WHAT CAUSED THE MILLENNIUM BRIDGE VIBRATING?

Tianjian Ji
University of Manchester Institute of Science and Technology

The unexpected swaying and torsional movements of the London Millennium Bridge brought a mechanics problem, the equilibrium of a cable and rigid-body system, for investigation. This article studies the static characteristics of a cable and rigid-body system, which may explain what was the first push to cause the vibrations of the bridge. As the bridge system allows geometrical movements in the horizontal and rotational directions when it is subjected to asymmetrically distributed **vertical** loads, the pedestrian loading can induce such movements.

The first section studies the static characteristics of a rigid-body hung by cables. It is found that an asymmetrically applied vertical load will induce horizontal and rotational movements of the body hung by inclined cables. A physical model is provided to demonstrate the phenomenon. The second section shows that the movements of the cross-sectional model of the bridge are the same as that of the cable and rigid-body system. Then parametric studies are conducted to examine the effects of several parameters on the movements in the third section. The main conclusions and discussions are given in the fourth section.

1. A Cable and Rigid-Body System

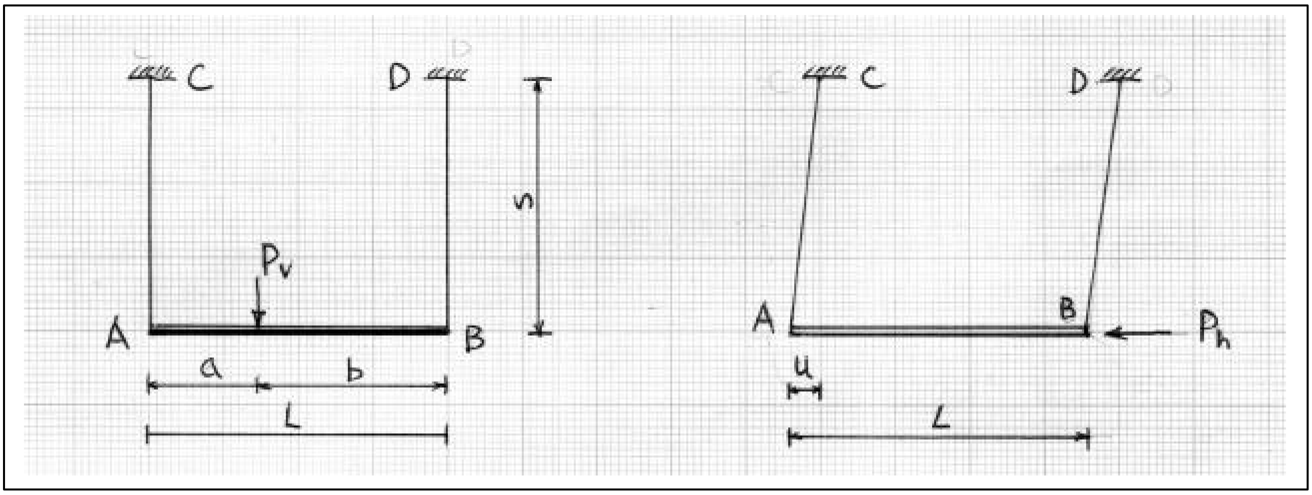
The static and dynamic behaviour of a suspended cable and cable suspended roofs have already been studied [1,2,3] and there are many applications of cables in structures, such as cable suspended bridges and roofs. However, there is little information about a combined cable and rigid-body system. The investigation of the basic behaviour of such a system may lead to an understanding of the movements of the Millennium Bridge.

1.1 A rigid-body hung by two vertical cables

1.1.1 Response to a vertical load

Fig.1 shows a simple cable and rigid body system where the rigid-body is hung by two vertical cables. If a concentrated vertical load is applied on the rigid body, the increments of the cable forces and the elongation of the cables are respectively

$$\Delta V_A = P_v b / L \qquad \Delta V_B = P_v a / L \qquad \Delta_A = \frac{P_v bS}{EAL} \qquad \Delta_B = \frac{P_v aS}{EAL} \tag{1}$$



(a) (b)

Fig.1: A rigid-body hung by two vertical cables

where *EA* is the tension stiffness of the cables. The rotation of the rigid body due to the load P_v is

$$b = \frac{\Delta_A - \Delta_B}{L} = \frac{P_v S(b-a)}{EAL^2}$$

The above equation indicates that a rotation of the rigid body will occur when the load P_v is not applied at the centre of the body. If the cables are considered to be inextensible, *i.e.* $EA \to \infty$. Δ_A, Δ_B and b become nil. Therefore, these are *elastic deformations*.

1.1.2 Response to a horizontal load

If a horizontal load P_h is applied on the rigid-body as shown in Fig.1b, a horizontal displacement of the rigid body occurs in order to reach an equilibrium position. The displacement can be determined based on the equilibrium condition in the horizontal direction

$$u = \frac{P_h S}{2T} \tag{3}$$

where *T* is the initial cable force before the load P_h is applied. Eq.3 shows that the horizontal displacement is proportional to the load P_h, the length of the cable *S* and the inverse of the cable force *T*. The displacement u in the studied case is considered as the *geometric deformation or movement*, which is independent of the material properties of the cables.

1.2 A rigid-body hung by two symmetrically inclined cables

1.2.1 Response to a vertical load

Fig.2a shows that a rigid body is hung by two symmetrically inclined cables and q denotes the angle between the cable force and its vertical component. When $q = 0$, the cables become vertical to the rigid-body which has been studied in the last subsection. The system is in an equilibrium state subjected to the self-weight *W*. A concentrated load P_v is now applied at a distance of *a* from point A (or *x* from the centre). Assume that the system remains equilibrium at the given configuration (Fig.2a), the increments of the cable force can then be worked out based on the equilibrium conditions.

The increments of the vertical components of the cable forces are:

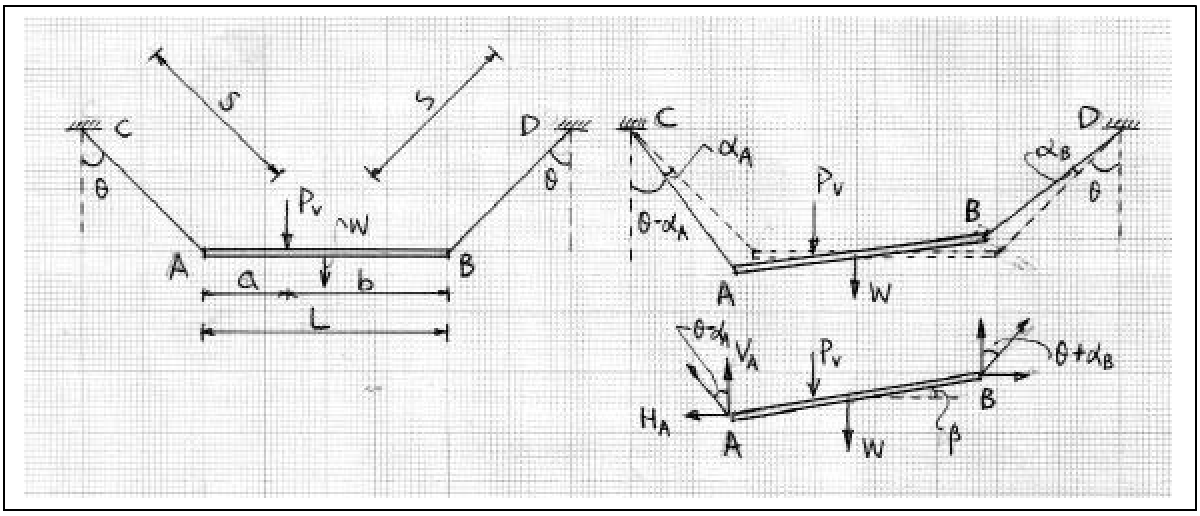
$$\Delta V_A = P_V \frac{b}{L} \qquad \Delta V_B = P_V \frac{a}{L} \tag{4}$$

The increments of the horizontal components can be determined directly from the vertical components:

$$\Delta H_A = \Delta V_B \tan q = P_V \frac{b}{L} \tan q \qquad \Delta H_B = \Delta V_B \tan q = P_V \frac{a}{L} \tan q \tag{5}$$

Checking the equilibrium condition in the horizontal direction gives

$$\Delta H_{AB} = \Delta H_A - \Delta H_B = \frac{P_V\,(b-a)}{L}\tan q = \frac{P_V\,(L-2a)}{L}\tan q \tag{6}$$



(a) (b)

Fig.2: A rigid-body hung by two symmetrically inclined cables

Eq.6 indicates that the system remains equilibrium, shown in Fig.2a, only if $a{=}L/2$, i.e. the load P_v is applied at the centre of the rigid-body. Otherwise, the inequilibrium horizontal force (Eq.6) will alter the configuration (Fig.2a) to a new position in which equilibrium can be achieved as shown in Fig.2b.

The new configuration can be completely described using two independent parameters as the system has two degrees of freedom. a_A and a_B are chosen to describe the movements of the rigid-body around the two fixed symmetric points. The two parameters can be determined based on the equilibrium equations and the geometrical compatibility condition.

Considering the equilibrium of the body at the new position shown in Fig.2b, we have

$$Pb\cos b + 0.5WL\cos b - V_A L\cos b - H_A L\sin b = 0 \tag{7}$$
$$Pa\cos b + 0.5WL\cos b - V_B L\cos b + H_B L\sin b = 0 \tag{8}$$
$$H_A - H_B = 0 \tag{9}$$

where V_A, V_B, H_A and H_B are the vertical and horizontal components of the cable forces subject to the initial static self-weight W and the applied load P_v. The two vertical components of the cable forces can be worked out from Eqs.(7 and 8):

$$V_A = P_V\frac{b}{L} + \frac{W}{2} - H_A\tan b \qquad\qquad V_B = P\frac{a}{L} + \frac{W}{2} + H_B\tan b \tag{10}$$

As the horizontal and vertical components in each cable are not independent, the horizontal components can be written as a function of the vertical ones as follows (see Fig.2b):

$$H_A = V_A \tan(\theta - \alpha_A) \qquad H_B = V_B \tan(\theta + \alpha_B) \tag{11}$$

Substituting Eq.10 into Eq.11 then Eq.11 into Eq.9 yields

$$\frac{(1+\frac{2b}{L}\frac{P_V}{W})\tan(\theta-\alpha_A)}{1+\tan\beta\,\tan(\theta-\alpha_A)} = \frac{(1+\frac{2a}{L}\frac{P_V}{W})\tan(\theta+\alpha_B)}{1-\tan\beta\,\tan(\theta+\alpha_B)} \tag{12}$$

The geometrical compatibility condition requires that the projection of the total length of the new configuration (Fig.2b) on the *x*-axis should be the same as that of the original configuration (Fig.2a). Thus

$$S\sin(\theta-\alpha_A) + S\sin(\theta+\alpha_B) + L\cos\beta = 2S\sin\theta + L \tag{13}$$

The angle β in Eqs.(12 and 13) is not an independent argument, which can be expressed by α_A and α_B as follows:

$$\beta = ArcSin\{\frac{2S}{L}[\sin(\alpha_B/2)\sin(\theta+\alpha_B/2)+\sin(\alpha_A/2)\sin(\theta-\alpha_B/2)]\} \tag{14}$$

Eq.13 implies that the cables are inextensible, as the elongation of the cables is not considered. Substituting Eq.14 into Eqs.(12 and 13) and solving the simultaneous equations (12 and 13), we can obtain the answers of α_A and α_B. The two angles are functions of the cable length S, the length of the rigid-body L, the initial slopθ, the location of the load a and the ratio of the applied load to the self-weight P_V/W. The rotation of the rigid-body is given in Eq.14. The horizontal and vertical movements at the two ends of the rigid-body are respectively

$$u_A = S[\sin(\theta)-\sin(\theta-\alpha_A)] \qquad u_B = S[\sin(\theta+\alpha_B)-\sin(\theta)] \tag{15}$$

$$v_A = S[\cos(\theta-\alpha_A)-\cos(\theta)] \qquad v_B = S[\cos(\theta)-\cos(\theta+\alpha_B)] \tag{16}$$

1.2.2 Response to a horizontal load

Consider that the system is subjected to a horizontal load P_h and assume that the original equilibrium position is held, we can write two equilibrium equations:

$$\begin{aligned} -T_A\sin\theta + T_B\sin\theta - P_h &= 0 \\ T_A\cos\theta + T_B\cos\theta - W &= 0 \end{aligned} \tag{17}$$

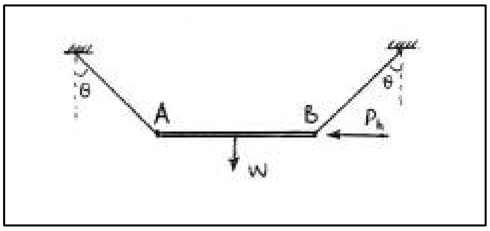


Fig.2c

where T_A and T_B are the cable forces at A and B, which can be obtained directly from Eq.17 that

$$T_A = \frac{1}{2}\left(\frac{W}{\cos\theta} - \frac{P_h}{\sin\theta}\right) \qquad T_B = \frac{1}{2}\left(\frac{W}{\cos\theta} + \frac{P_h}{\sin\theta}\right) \tag{18}$$

Eq.18 indicates that the system subject to P_h can remain the equilibrium at its original position providing that $W\tan\theta > P_h$ as shown in Fig.2c.

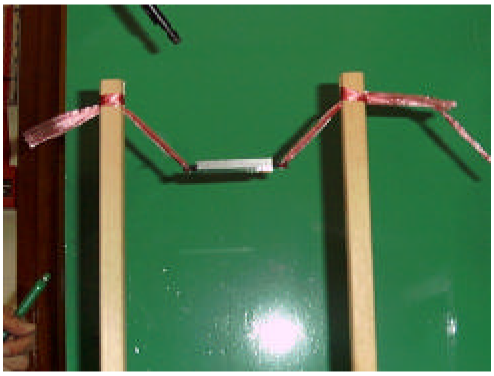
The above study of the rigid-body hung by the two symmetrically inclined cables indicates that:

- The asymmetrically applied vertical load can induce geometrical movements of the body in the horizontal (Eq.15) and rotational (Eq.14) directions
- When the location of the load changes, the equilibrium position of the rigid-body will move accordingly.
- The system has no geometrical movements when it is subjected to a horizontal load.

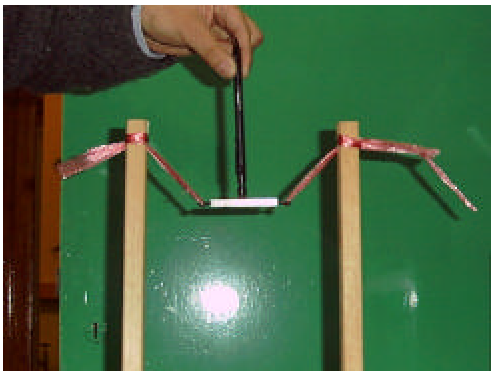
1.3 A demonstration

How the configuration of the system alters to respond to the change of the location of a static vertical load can be physically demonstrated. A simple model is built to show the derived motions in Fig.3. A small piece of steel plate represents the rigid-body and two strings of the same length represent the inextensible cables.

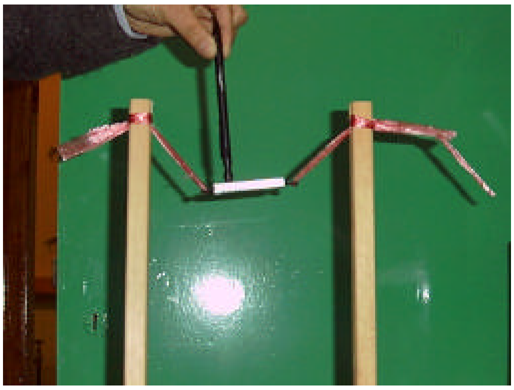
When a static vertical load is applied at the centre of the plate, the equilibrium configuration (Fig.3b) is the same as that before any external loading is applied (Fig.3a). When the load moves to a quarter of the plate, the plate moves horizontally and rotationally from the original position to a new position and reaches an equilibrium state (Fig.3c). When the load moves to the left end of the plate, the plate moves further to the left side (Fig.3d).



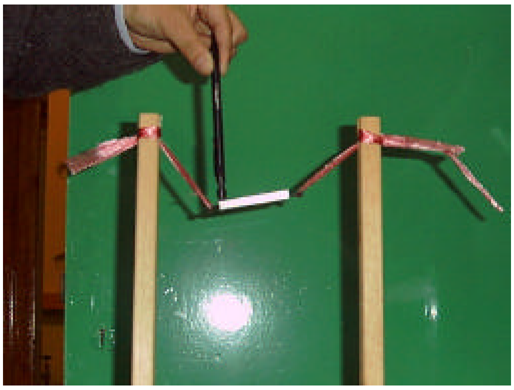
(a) Equilibrium position before loading



(b) A concentrated static load is applied at the centre of the plate



c) A concentrated static load is applied at a quarter of the plate



d) A concentrated static load is applied at the end of the plate

Fig.3: The equilibrium configurations of the cable and rigid-body system subject to a static vertical load at three typical locations

2. A Cross-Sectional Model of the Millennium Bridge

Several assumptions are introduced in order to grasp the main characteristics of the bridge, simplify the mathematical model and obtain qualitative conclusions. A sketch of the cross-section of the bridge at its main span is shown in Fig.4a [4]. If a straight line links the supports of each set of cable, then the points C and D will appear in the cross section.

The assumptions introduced and their limitations are given as follows:

1. The swaying, vertical and rotational movements of the main span of the bridge can be approximately represented by that of a typical slice of the bridge

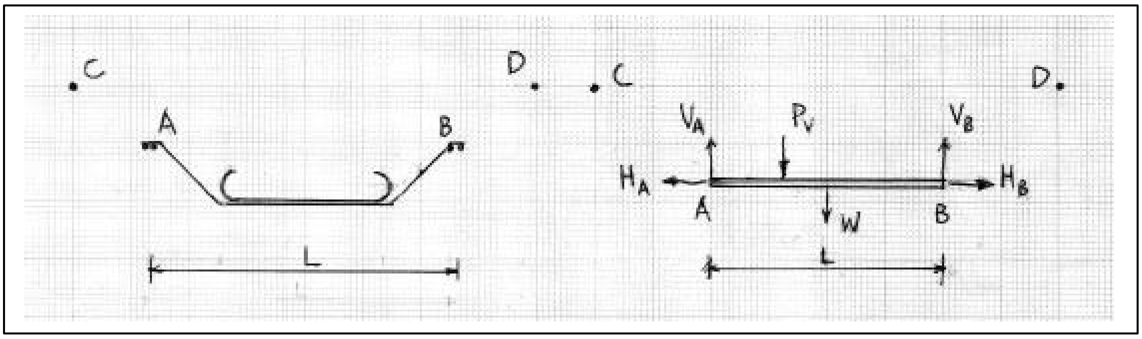
It is not an ideal plane strain problem as the cross-section of the bridge varies gradually along the span and the piers provide supports at the ends of the main span of the bridge. Considering that the main span of the bridge is of 144 meters long, the supports may not effectively restrain the movements of the middle part of the bridge. This simplification allows to studying the motion using a 2D model, which should capture the main features of the movements of the bridge, but the movements obtained based on this assumption may be larger than they should be.

2. The cables are inextensible

When inextensible cables are considered, there is no elastic deformation and the vertical movement of the deck due to cable deformation is neglected. However, the vertical movement of the deck due to the rotation of the bridge is still included in the analysis.

3. The deck and its wing support are considered as a rigid-body in the sway and torsional movements of the bridge.

The deformations of the deck and its supporting wing are much smaller than the movements of the cables in the swaying and torsional directions. The assumption allows to focusing on the effect of the cables.



(a) (b)

Fig.4: A sketch of the cross-section of the bridge and its simplified model

4. The swaying and rotational movements of the rigid body (the deck and supporting wing) can be considered as rotations around two symmetric remote points.

The lateral movement (v) of the body and the rotation (b) of the body around its centre can be equivalently described by two other independent variables as the system has two degrees of freedom. The movements of the deck are restrained by two symmetrical sets of cable while the movements of the cables are confined by their supports on the piers. Therefore, the deck will follow the movements of the cables around the straight lines between the cable supports. In the cross section model, the movements of the rigid-body are considered to be determined by two rotation angles around the two given points C and D. However, the distance, S, between the rotation point C (D) and the ends of the body A (B) is unknown. Thus the distance is assumed and the effect of the distance will be evaluated. This assumption may limit the movement of the system.

Based on the above assumptions, the movements of the cross-section model of the bridge is the same as that of the rigid-body hung by the two symmetrically inclined cables studied in the last section (Figs. 2b and 4b). Each of the parameters in Eqs.(12,13) on the geometric movements can be evaluated numerically.

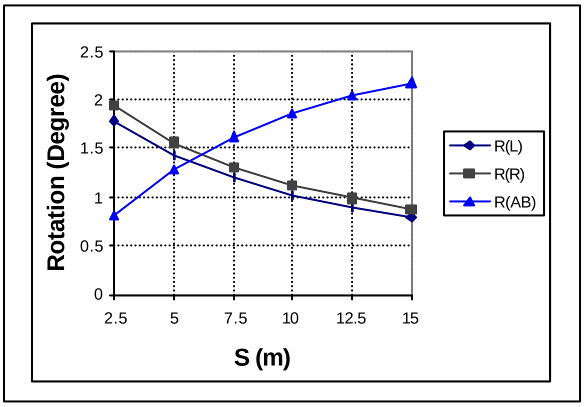
3. Evaluation of the Effects of Related Parameters

For the quantitative evaluation, it is assumed that $L=10m$ [4]. As the structure is symmetric and the deck has a width of 4 metres, the variation of the load location is given $3m \le a \le 5m$ or $0 \le x \le 2m$

The parameters in Eqs.(12 and 13) include the location of the concentrated load (a), the initial slop of the cables (q), the rotation length (S) and the ratio of the applied load to the self-weight (P_V / W). The movements considered are the rotation b of the rigid body (Eq.14) and the horizontal and vertical movements at the two ends of the rigid-body, A and B (Eqs.15 and 16).

1. The effect of the rotation distance S

Consider $q = p/4$, $x=1.0m$ and $P/W = 1$, the movements of the body can be determined as a function of the distance S. Figure 5 shows the rotations, a_A , a_B and b (R(L), R(R) and R(AB) in the Figure), and movements, $u(A)$, $u(B)$, $v(A)$ and $v(B)$, at A and B.



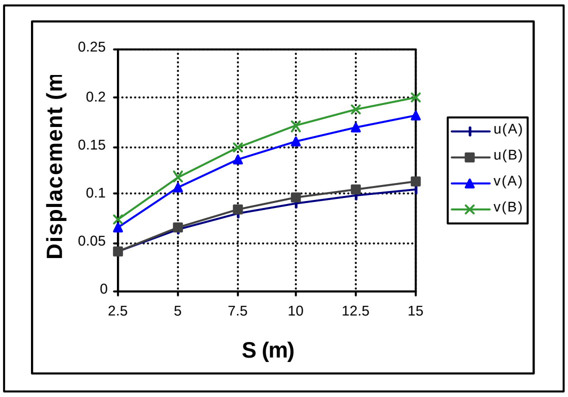


Fig.5: The effect of the rotation distance

It can be observed from Fig.5 that

- As the increase of the rotation distance S, the rotation angles a_A and a_B decrease but the rotation of the body b increases.
- The horizontal and vertical movements at the two ends of the body increase as the distance S increases.

2. The effect of the initial angle q

Here S=7.5m, x=*1m* and $P/W = 1$ are assumed. Figure 6 provides the movements as functions of the angle q

When q is close to 30 degrees the horizontal movements of the body reach their maximum while at 45 degree the rotation of the body has a maximum. Further increasing the slopes (degrees) of the cables leads to decreases of both horizontal and rotational movements. When $q \geq 90^0$, the plane model becomes a mechanism as expected.

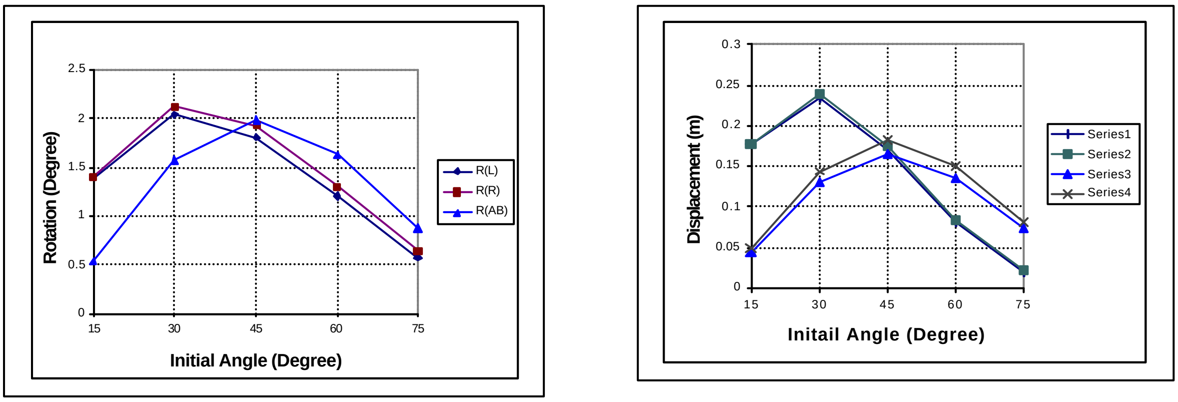


Fig.6: The effect of the initial angle

3. The effect of the location (x) of the applied loading

Let $q = p/4$, S=7.5 m and $P/W = 1$, we can find out the relations between the movements and the load location. Fig.7 shows that the rotations and displacements at the two ends of the body increase almost linearly to the distance between the location of the load and the centre of the deck.

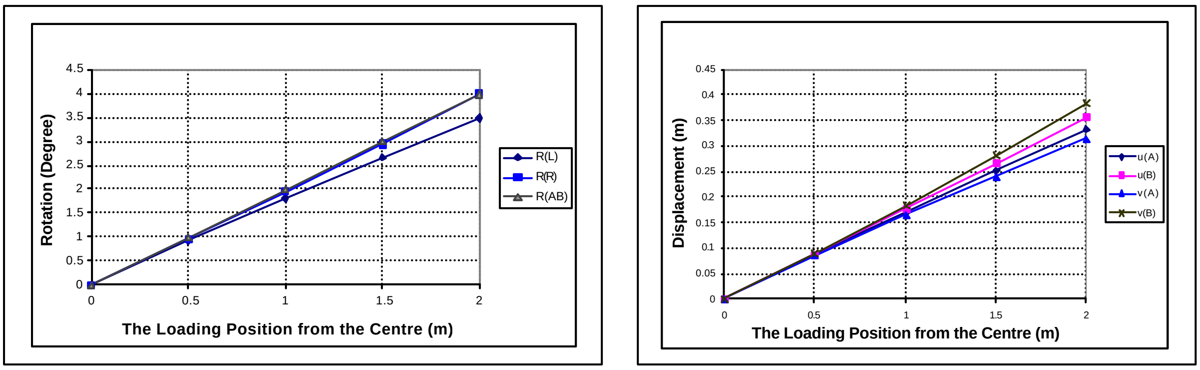


Fig.7: The effect of the loading locations

4. The effect of the ratio of the applied load to the self-weight P_v/W

It is assumed that $q = p/4$, S=7.5m and $x = 1.0m$ in examining the effect due to variation of the ratio of P_v / W. Fig.8 provides the results.

The movements of the body increase as the ratio becomes larger. When the ratio reaches a unit, the rotation of the body is about 2 degrees. Further increase of the ratio does not bring the same degree of changes. For instance when the ratio becomes 2, the rotation of the deck is about 2.6 degrees. The horizontal movements at A and B of the body show the same pattern as the rotation.

Figures (5-8) also show that a 1^0 rotation of the rigid body corresponds approximately to 0.1m movement of the body in the horizontal direction.

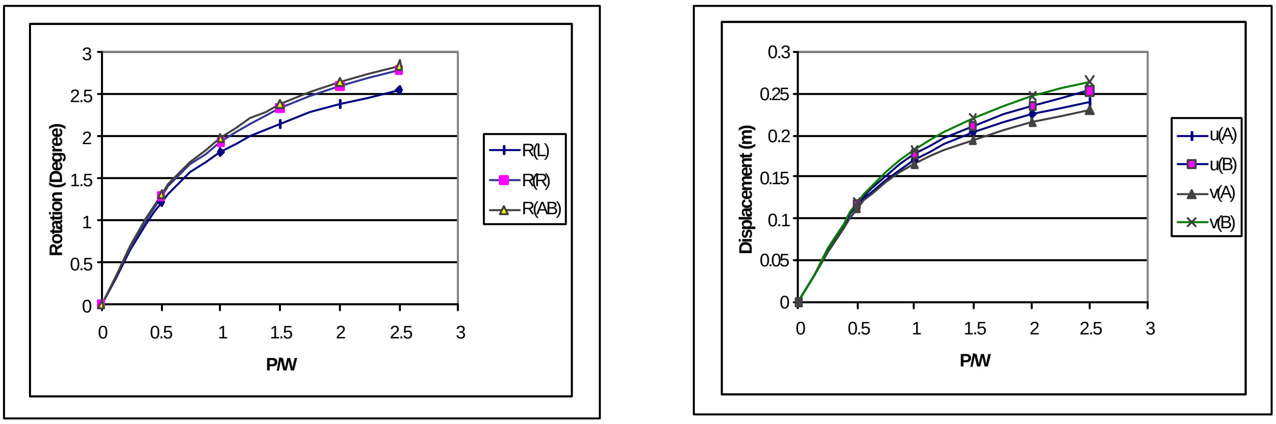


Fig.8: The effect of the ratio of the applied load to the self-weight

4. Discussions and Conclusions

The qualitative and approximate analysis of the movements of the Millennium Bridge is conducted in order to get a better understanding of the use of cables in structures and the cause of the movements. The main conclusions from the study include:

- The structural form of the Bridge allows the geometrical movements in the horizontal and rotational directions when the Bridge is subjected to asymmetrically distributed static vertical loads. Thus the pedestrian loading can induce such movements

When people move onto the bridge, the centre of the static vertical pedestrian loads does not always locate at the centre of the deck. For a small group of people, the ratio P_v / W is low and the bridge deck can balance the difference of the horizontal forces (Eq.6). Thus the bridge has no horizontal movement at this stage. When a larger crowd of people is involved, the difference of the horizontal cable force can not be completely balanced by the bridge deck. In this situation, the bridge has to move to a new position to reach equilibrium. When people move on the bridge, the centre of the vertical pedestrian load changes, which cause the variation of the difference of the horizontal cable forces. Consequently, the equilibrium position changes accordingly.

The bracing arranged under the deck will increase the lateral stiffness of the bridge and reduce the geometrical movements as the stiffened deck can transmit more unbalanced horizontal components of the cable forces.

Once a movement was initiated, the bridge would vibrate at its own frequency. This may be illustrated by considering a pendulum. A pendulum has a nature of swaying and sways at its own frequency once the swaying starts. An initial displacement of the mass, the base (hand) movement, an impact or any horizontal force can start the movement. It is not necessary that the force must be a harmonic with the frequency that is the same as the pendulum frequency.

- The movements become larger when more pedestrians are involved and/or when they move to one side of the bridge (deck).

To reproduce the bridge movements due to a large crowd of people on the openday experimentally, a small group of people involved in the test should move from one side of the deck to another, i.e. crossing the deck, rather than walking along the centre of the bridge.

It seems unavoidable in the normal use of the bridge that a large crowd of people walks on the bridge and many of them move to one side of the bridge due to any attractions over the river.

- When the pedestrians move, the centre of the static vertical loads alters and the equilibrium position of the Bridge changes accordingly, which initiates the vibration of the bridge.

When a structure vibrates, it will oscillate at one of its owns frequencies. Further studies on the vibration needs to consider the dynamic properties of the system, human dynamic loading and even the biodynamic properties of the human bodies.

- Conceptually, preventing the geometrical movements of the bridge can cure the problem identified in the study.

This may be difficult to be implemented in practice, but an understanding of the cause may provide a possibility to generate new thoughts.

There are two ways to deal with a ‘problem’. 1) Identify the cause(s) of the ‘problem’ and then eliminate the cause. However, this may not always be practicable in many situations. Thus an alternative way needs to be considered. 2) Reduce the effect(s) of the ‘problem’. As the users’ main concern is the effect rather than the cause, solutions by this way can also be effective. A combination of dampers and additional cross bracing as proposed for the Bridge should effectively reduce the vibration, but they would not prevent the static geometrical movements.

References

1. Irvine, H M, (1981), Cable Structures, The MIT Press, ISBN 0-262 09023-6
2. Buchholdt, H A, (1985), An Introduction to Cable Roof Structures, Cambridge University Press, ISBN 0 521 30263
3. Krishna, P, (1978), Cable-Suspended Roofs, McGraw-Hill Book Company, ISBN 0-07-035504-5.
4. http://www.arup.com/millenniumbridge/