$$

and viscosity of water at the wall temperature of 100 °C

$$\mu_w = 2.83 \times 10^{-4}\,\mathrm{kg\,m^{-1}\,s^{-1}}$$

Calculate the Reynolds number:

$$\begin{aligned} \mathrm{Re} &= \frac{\rho u_m D}{\mu} \\ &= \frac{983 \times 2 \times 0.05}{4.67 \times 10^{-4}} \\ &= 210\,493 \end{aligned}$$

therefore the flow is well into the turbulent region.

Calculate the Prandtl number:

$$\begin{aligned} \mathrm{Pr} &= \frac{c_p \mu}{\lambda} \\ &= \frac{4185 \times 4.67 \times 10^{-4}}{0.653} \\ &= 2.993 \end{aligned}$$

Then

$$
\begin{aligned}
\mathrm{Nu} &= 0.027\mathrm{Re}^{0.8}\mathrm{Pr}^{0.333}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \\
&= 0.027 \times 210\,493^{0.8} \times 2.993^{0.333} \times \left(\frac{4.67}{2.83}\right)^{0.14} \\
&= 0.027 \times 18\,138 \times 1.441 \times 1.073 \\
&= 757.2 \\
\frac{hD}{\lambda} &= 757.2 \\
h &= \frac{757.2 \times 0.653}{0.05} \\
&= 9889\,\mathrm{W\,m^{-2}\,K^{-1}} \\
\dot{q} &= h\left(T_w - T_f\right) \\
&= 9889 \times (100 - 60) \\
&= 395\,597\,\mathrm{W\,m^{-2}}
\end{aligned}
$$

The area per meter of pipe is,

$$
\begin{aligned}
A &= \pi D h \\
&= \pi \times 0.05 \times 1 \\
&= 0.1571\,\mathrm{m^2\,m^{-1}} \\
\dot{Q} &= \dot{q}A \\
&= 395\,567 \times 0.1571 \\
&= 62\,112\,\mathrm{W\,m^{-1}}
\end{aligned}
$$

The above example demonstrates the use of a convection heat transfer correlation. There are many such correlations, developed for different geometrical situations e.g. pipes, ducts, flat plates, and applicable over different ranges of conditions e.g. laminar or turbulent flow, range of Prandtl number. The important thing is to know that, to use a correlation, you must be sure that it is applicable to your particular system, and that you are operating within its range of applicability. You should also bear in mind that their accuracy is perhaps only $\pm 25\%$.

4.5.2 Forced Convection, Laminar Flow over a flat Plate

Local Nusselt number, Nu_x,

$$\mathrm{Nu}_x = 0.332\mathrm{Re}_x^{0.5}\mathrm{Pr}^{0.333}$$

Average Nusselt number, Nu_m, over a distance L

$$\mathrm{Nu}_m = 0.664\mathrm{Re}_L^{0.5}\mathrm{Pr}^{0.333}$$

Range of application: $\mathrm{Re}_L < 5 \times 10^5$, $0.6 < \mathrm{Pr} < 10$, constant wall temperature.

4.5.3 Forced Convection, Turbulent Flow over a Flat Plate

Local Nusselt number, Nu_x,

$$\mathrm{Nu}_x = 0.029\mathrm{Re}_x^{0.8}\mathrm{Pr}^{0.43}$$

Average Nusselt number, Nu_m, over a distance L

$$\mathrm{Nu}_m = 0.036\left(\mathrm{Re}_L^{0.8} - 9200\right)\mathrm{Pr}^{0.43}\left(\frac{\mu_b}{\mu_w}\right)^{0.25}$$

Range of application: $2 \times 10^5 < \mathrm{Re}_L < 5.5 \times 10^6$, $0.7 < \mathrm{Pr} < 380$, $0.26 < \mu_b/\mu_w < 3.5$, constant wall temperature.

4.5.4 Forced Convection, Fully Developed Laminar Flow in Smooth, circular Pipes

$$Nu = 4.364$$

Range of application: $\mathrm{Re} < 2300$, constant wall heat flux along the pipe.

$$Nu = 3.66$$

Range of application: $\mathrm{Re} < 2300$, constant wall temperature along the pipe.

4.5.5 Forced Convection, Entrance Region of a Circular Tube

$$\mathrm{Nu}_m = 3.66 + \frac{0.0668\mathrm{Gz}}{1 + 0.04\mathrm{Gz}^{0.667}}$$

where $\mathrm{Gz} = \mathrm{Re}\,\mathrm{Pr}/\left(L/D\right) =$ Graetz number and L is distance from inlet.

Range of application: $\mathrm{Gz} < 100$, constant wall temperature

4.5.6 Forced Convection, Fully Developed Turbulent Flow in Smooth, Circular Pipes

Colburn Equation

$$\mathrm{Nu} = 0.023\mathrm{Re}^{0.8}\mathrm{Pr}^{0.333}$$

Range of application: $0.7 < \mathrm{Pr} < 160$, $\mathrm{Re} > 10\,000$, $L/D > 60$, smooth pipes.

Dittus-Boelter Equation

$$\mathrm{Nu} = 0.023\mathrm{Re}^{0.8}\mathrm{Pr}^{n}$$

where $n = 0.4$ for heating and 0.3 for cooling.

Range of application: $0.7 < \mathrm{Pr} < 160$, $\mathrm{Re} > 10\,000$, $L/D > 60$, smooth pipes.

Sieder and Tate Equation

$$\text{Nu} = 0.027\text{Re}^{0.8}\text{Pr}^{0.333}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

Range of application: $0.7 < \text{Pr} < 16\,700$, Re $> 10\,000$, $L/D > 60$, smooth pipes.

Petukhov Equation

$$\text{Nu} = \frac{\text{Re}\,\text{Pr}}{1.07 + 12.7\left(\text{Pr}^{0.667} - 1\right)\left(\dfrac{f}{8}\right)^{0.5}}\left(\frac{f}{8}\right)\left(\frac{\mu_b}{\mu_w}\right)^{n}$$

where

$$\begin{aligned} n &= 0.11 && \text{for heating with uniform } T_w \\ &= 0.25 && \text{for cooling with uniform } T_w \\ &= 0.0 && \text{for uniform wall heat flux or for gases} \end{aligned}$$

and f is the friction factor.

Range of application: $10^4 <$ Re $< 5 \times 10^6$, $0.5 <$ Pr < 200 with 5-6% error, $0.5 <$ Pr < 2000 with 10% error, $0.025 < \mu_b/\mu_w < 12.5$.

For non-circular ducts, the tube diameter in the above correlations can be replaced by the hydraulic diameter, D_h, of the non-circular duct, defined as

$$D_h = \frac{4A_c}{P}$$

where A_c is the cross sectional area and P is the wetted perimeter.

4.6 Problems

1. A refrigerated lorry container is being designed. The container of the lorry is 7 m long, 3 m high and 2 m wide. A worst case scenario is assumed, that the lorry is travelling at 60 miles per hour on a hot sunny day with the ambient air temperature at 30 °C.

   (a) Calculate the rate of convective heat transfer to the top and two sides of the lorry container, if their surfaces are at 18 °C. (Ignore the lorry's cab, underside and back doors.) Use the following correlation to calculate the average convection heat transfer coeffcient for air flowing over a flat surface of length L:

   $$h = \frac{\lambda}{L}\left(0.0125\mathrm{Re}_L^{0.8} - 287\right)$$

   Use the following physical properties of air at 30 °C: $\lambda = 0.026\,\mathrm{W\,m^{-1}\,K^{-1}}$, $\rho = 1.163\,\mathrm{kg\,m^{-3}}$, $\mu = 1.86 \times 10^{-5}\,\mathrm{Pa\,s}$.

   (b) Solar radiation falls onto the top and one side of the lorry at a flux of $500\,\mathrm{W\,m^{-2}}$. Calculate the total cooling duty required to maintain the container's steady state internal temperature.

   (c) The walls of the container are made of insulation (thermal conductivity of $0.11\,\mathrm{W\,m^{-1}\,K^{-1}}$) sandwiched between two stainless steel sheets, each of 5 mm thickness and with a thermal conductivity of $19\,\mathrm{W\,m^{-1}\,K^{-1}}$.

   (d) If the outer surface of the lorry walls are at 18 °C, and the inside wall of the chamber is to be maintained at 2 °C, calculate the thickness of the insulation required. Assume heat enters the lorry container only through the top and two sides (ignore the front, back and underside).

2. Energy is generated at a constant rate of 104 W m$-$3 throughout a slab of thickness 10 cm and constant thermal conductivity $1.2\,\mathrm{W\,m^{-1}\,K^{-1}}$. The boundary surface on one side is insulated so that there is no heat flow across this boundary, and on the other dissipates heat by convection into a fluid at temperature 20 °C with a heat transfer coefficient $74\,\mathrm{W\,m^{-2}\,K^{-1}}$. Calculate the steady state temperature at both surfaces of the slab, and the heat flux from the slab into the fluid. Sketch the temperature profile and heat flux profile through the slab. (N.B. The heat flux can be calculated in three ways - from the temperature profile at the slab surface, from the convection heat transfer equation, and one other way - what is it?)

   [$T(0) = 75.18$ °C, $T(L) = 33.51$ °C, $Q = 1000\,\mathrm{W\,m^{-2}}$]

3. A wall of thickness L and constant thermal conductivity λ is irradiated with gamma rays at the surface $x = 0$. The gamma rays attenuate as they pass through the wall, and in so doing convert their energy into heat. This volumetric heat generation varies through the wall at a rate given by $G(x) = G_0 \exp(-\beta x)$ where G_0 and β are constants. The boundary surface at $x = 0$ is kept thermally insulated, and the boundary surface at $x = L$ dissipates heat by convection with convection heat transfer coefficient h into a fluid at temperature T_f. Develop an expression for the steady state heat flux throughout the wall as a function of distance x, and for the wall surface temperature at $x = L$.

   [$\dot{q} = \frac{G_0}{\beta}(1 - \exp(-\beta x)), T(L) = T_f + \frac{G_0}{h\beta}(1 - \exp(-\beta L))$]

4. For the system described in the question above, calculate the convection heat flux and the wall surface temperature at $x = L$, for $G_0 = 104\,\mathrm{W\,m^{-3}}$, $\beta = 3\,\mathrm{m^{-1}}$, $L = 0.5\,\mathrm{m}$, $\lambda = 2.5\,\mathrm{W\,m^{-1}\,K^{-1}}$, $h = 70\,\mathrm{W\,m^{-2}\,K^{-1}}$, $T_f = 15\,^\circ\mathrm{C}$.

5. A wire of radius R and constant thermal conductivity λ carrying current generates heat at a constant rate throughout of $g\,\mathrm{W\,m^{-3}}$. The heat is dissipated by convection into the ambient air at a temperature T_f with a heat transfer coefficient h.

   (a) Show that the expression for the steady state temperature profile throughout the wire is given by

   $$T(r) = \frac{g}{4\lambda}R^2\left(1 - \left(\frac{r}{R}\right)^2\right) + \frac{gR}{2h} + T_f$$

   (b) Derive an expression for the heat flux profile throughout the wire. Sketch the temperature profile and heat flux profile.

   (c) For the above wire and the following data: $g = 6 \times 10^6\,\mathrm{W\,m^{-3}}$, $R = 1.5\,\mathrm{mm}$, $T_f = 10\,^\circ\mathrm{C}$, $\lambda = 2.5\,\mathrm{W\,m^{-1}\,K^{-1}}$, $h = 45\,\mathrm{W\,m^{-2}\,K^{-1}}$; calculate the temperature at the surface of the wire.

Chapter 5

Applications of Heat Transfer in Chemical Engineering

Contents

5.1 Introduction

We started this course by stating that heat transfer is one of the most useful things you will ever learn, for several reasons, including that many processing situations involve heat transfer, and that it is an excellent example of transport phenomena. Next year you will do a course called Momentum, Heat and Mass Transfer, which will aim to bring these three topics together and show that, mathematically, they have the same form. That course will then concentrate on the mathematics, so that you become competent at solving problems for any of the three transport phenomena. But you will cope with that course much better if you have a good grasp of the physical situations that all this maths describes. That is why, throughout this course, we have tried to highlight what is happening physically, recognising that the maths is simply a tool to describe the physical situation quantitatively. (Show an overhead from Momentum, Heat and Mass Transfer.)

But the other, more immediately practical reason that heat transfer will be so useful to you is that it applies to many processing situations of practical interest. So for the rest of this course, we are going to apply the heat transfer principles that you have learned to typical industrial problems.

One such problem is heat losses from insulated pipes carrying hot fluids. We know that for steady state heat transfer through a composite slab, we simply add the resistances to heat transfer, both for conduction through the slab, and for convection at the surface. The resistance to heat transfer for conduction through a slab is given by $R = 1/UA = x/Al$, and for convection is given by $R = 1/hA$. But with cylinders and spheres, the area increases with radius, so the expression for the thermal resistance is more complicated, and had to be derived by solving Laplace's equation, as we did in Lecture 10. Today we will apply this to heat loss through composite cylinders, in particular, insulated pipes, and we will learn why sometimes adding insulation to pipes is counter-productive, as it can actually increase heat loss.

5.2 Heat Loss through Composite Cylinders

As noted earlier, cylindrical systems are of great interest to process engineers, as pipes are cylindrical, and much of a process plant is taken up with pipework. Often these pipes are carrying hot fluids, such as steam, and often they are insulated to minimise heat losses. It is important that we are able to calculate the amount of insulation required, and that requires us to be able to calculate the heat loss across composite coaxial cylinders i.e. several cylindrical layers in series, as shown in Figure 5.1.

The clearest way to approach this is to utilise the thermal resistance concept introduced earlier, that the resistance to heat flow is determined by summing the individual resistances across each layer, along with the convection heat transfer resistances on the inside and outside of the pipe. This will only work for steady state conditions, as the thermal resistance analogy only applies for steady state.

$$\begin{aligned}\dot{Q} &= \frac{T_a - T_b}{R} \\ &= \frac{T_a - T_b}{R_a + R_1 + R_2 + R_b}\end{aligned}$$

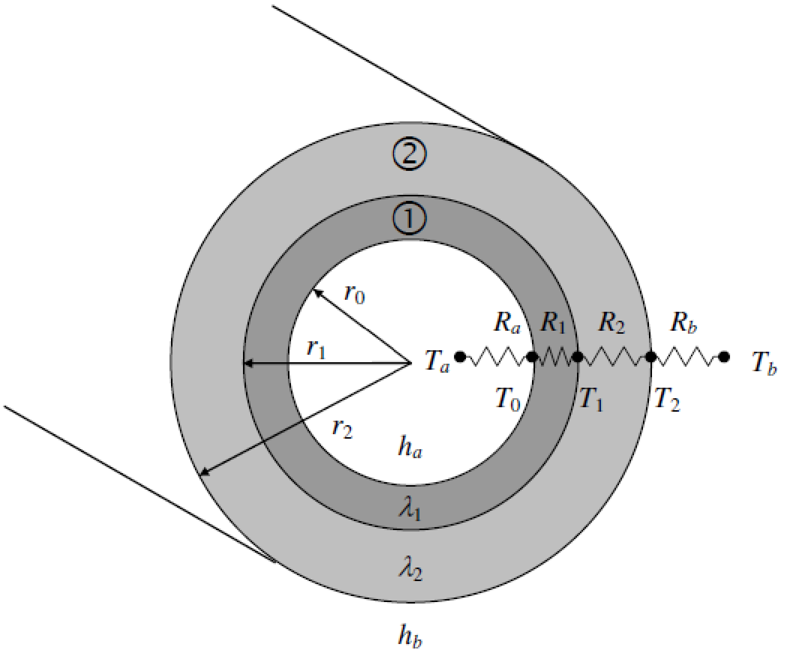


Figure 5.1: Radial heat flow through a hollow composite cylinder with layers in perfect thermal contact.

For convection heat transfer, we know that the resistance to heat transfer is given by

$$R = \frac{1}{hA}$$

Therefore

$$R_a = \frac{1}{h_a 2\pi r_0 H}$$
$$R_b = \frac{1}{h_b 2\pi r_2 H}$$

We have previously derived the temperature profile and heat flux across a cylindrical wall, and found that the resistance to heat transfer was given by

$$R = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi H\lambda}$$

Therefore, for each of the two layers in the composite cylinder,

$$R_1 = \frac{\ln\left(\frac{r_1}{r_0}\right)}{2\pi H\lambda_1}$$
$$R_2 = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi H\lambda_2}$$

We can thus calculate each of the resistances, add them together, and divide the temperature difference by the sum to get the heat flow rate.

$$R = \frac{1}{UA} = \frac{1}{h_a 2\pi r_0 H} + \frac{\ln\left(\frac{r_1}{r_0}\right)}{2\pi H\lambda_1} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi H\lambda_2} + \frac{1}{h_b 2\pi r_2 H}$$

In calculations, the overall heat transfer coefficient is usually based on the outer tube surface, in this case, r_2. The overall heat transfer coefficient can be defined as

$$\begin{aligned}
U_oA_o &= \frac{1}{\dfrac{1}{h_a 2\pi r_0 H} + \dfrac{\ln\left(\dfrac{r_1}{r_0}\right)}{2\pi H\lambda_1} + \dfrac{\ln\left(\dfrac{r_2}{r_1}\right)}{2\pi H\lambda_2} + \dfrac{1}{h_b 2\pi r_2 H}} \\
U_o &= \left[A_o\left(\frac{1}{h_a 2\pi r_0 H} + \frac{1}{2\pi H\lambda_1}\ln\left(\frac{r_1}{r_0}\right) + \frac{1}{2\pi H\lambda_2}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{h_b 2\pi r_2 H}\right)\right]^{-1} \\
&= \left(\frac{2\pi r_2 H}{h_a 2\pi r_0 H} + \frac{2\pi r_2 H}{2\pi H\lambda_1}\ln\left(\frac{r_1}{r_0}\right) + \frac{2\pi r_2 H}{2\pi H\lambda_2}\ln\left(\frac{r_2}{r_1}\right) + \frac{2\pi r_2 H}{h_b 2\pi r_2 H}\right)^{-1} \\
&= \left(\frac{r_2}{h_a r_0} + \frac{r_2}{\lambda_1}\ln\left(\frac{r_1}{r_0}\right) + \frac{r_2}{\lambda_2}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{h_b}\right)^{-1}
\end{aligned}$$

Or often in terms of the diameters,

$$U_o = \left[\frac{D_2}{D_0}\frac{1}{h_a} + \frac{D_2}{2\lambda_1}\ln\left(\frac{D_1}{D_0}\right) + \frac{D_2}{2\lambda_2}\ln\left(\frac{D_2}{D_1}\right) + \frac{1}{h_b}\right]^{-1}$$

5.2.1 Critical Thickness of Insulation, and the Paradox of cylindrical Insulation

Now, consider what happens when we add a layer of insulation to a cylindrical pipe. The resistance to heat transfer contributed by the insulation is given by

$$R_{\text{ins}} = \frac{\ln\left(\dfrac{r_2}{r_1}\right)}{2\pi H\lambda_{\text{ins}}}$$

Clearly as r_2 increases (and r_1 stays constant), the resistance to heat transfer increases, which is what you would expect.

Consider also the resistance to convection heat transfer at the surface, given by

$$R_b = \frac{1}{h_b 2\pi r_2 H}$$

Here, as r_2 increases, the resistance to heat transfer actually decreases, as the area for heat transfer increases.

So as we add insulation to a pipe, the resistance to conduction heat transfer through the insulation increases, but the resistance to convection heat transfer at the surface decreases. At some point, there will be a minimum resistance, which would correspond to a maximum heat loss. So the paradox of cylindrical insulation is that, it is possible, by adding insulation to a pipe, to actually increase the heat loss!

It is important, therefore, to know at what thickness of insulation you are actually increasing heat loss, so that you can be sure that you put on a greater thickness than this. But in some situations, we can actually use this to our advantage, for example when we want to maximise heat loss to avoid overheating. So electric wires and power cables often have a coating of just the right thickness to maximise cooling.

The radius at which heat loss is a maximum (resistance to heat transfer is at a minimum) is called the critical radius of insulation, and is found by finding the radius at which the differential of the resistance to heat transfer with respect to radius is equal to zero. Consider the heat transfer just through the insulation and into the ambient fluid at temperature T_b. The resistance to heat transfer is given by

$$\begin{aligned} R_{\text{tot}} &= R_{\text{ins}} + R_b \\ &= \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi H \lambda_{\text{ins}}} + \frac{1}{h_b 2\pi r_2 H} \\ &= \frac{\ln\left(\frac{r_2}{r_1}\right) + \frac{\lambda_{\text{ins}}}{h_b r_2}}{2\pi H \lambda_{\text{ins}}} \end{aligned}$$

The heat transfer rate clearly varies as we increase the thickness of insulation, i.e. as we increase r_2. To find when it reaches a maximum, we differentiate the above equation and set it to zero:

$$\frac{\mathrm{d}\, R_{\text{tot}}}{\mathrm{d}\, r_2} = \frac{1}{2\pi H \lambda_{\text{ins}}}\left(\frac{1}{r_2} - \frac{\lambda_{\text{ins}}}{h_b r_2^2}\right) = 0$$

The above expression will only equal zero when the term in the right hand bracket equals zero. So heat loss is as a maximum when

$$\frac{1}{r_2} - \frac{\lambda_{\text{ins}}}{h_b r_2^2} = 0$$

i.e. when

$$\begin{aligned} \frac{1}{r_2} &= \frac{\lambda_{\text{ins}}}{h_b r_2^2} \\ r_2 &= \frac{\lambda_{\text{ins}}}{h_b} \end{aligned}$$

Or, writing the equation more generally

$$r_c = \frac{\lambda_{\text{ins}}}{h_o}$$

where r_c is the critical radius of insulation and h_o is the outside heat transfer coefficient.

So if the radius of insulation on a pipe is equal to the critical radius, heat loss will be maximised. Above the critical radius, adding more insulation reduces the heat loss. Below the critical radius, adding more insulation increases the heat loss. This applies particularly to small pipes; hence small pipes are often left uninsulated. It also applies to cables and wires, but here we use it to advantage to maximise heat loss (as cooler cables offer less electrical resistance).

Also, if insulation becomes wet, this tends to increase its thermal conductivity. As λ_{ins} increases, so does r_c – in other words, the critical radius increases. It may be possible that insulation which is initially thicker than the critical radius becomes equal to the critical radius when wet - which would mean that wet insulation was worse than having no insulation at all. Again, the tutorial problems give an example of this.

The critical radius of insulation of a sphere is given by

$$r_c = \frac{2\lambda_{\text{ins}}}{h_o}$$

You should be able to derive this, following the same logic as used for the cylindrical case. To do so, you would need to use the surface area of a sphere ($= 4\pi r^2$) and the thermal resistance across a spherical shell. You also need to know this for the coursework, so by way of reminder, the thermal resistance across a hollow sphere is given by:

$$R_{\text{sphere}} = \frac{1}{4\pi\lambda}\frac{r_2 - r_1}{r_2 r_1}$$

Note that the critical radius is independent of the thickness of the insulation.

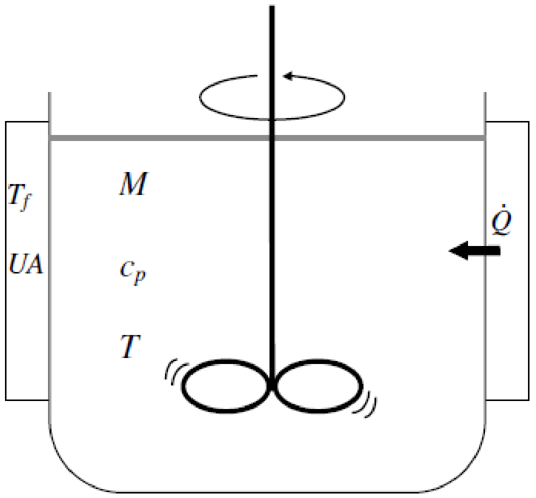
5.3 Transient Heating and Cooling of Mixing Vessels and of Solids of High λ

Another problem that arises frequently in industry is the transient heating or cooling of objects, for example, how long will it take to heat or cool an object or a batch of liquid?

Consider a mixing vessel containing a mass M of liquid (specific heat capacity = C_p) at initial temperature T_0, being heated by an external heating source at temperature T_f according to the general heat transfer equation

$$\dot{Q} = UA\,(T_f - T)$$

How will the temperature of the liquid change with time? How long will it take to achieve a target temperature?



We answer these questions by assuming the liquid to be well mixed (i.e. temperature is even throughout) and by considering an energy balance over a time interval $\Delta\,t$:

$$\begin{array}{ccccc} \left(\begin{array}{c}\text{Energy content}\\ \text{at time } t + \Delta\,t\end{array}\right) & = & \left(\begin{array}{c}\text{Energy content}\\ \text{at time } t\end{array}\right) & + & \left(\begin{array}{c}\text{Energy input by heat}\\ \text{transfer over time interval } \Delta\,t\end{array}\right) \\ MC_pT\,(t + \delta\,t) & = & MC_pT\,(t) & + & \dot{Q}\Delta\,t \end{array}$$

Rearranging

$$\dot{Q} = \frac{MC_p\left(T\left(t+\Delta t\right)-T\left(t\right)\right)}{\Delta t} = MC_p\frac{\left(T\left(t+\Delta t\right)-T\left(t\right)\right)}{\Delta t}$$

As Δt goes to zero, the right hand side becomes the differential of T with respect to time. Also, substituting in for $\dot{Q}$ gives

$$\dot{Q} = UA\left(T_f - T\right) = MC_p\frac{\mathrm{d}T}{\mathrm{d}t}$$

in other words, the rate of heat transfer at any moment is balanced by the rate of change of temperature. To find an explicit expression for temperature at time t, we need to rearrange and integrate:

$$\begin{aligned}
\frac{\mathrm{d}T}{T_f - T} &= \frac{UA}{MC_p}\mathrm{d}t \\
\int_{T_0}^{T(t)}\frac{\mathrm{d}T}{T_f - T} &= \int_0^t\frac{UA}{MC_p}\mathrm{d}t \\
\left[-\ln\left(T_f - T\right)\right]_{T_0}^{T(T)} &= \frac{UA}{MC_p}t \\
-\ln\left(\frac{T_f - T\left(t\right)}{T_f - T_0}\right) &= \frac{UA}{MC_p}t \\
\frac{T_f - T\left(t\right)}{T_f - T_0} &= \exp\left[-\frac{UA}{MC_p}t\right] \\
&= \exp\left[-\frac{t}{\tau}\right]
\end{aligned}$$

where $\tau = MC_p/UA$ is the time constant for the system.

Note that the left hand side expression is a dimensionless temperature expression which varies from 1 initially to zero after an infinitely long time. So the temperature of the vessel contents approaches the heating temperature by exponential decay. The analysis applies whether we are heating or cooling, provided the heat source maintains a constant temperature.

The above analysis would apply, for example, to a steam jacketed vessel, in which the steam temperature stays constant throughout the heating process. Note that we could heat the vessel contents in other ways, for example, by direct steam injection (which would alter the mass of the liquid as it heated), or by direct electrical heating (in which case heat transfer would be at a constant rate and the temperature would increase linearly with time), or by a heating fluid which is simultaneously cooling so that its temperature is also changing.

The above analysis also applies to heating (or cooling) of a solid of very high thermal conductivity (so that the temperature throughout can be considered uniform, equivalent to the assumption above that the liquid is well mixed). This type of analysis is called a lumped system analysis. We can determine whether a lumped system analysis is appropriate for a solid, i.e. whether the temperature distribution in the solid is sufficiently uniform for this sort of analysis, by defining a dimensionless number called the Biot number, Bi, as

$$\mathrm{Bi} = \frac{hL_s}{\lambda_s}$$

where λ_s is the thermal conductivity of the solid and L_s is the characteristic length, given by

$$L_s = \frac{V}{A}$$

i.e. volume divided by surface area for heat transfer. For a sphere, the characteristic length is

$$L_s = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3} = \frac{D}{6}$$

and for a slab heated on both sides it is half the thickness of the slab, $L/2$.

The lumped system analysis will then be accurate within about 5% if Bi < 0.1.

The physical significance of the Biot number becomes clearer if it is written as

$$\text{Bi} = \frac{h}{\frac{\lambda_s}{L_s}}$$

where it is clearly the ratio of heat transfer by convection to the solid to heat transfer by conduction within the solid. So, if heat transfer within the solid is very fast, the temperature within the solid will be fairly uniform, and the Biot number will be small.

The analysis also applies to a fluid flowing along a pipe, in which case time can be replaced by distance down the pipe, resulting in

$$\frac{T_f - T(x)}{T_f - T_0} = \exp\left[-\frac{UA}{\dot{M}C_p}x\right] = \exp\left[-\frac{x}{X}\right]$$

where $X = \dot{M}C_p/UA$ with units of m, where A is the surface area for heat transfer per meter of pipe.

5.3.1 Example

An aluminium plate ($\lambda = 160\,\mathrm{W\,m^{-1}\,K^{-1}}$, $\rho = 2790\,\mathrm{kg\,m^{-3}}$, $C_p = 880\,\mathrm{J\,kg^{-1}\,K^{-1}}$) of thickness $L = 3\,\mathrm{cm}$ and at a uniform intial temperature of 225 °C is suddenly immersed into a well-stirred fluid maintained at a constant temperature of 25 °C. The heat transfer coefficient between the plate and the fluid is $320\,\mathrm{W\,m^{-2}\,K^{-1}}$. Determine the time required for the centre of the plate to cool to 50 °C.

A lumped system analysis is appropriate if Bi < 0.1. The characteristic dimension is given by

$$L_s = \frac{\text{Volume}}{\text{Area}} = \frac{LA}{2A} = \frac{L}{2} = 1.5\,\mathrm{cm}$$

Therefore

$$\text{Bi} = \frac{hL_s}{\lambda_s} = \frac{320 \times 1.5 \times 10^{-2}}{160} = 3 \times 10^{-2}$$

which is less than 0.1, so the lumped system analysis is applicable. The temperature at time t is therefore given by

$$\frac{T_f - T(t)}{T_f - T_o} = \exp\left(-\frac{t}{\tau}\right)$$

where

$$\begin{aligned}\tau &= \frac{\rho V C_p}{UA} \\ &= \frac{\rho C_p L_s}{h} \\ &= \frac{2790 \times 880 \times 1.5 \times 10^{-2}}{320} \\ &= 115.1\,\mathrm{s}\end{aligned}$$

Then, putting the temperatures in:

$$\begin{aligned}\frac{25-50}{25-225} &= \exp(-t/115.1) \\ 0.125 &= \exp(-t/115.1) \\ -2.079 &= -\frac{t}{115.1} \\ t &= 239.3\,\mathrm{s} \\ &\approx 4\,\mathrm{minutes}\end{aligned}$$

5.4 Problems

1. The contents of a jacketed fermentation vessel with a capacity of 30 m^3 of liquid are to be heated using steam condensing in the jacket at 121 °C. The heat transfer area of the jacketed vessel is 25 m^2, and the overall heat transfer coefficient is 1500 $\mathrm{W\,m^{-2}\,K^{-1}}$. The density of the fermentation broth is 1020 $\mathrm{kg\,m^{-3}}$, its specific heat capacity is 3880 $\mathrm{J\,kg^{-1}\,K^{-1}}$, and the broth is well mixed. The initial temperature of the broth is 15 °C. Ignoring the thermal resistance and heat capacity of the vessel walls, calculate the time (in minutes) required to raise the temperature of the broth to 100 °C.

   [85 minutes]

2. After allowing for the heat generated by respiration, a fermenter loses heat at a rate of 8 kW, which must be replaced to maintain the temperature in the fermenter at 37 °C. This heat duty is to be supplied by a thin helical copper coil of internal diameter 1.5 cm passing through the fermenter, carrying hot water at 90 °C. The average velocity of water in the tube is 1.8 $\mathrm{m\,s^{-1}}$. Calculate the heat transfer coefficient in the tube using the Seider and Tate correlation

$$\mathrm{Nu} = 0.027\mathrm{Re}^{0.8}\mathrm{Pr}^{0.333}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

   If the heat transfer coefficient on the outside of the tube is 1500 $\mathrm{W\,m^{-2}\,K^{-1}}$, calculate the overall heat transfer coefficient, heat flow rate per unit length of pipe, and the length of pipe required to achieve the required heat transfer duty. Ignore the thickness and thermal resistance of the copper pipe, and assume the pipe wall temperature to be 70 °C.

   Thermophysical properties of water at 90 °C,

$$\begin{aligned}\lambda &= 0.676\,\mathrm{W\,m^{-1}\,K^{-1}}\\ \rho &= 965\,\mathrm{kg\,m^{-3}}\\ \mu &= 316\times10^{-6}\,\mathrm{Pa\,s}\\ C_p &= 4207\,\mathrm{J\,kg^{-2}\,K^{-1}}\end{aligned}$$

   and viscosity of water at 70 °C $= 405\times10^{-6}$ Pa s

   [12 622 $\mathrm{W\,m^{-2}\,K^{-1}}$, 1341 $\mathrm{W\,m^{-2}\,K^{-1}}$, 3349 W, 2.39 m]

3. (a) Explain why the steady state rate of heat transfer through several layers in series is the same through each layer, even if their thermal conductivities differ, and how this is achieved.

   (b) A cylindrical fermenter is to be maintained at 37 °C. The fermenter is 4 m tall and 1.5 m in diameter, and is made of 5 mm thick stainless steel ($\lambda = 19\,\mathrm{W\,m^{-1}\,K^{-1}}$).

   Calculate the steady state rate of heat loss to the surrounding air at 10 °C, if the liquid side heat transfer coefficient is 150 $\mathrm{W\,m^{-2}\,K^{-1}}$ and the air side heat transfer coefficient is 10 $\mathrm{W\,m^{-2}\,K^{-1}}$. Assume the fermenter is full of liquid, and that heat loss occurs through the sides, top and bottom.

   (c) Estimate the reduction in heat loss that would occur if the fermenter were completely insulated with a 1 cm thick layer of glass wool insulation ($\lambda = 0.038\,\mathrm{W\,m^{-1}\,K^{-1}}$). Ignore the curvature of the fermenter.

   [5.65 kW, 1.63 kW]

4. A 0.008 $\mathrm{m^3}$ laboratory fermenter is to be maintained at 37 °C. The organisms generate heat in the broth at a rate of $10^4\,\mathrm{W\,m^{-3}}$. The fermenter is cylindrical, made of stainless steel, with a surface area for heat transfer of 0.24 $\mathrm{m^2}$. The fermenter loses heat into ambient air at 10 °C, with a convection heat transfer coefficient of $20\,\mathrm{W\,m^{-2}\,K^{-1}}$.

   (a) Calculate the rate of heat transfer from the fermenter. Ignore the thermal resistance of the liquid-side heat transfer and of the stainless steel walls.

   (b) Considering the heat generated by the organisms, show that the rate at which heat must be supplied to maintain the temperature of the fermenter is about 50 W.

   The above fermentation is scaled up to a 100 $\mathrm{m^3}$ fermenter with a surface area for heat transfer of 55 $\mathrm{m^2}$.

   (c) Assuming the same ambient air temperature, convection heat transfer coefficient and rate of heat generation as above, show that the rate at which heat must be removed from the large scale fermenter, to maintain the temperature at 37 °C, is 970 kW.

   (d) The heat is removed from the fermenter using a thin coiled copper tube through which chilled water at 4 °C is pumped. The internal diameter of the tube is 2 cm, and the average velocity of water in the tube is $2\,\mathrm{m\,s^{-1}}$. Using the following correlation, calculate the convection heat transfer coefficient of the water flowing through the coil. Assume the water temperature at the tube wall is 37 °C.

$$\mathrm{Nu} = 0.027\mathrm{Re}^{0.8}\mathrm{Pr}^{0.333}\left(\frac{\mu_b}{\mu_w}\right)^{0.14}$$

   (e) If the heat transfer coefficient on the outside of the tube is $1800\,\mathrm{W\,m^{-2}\,K^{-1}}$, calculate the overall heat transfer coefficient, ignoring the thermal resistance of the tube wall.

   (f) Calculate the length of copper tube required to achieve the necessary heat transfer duty, if the area per metre of pipe is 0.0628 $\mathrm{m^2}$.

   (g) Why must heat be removed from the larger fermenter, but supplied to the smaller one?

   Thermophysical properties of water at 4 °C

$$\begin{aligned}\lambda &= 0.576\,\mathrm{W\,m^{-1}\,K^{-1}}\\ \rho &= 1000\,\mathrm{kg\,m^{-3}}\\ \mu &= 1561\times10^{-6}\,\mathrm{Pa\,s}\\ C_p &= 4210\,\mathrm{J\,kg^{-2}\,K^{-1}}\end{aligned}$$

   and viscosity of water at 37 °C $= 687\times10^{-6}\,\mathrm{Pa\,s}$

   [129.6 W, 9.703×10^5 W, $6594\,\mathrm{W\,m^{-2}\,K^{-1}}$, $1414\,\mathrm{W\,m^{-2}\,K^{-1}}$, 332 m]