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Free vibration of rectangular plates with continuously distributed spring-mass

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Abstract

The vibratory characteristics of rectangular plates attached with continuously and uniformly distributed spring-mass in a rectangular region are studied which may represent free vibration of a human–structure system. Firstly, the governing differential equations of a plate with uniformly distributed spring-mass are developed. When the spring-mass fully occupies the plate, the natural frequencies of the coupled system are exactly solved and a relationship between the continuous system and a series of discrete two degrees-of-freedom system is provided. The degree of frequency coupling is defined. Then, the Ritz–Galerkin method is used to derive the approximate solution, when the spring-mass is distributed on a part of the plate, by using the Chebyshev polynomial series to construct the admissible functions. Comparative studies demonstrate the high accuracy and wide applicability of the proposed method. Finally, the frequency and modal characteristics of the plate partially occupied by distributed spring-mass are numerically analysed. It has been observed that both the natural frequencies and the modes appear in pairs. Moreover, a parametric study is performed for rectangular plates with three edges simply supported and one edge free. The effects of occupation size and position of the distributed spring-mass on natural frequencies of the coupled system are studied in detailed. The present investigation provides an improved understanding of human–structure interaction, such as grandstands or floors occupied by a stationary crowd.

Keywords: Vibratory characteristics; Human–structure interaction; Frequency coupling; Rectangular plates; Distributed spring-mass; Ritz–Galerkin solution

1. Introduction

It has been experimentally identified that a person or a crowd in stationary acts at least as a mass-spring-damper on a structure where the interaction between human and the structure needs to be considered

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(Ellis and Ji, 1997). This finding has led to further research of human–structure interaction (Ellis and Ji, 1996; Ji, 2003; Ji and Ellis, 1999). When a crowd of stationary people presents on a grandstand or a long-span floor, the interaction between human and structure can be represented using discrete or continuous models. The simplest discrete model is a two degrees-of-freedom (TDOF) system in which the crowd is modelled as a single degree-of-freedom (SDOF) system and the structure as another SDOF system. The advantage of the model is the simplicity of analysis. On the other hand, it neglects the effect of all higher modes. The contribution of the higher modes may be not significant, but it needs to be validated. A simple continuous model is to consider a grandstand or a floor as an elastic plate while the occupants are modelled as a continuously distributed spring-mass on the plate. In such a case, all modes of vibration of the human-plate system can be considered. Therefore, the continuous model is more close to the actual situation than a discrete TDOF system. For such an engineering background, this paper investigates the dynamic characteristics of a rectangular plate with continuously and uniformly distributed spring-mass.

It is well known in structural dynamics that an undamped analysis is the ground work of a damped analysis. In general, the dynamic response analysis of a damped system can be carried out on the basis of the free vibration analysis of the undamped system. Searching the literature, no single paper is found to study the free vibration characteristics of a plate attached by continuously distributed spring-mass. Some papers studied the free vibration of rectangular plates with elastic/rigid point-supports, based on the energy method, using different approaches such as spline finite strip method (Fan and Cheung, 1984), finite layer method (Zhou et al., 2000), the Lagrangian multiplier (Kim and Dickinson, 1987), the static beam functions (Cheung and Zhou, 1999; Zhou, 2002), the orthogonal polynomials (Liew et al., 1994) and the exact solution (Gershgorin, 1933; Bergman et al., 1993; Zhou and Ji, submitted for publication). Moreover, some papers studied the free vibration of rectangular plates with elastic/rigid concentrated masses by using different methods such as the exact solution (Bergman et al., 1993; Li, 2001, 2003; Zhou and Ji, submitted for publication), the optimal Rayleigh–Ritz method (Avalos et al., 1994) and the mode expansion method (Avalos et al., 1993; Wu and Luo, 1997; Wu et al., 2003; Chiba and Sugimoto, 2003). McMillan and Keane (1996, 1997) investigated the possibility of using attached masses to control the vibration of rectangular plates. Recently, Wong (2002) applied the Rayleigh–Ritz method to study the effect of distributed-mass on the vibration behaviour of a simply supported plate. Kopmaz and Telli (2002) studied the free vibration of simply supported rectangular plates carrying distributed-mass by using the Galerkin method. In addition, Avalos et al. (1989) studied the free vibration of simply supported rectangular plates partially embedded in an elastic foundation.

In the present study, the free vibration of rectangular plates with uniformly distributed spring-mass in a rectangular region is investigated. The relationship between continuous and discrete human–structure systems is demonstrated. A combination of the Ritz method and the Galerkin method is applied to derive the eigenvalue equation. The method is applicable for rectangular plates with arbitrary boundary conditions. Numerical results with high accuracy have been obtained. Some unique dynamic characteristics of a plate and distributed spring-mass system have been observed and discussed.

2. Plates fully occupied by uniformly distributed spring-mass

Consider an arbitrarily shaped uniform plate fully occupied by uniformly distributed spring-mass. The mass and stiffness per unit area of the spring-mass are m and k, respectively. Assuming that the dynamic deflection of the plate is $w(x, y, t)$ and the dynamic displacement of the distributed-mass is $z(x, y, t)$. The governing differential equations of free vibration of the coupled system are, respectively,

$$m\frac{\partial^2 z(x,y,t)}{\partial t^2} = -k[z(x,y,t) - w(x,y,t)] \tag{1}$$

$$D\Delta\Delta w(x,y,t) + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} = k[z(x,y,t) - w(x,y,t)] \tag{2}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $D = \frac{Eh^3}{12(1-v^2)}$ is the flexural stiffness of the plate, h is the plate thickness, E is Young's modulus and v is Poisson's ratio.

The solutions of Eqs. (1) and (2) can be written in the following form:

$$z(x,y,t) = Z(x,y)\mathrm{e}^{-\mathrm{j}\omega t}, \quad w(x,y,t) = W(x,y)\mathrm{e}^{-\mathrm{j}\omega t} \tag{3}$$

where $\mathrm{j} = \sqrt{-1}$ and ω is the natural circular frequency of the coupled system. Substituting Eq. (3) into Eq. (1), the relationship between the distributed-mass mode $Z(x, y)$ and the plate mode $W(x, y)$ can be obtained

$$Z(x,y) = \frac{1}{1-(\omega/\varpi)^2}W(x,y) \tag{4}$$

where $\varpi = \sqrt{k/m}$ is the natural circular frequency of the spring-mass. Eq. (4) shows that the mode of the distributed-mass is proportional to the mode of the plate. The amplitude ratio of the distributed-mass mode to the plate mode equals to $1/[1-(\omega/\varpi)^2]$. This means that if the natural frequency of the coupled system is smaller than that of the spring-mass, the distributed-mass and the plate move in the same direction. However, if the natural frequency of the coupled system is larger than that of the spring-mass, the distributed-mass and the plate move in the opposite directions. Moreover, Eq. (4) shows that the closer the natural frequency of the coupled system to that of the spring-mass, the larger the amplitude ratio of the plate mode and the distributed-mass mode. Substituting Eqs. (3) and (4) into Eq. (2) gives

$$\Delta\Delta W(x,y) - \frac{\rho h}{D}\Lambda W(x,y) = 0 \tag{5}$$

in which,

$$\Lambda = \frac{1+\mu-(\omega/\varpi)^2}{1-(\omega/\varpi)^2}\omega^2 \tag{6}$$

where $\mu = m/(\rho h)$ is the mass ratio of the spring-mass to the plate.

The solution of Eq. (6) is dependent on the sign of eigen-parameter Λ. It is seen from Eqs. (5) and (6) that when $(\omega/\varpi)^2 > 1+\mu$ or $(\omega/\varpi)^2 < 1$, i.e. $\Lambda > 0$, the solution of Eq. (5) is identical to that of free vibration of the bare plate if Λ is regarded as the square of the natural frequency of the bare plate. When $1 < (\omega/\varpi)^2 < 1+\mu$, i.e. $\Lambda < 0$, the solution is the same as the homogeneous solution of the static plate on an elastic foundation with the stiffness $-\rho h\Lambda$. Whereas when $(\omega/\varpi)^2 = 1+\mu$, i.e. $\Lambda = 0$, the solution is just the homogeneous solution of the static plate. In such a case, the elastic force of the distributed spring on the plate just counteracts the inertia force of the plate. It is obvious that the free vibration solution of Eq. (6) exists only for $\Lambda > 0$. In such a case, the modes of the coupled system are the same as those of the bare plate and the eigenvalue Λ equals to the square of the natural frequency ω_p of the bare plate, i.e.

$$\omega_p^2 = \frac{1+\mu-(\omega/\varpi)^2}{1-(\omega/\varpi)^2}\omega^2 \tag{7}$$

It is well known that there is an infinite number of natural frequencies ω_p for a bare plate, which is denoted by ω_{pj} ($j = 1, 2, 3, \ldots, \infty$). Each ω_{pj} corresponds to a mode of the bare plate. Solving Eq. (7) leads to a pair of natural frequencies, ω_{1j} and ω_{2j}, of the coupled system corresponding to each ω_{pj}:

$$\begin{aligned} \omega_{1j}^2 &= \frac{1}{2}\left\{(1+\mu)\varpi^2 + \omega_{pj}^2 - \sqrt{[(1+\mu)\varpi^2 + \omega_{pj}^2]^2 - 4\omega_{pj}^2\varpi^2}\right\} \\ \omega_{2j}^2 &= \frac{1}{2}\left\{(1+\mu)\varpi^2 + \omega_{pj}^2 + \sqrt{[(1+\mu)\varpi^2 + \omega_{pj}^2]^2 - 4\omega_{pj}^2\varpi^2}\right\} \end{aligned} \tag{8}$$

This pair of coupled natural frequencies corresponds to one and the same mode shape of the bare plate. The following relationships can be demonstrated from Eq. (8) (Ellis and Ji, 1997)

$$\omega_{1j}\omega_{2j} = \omega_{pj}\varpi, \quad \omega_{1j} < (\omega_{pj}, \varpi) < \omega_{2j} \tag{9}$$

Eq. (9) shows that

- The product of the natural frequencies of the coupled system is equal to that of the independent plate and distributed spring-mass.

- The first natural frequency of the coupled system is always lower than the smaller one of the two independent systems. However, the second natural frequency of the coupled system is always higher than the bigger one of the two independent systems.

It is observed from Eq. (8) that the frequency solution of the coupled system is equivalent to that of a discrete TDOF system, as shown in Fig. 1, with the following parameters:

$$M_s = mA, \quad M_p = \rho hA, \quad K_s = kA, \quad K_{pj} = M_p\omega_{pj}^2 \tag{10}$$

where A is the area of the plate.

From the above analysis, it can be concluded that when uniformly distributed spring-mass fully occupies the plate, the vibratory characteristics of the plate and distributed spring-mass system can be exactly represented by a series of discrete TDOF systems. It is seen that for the studied case, two masses and the stiffness of the upper mass are constants and independent of the order of the TDOF systems. However, the stiffness K_{pj} of the base mass varies with the order j of the TDOF systems. Therefore, with the increase of the order j, ω_{pj} and K_{pj} monotonically increase.

Eq. (8) can be rewritten in the following form:

$$\begin{aligned}
\omega_{1j}^2 &= \frac{\omega_{pj}^2}{2}\left\{(1+\mu)\frac{\varpi^2}{\omega_{pj}^2}+1-\sqrt{\left[(1+\mu)\frac{\varpi^2}{\omega_{pj}^2}+1\right]^2-4\frac{\varpi^2}{\omega_{pj}^2}}\right\}\\
\omega_{2j}^2 &= \frac{\omega_{pj}^2}{2}\left\{(1+\mu)\frac{\varpi^2}{\omega_{pj}^2}+1+\sqrt{\left[(1+\mu)\frac{\varpi^2}{\omega_{pj}^2}+1\right]^2-4\frac{\varpi^2}{\omega_{pj}^2}}\right\}
\end{aligned} \tag{11}$$

As $\varpi = \sqrt{k/m}$ is a constant, ϖ/ω_{pj} is a small amount for a larger j. In such a case, the following one-order approximation about $(\varpi/\omega_{pj})^2$ can be obtained

$$\sqrt{\left[(1+\mu)\frac{\varpi^2}{\omega_{pj}^2}+1\right]^2-4\frac{\varpi^2}{\omega_{pj}^2}} \approx 1+(\mu-1)\frac{\varpi^2}{\omega_{pj}^2} \tag{12}$$

Substituting the above equation into Eq. (11) gives

$$\omega_{1j} \approx \varpi, \quad \omega_{2j} \approx \omega_{pj} \tag{13}$$

Eq. (13) indicates that for a sufficiently higher mode, the first natural frequency of the TDOF system is close to the natural frequency of the spring-mass and the second is close to that of the bare plate.

According to the above analysis, the natural frequencies of the coupled system can be classified into two groups. With the increase of the pair order, one group of natural frequencies (the first natural frequencies in every pair) monotonically approaches to the natural frequency of the spring-mass from the lower side. Correspondingly, the other group of natural frequencies (the second natural frequencies in every pair) monotonically approaches to that of the bare plate from the upper side.

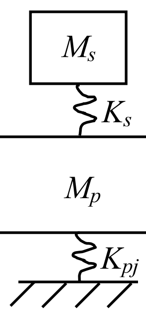


Fig. 1. A series of TDOF systems representing a plate fully occupied by uniformly distributed spring-mass ($j = 1, 2, \ldots, \infty$).

For the given uncoupled frequencies ϖ and ω_{pj}, the coupled frequencies ω_{1j} and ω_{2j} are determined by the mass ratio μ. It is useful in engineering practice to define the degree of frequency coupling of the structure-human interaction. The stronger the frequency coupling, the farther away the natural frequencies of the coupled system from those of the uncoupled systems. This can be explained from Eq. (9). If $\varpi \leqslant \omega_{pj}$, then $\omega_{1j}/\varpi < 1$ and $\omega_{2j}/\omega_{pj} > 1$. Eq. (11) can be rewritten as follows:

$$\begin{aligned}(\omega_{1j}/\varpi)^2 &= \frac{1}{2}\left\{1+\mu+(\omega_{pj}/\varpi)^2 - \sqrt{[1+\mu+(\omega_{pj}/\varpi)^2]^2 - 4(\omega_{pj}/\varpi)^2}\right\}\\ (\omega_{2j}/\omega_{pj})^2 &= \frac{1}{2}\left\{(1+\mu)(\varpi/\omega_{pj})^2 + 1 + \sqrt{[(1+\mu)(\varpi/\omega_{pj})^2+1]^2 - 4(\varpi/\omega_{pj})^2}\right\}\end{aligned} \tag{14}$$

If $\varpi \geqslant \omega_{pj}$, then $\omega_{1j}/\omega_{pj} < 1$ and $\omega_{2j}/\varpi > 1$. Eq. (11) can be rewritten as follows:

$$\begin{aligned}(\omega_{1j}/\omega_{pj})^2 &= \frac{1}{2}\left\{(1+\mu)(\varpi/\omega_{pj})^2 + 1 - \sqrt{[(1+\mu)(\varpi/\omega_{pj})^2+1]^2 - 4(\varpi/\omega_{pj})^2}\right\}\\ (\omega_{2j}/\varpi)^2 &= \frac{1}{2}\left\{1+\mu+(\omega_{pj}/\varpi)^2 + \sqrt{[1+\mu+(\omega_{pj}/\varpi)^2]^2 - 4(\omega_{pj}/\varpi)^2}\right\}\end{aligned} \tag{15}$$

Eqs. (14) and (15) can be, respectively, used to evaluate the degree of frequency coupling. It is seen from Eq. (14) that when $\varpi \leqslant \omega_{pj}$, the larger the deviation of the frequency ratio ω_{1j}/ϖ (or ω_{2j}/ω_{pj}) from 1, the stronger the frequency coupling, as shown in Fig. 2a. It is seen from Eq. (15) that when $\varpi \geqslant \omega_{pj}$, the larger the deviation of the frequency ratio ω_{1j}/ω_{pj} (or ω_{2j}/ϖ) from 1, the stronger the frequency coupling, as shown in

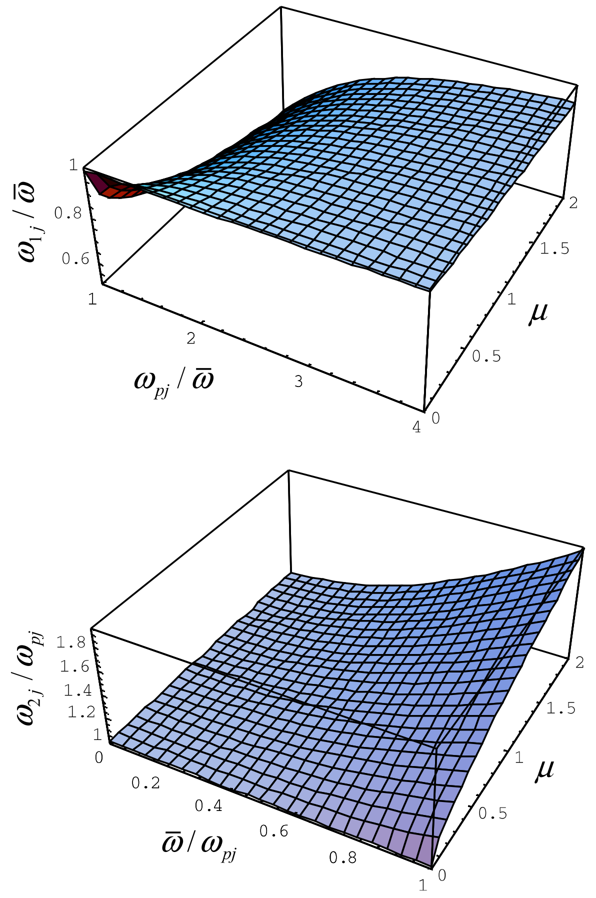


Fig. 2a. Degree of frequency coupling when $\varpi \leqslant \omega_{pj}$. The smaller the frequency ratio ω_{1j}/ϖ or the larger the frequency ratio ω_{2j}/ω_{pj}, the stronger the frequency coupling.

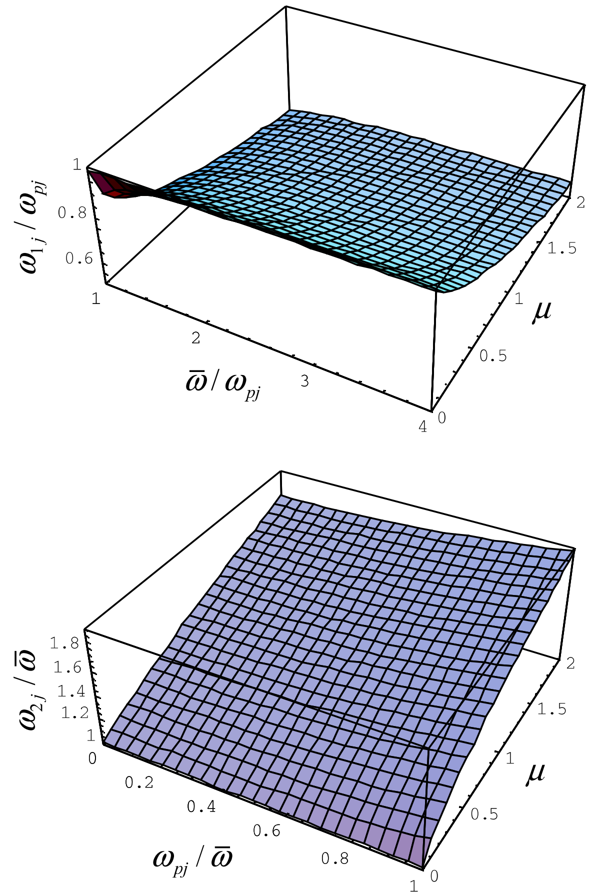


Fig. 2b. Degree of frequency coupling when $\varpi \geqslant \omega_{pj}$. The smaller the frequency ratio ω_{1j}/ω_{pj} or the larger the frequency ratio ω_{2j}/ϖ, the stronger the frequency coupling.

Fig. 2b. It can be observed from Fig. 2 that the closer the natural frequencies of the two uncoupled systems (ϖ/ω_{pj} closes to 1), the stronger the frequency coupling, especially for a bigger mass ratio μ. Moreover, the degree of the frequency coupling when $\varpi < \omega_{pj}$ is more sensitive to the frequency ratio ϖ/ω_{pj} than that when $\varpi > \omega_{pj}$.

3. Rectangular plates partially occupied by uniformly distributed spring-mass

The above analysis is suitable for an arbitrarily shaped plate with any boundary conditions when the plate is fully occupied by the uniformly distributed spring-mass. However, if a plate is partially occupied by the distributed spring-mass, the exact solution of free vibration of the coupled system cannot be obtained. Approximate solutions are thus studied. Consider a rectangular plate with uniformly distributed spring-mass in a rectangular region having the aspect sizes $2a_1$ and $2b_1$, as shown in Fig. 3. The mass and stiffness per unit area of the spring-mass are m and k, respectively. In the present study, the Ritz–Galerkin solution is developed to analyse the coupled vibration of the distributed spring-mass and the rectangular plate with general boundary conditions. In other words, the Ritz method is applied to the energy functional of the rectangular plate and the Galerkin method is used to the motion equation of the distributed-mass. This method is based on such a model that the spring actions are replaced by forces. In such a case, the plate and the distributed-mass can be separated into two systems. One is the plate under the spring-force and the other is the distributed-mass under the spring-force. The spring-forces on the plate and distributed-mass are equal in magnitude but opposite directions.

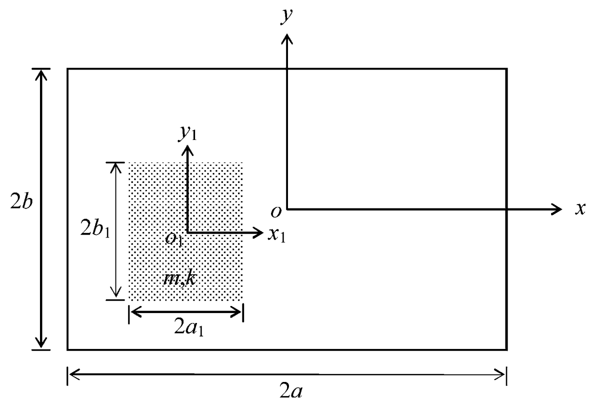


Fig. 3. A rectangular plate with uniformly distributed spring-mass on a rectangular region.

Taking the isolated plate (the spring-force is considered to be the external force), the kinetic energy T_p, the potential energy U_p of the plate and the work W_s of the external force (spring-force) on the plate can be written as follows:

$$T_p = \frac{1}{2}\rho h \int_{-b}^{b}\int_{-a}^{a}\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y,$$
$$U_p = \frac{1}{2}D \int_{-b}^{b}\int_{-a}^{a}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y, \tag{16}$$
$$W_s = k\int_{-b_1}^{b_1}\int_{-a_1}^{a_1} w(x,y,t)r(x_1,y_1,t)\,\mathrm{d}x_1\,\mathrm{d}y_1$$

where $r(x_1, y_1, t)$ is the relative displacement of the distributed-mass to the plate as follows:

$$r(x_1,y_1,t) = z(x_1,y_1,t) - w(x,y,t) \tag{17}$$

In Eq. (16), two sets of Cartesian coordinate systems $oxyz$ and $o_1x_1y_1z$ are used. One set is used to describe the plate deflection and the other set is used to describe the distributed-mass displacement. They have the following relation:

$$x = x_0 + x_1, \quad y = y_0 + y_1 \tag{18}$$

where (x_0, y_0) is the coordinates of the centre of the distributed spring-mass in the plate coordinate system. Taking the following non-dimensional coordinates and parameters:

$$\begin{gathered}\xi = x/a, \quad \eta = y/b, \quad \xi_1 = x_1/a_1, \quad \eta_1 = y_1/b_1, \quad \xi = \xi_0 + \alpha\xi_1\\ \eta = \eta_0 + \beta\eta_1, \quad \xi_0 = x_0/a, \quad \eta_0 = y_0/b, \quad \gamma = a/b, \quad \alpha = a_1/a\\ \beta = b_1/b, \quad K = kb^4/D, \quad \overline{\lambda}^2 = \rho h b^4\overline{\omega}^2/D, \quad \lambda = \omega b^2\sqrt{\rho h/D}\end{gathered} \tag{19}$$

and substituting Eq. (3) into Eq. (16) gives the maximum kinetic energy $T_p^{\max}$, the maximum potential energy $U_p^{\max}$ and the maximum work $W_s^{\max}$ in a vibratory period, respectively,

$$T_p^{\max} = \frac{1}{2}\rho hab\omega^2\int_{-1}^{1}\int_{-1}^{1} W^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$U_p^{\max} = \frac{1}{2}\frac{aD}{b^3}\int_{-1}^{1}\int_{-1}^{1}\left\{\left(\frac{\partial^2 W}{\gamma^2\partial\xi^2}+\frac{\partial^2 W}{\partial\eta^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\gamma^2\partial\xi^2}\frac{\partial^2 W}{\partial\eta^2}-\left(\frac{\partial^2 W}{\gamma\partial\xi\partial\eta}\right)^2\right]\right\}\mathrm{d}\xi\,\mathrm{d}\eta \tag{20}$$
$$W_s^{\max} = a_1b_1k\int_{-1}^{1}\int_{-1}^{1} W(\xi,\eta)R(\xi_1,\eta_1)\mathrm{d}\xi_1\,\mathrm{d}\eta_1$$

where

$$R(\xi_1,\eta_1)=Z(\xi_1,\eta_1)-W(\xi,\eta) \tag{21}$$

The total energy Π of the plate under the spring-force can be expressed as follows:

$$\Pi=T_p^{\max}-U_p^{\max}+W_s^{\max} \tag{22}$$

Assuming that

$$W(\xi,\eta)=\sum_{i=1}^{I}\sum_{j=1}^{J}c_{ij}\varphi_i(\xi)\psi_j(\eta),\quad R(\xi_1,\eta_1)=\sum_{m=1}^{M}\sum_{n=1}^{N}d_{mn}p_m(\xi_1)q_n(\eta_1) \tag{23}$$

where c_{ij} and d_{mn} are the unknown constants, $\varphi_i(\xi)$, $\psi_j(\eta)$, $p_m(\xi_1)$ and $q_n(\eta_1)$ are the admissible functions which will be determined later. The coefficients c_{ij} should be chosen so as to make the functional Π a minimum, from which it follows that

$$\frac{\partial\Pi}{\partial c_{ij}}=0\quad(i=1,2,3,\ldots,I;j=1,2,3,\ldots,J) \tag{24}$$

This is a group of $I\times J$ linear equations with $I\times J+M\times N$ unknown coefficients c_{ij} ($i=1,2,3,\ldots,I$; $j=1,2,3,\ldots,J$) and d_{mn} ($m=1,2,3,\ldots,M$; $n=1,2,3,\ldots,N$) as follows:

$$\sum_{\bar{i}=1}^{I}\sum_{\bar{j}=1}^{J}[B_{\bar{i}\bar{j}ij}-\lambda^2E_{\bar{i}i}^{(0,0)}F_{\bar{j}j}^{(0,0)}]c_{\bar{i}\bar{j}}-K\alpha\beta\sum_{m}^{M}\sum_{n}^{N}G_{mi}H_{nj}d_{mn}=0\quad(i=1,2,3,\ldots,I;j=1,2,3,\ldots,J) \tag{25}$$

where

$$\begin{aligned}
&B_{\bar{i}\bar{j}ij}=E_{\bar{i}i}^{(2,2)}F_{\bar{j}j}^{(0,0)}/\gamma^4+E_{\bar{i}i}^{(0,0)}F_{\bar{j}j}^{(2,2)}+\nu\left(E_{\bar{i}i}^{(0,2)}F_{\bar{j}j}^{(2,0)}+E_{\bar{i}i}^{(2,0)}F_{\bar{j}j}^{(0,2)}\right)/\gamma^2+2(1-\nu)E_{\bar{i}i}^{(1,1)}F_{\bar{j}j}^{(1,1)}/\gamma^2\\
&E_{\bar{i}i}^{(u,v)}=\int_{-1}^{1}\frac{d^u\varphi_{\bar{i}}(\xi)}{d\xi^u}\frac{d^v\varphi_i(\xi)}{d\xi^v}d\xi,\quad F_{\bar{i}i}^{(u,v)}=\int_{-1}^{1}\frac{d^u\psi_{\bar{j}}(\eta)}{d\eta^u}\frac{d^v\psi_j(\eta)}{d\eta^v}d\eta\qquad(u,v=0,1,2)\\
&G_{mi}=\int_{-1}^{1}p_m(\xi_1)\varphi_i(\xi_0+\alpha\xi_1)d\xi_1,\quad H_{nj}=\int_{-1}^{1}q_n(\eta_1)\psi_j(\eta_0+\beta\eta_1)d\eta_1
\end{aligned} \tag{26}$$

It should be mentioned that the functional minimum in Eq. (24) only corresponds to the coefficients c_{ij} but not to the coefficients d_{mn} because d_{mn} are concerned with the spring-force for the isolated plate.

We further consider the isolated distributed-mass system. Substituting Eq. (21) into Eq. (4) gives

$$\lambda^2[W(\xi_0+\alpha\xi_1,\eta_0+\beta\eta_1)+R(\xi_1,\eta_1)]-\bar{\lambda}^2R(\xi_1,\eta_1)=0 \tag{27}$$

Substituting Eq. (26) into the above equation gives

$$\lambda^2\sum_{i=1}^{I}\sum_{j=1}^{J}c_{ij}\varphi_i(\xi_0+\alpha\xi_1)\psi_j(\eta_0+\beta\eta_1)+\lambda^2\sum_{m=1}^{M}\sum_{n=1}^{N}d_{mn}p_m(\xi_1)q_n(\eta_1)-\frac{K}{\mu}\sum_{m=1}^{M}\sum_{n=1}^{N}d_{mn}p_m(\xi_1)q_n(\eta_1)=0 \tag{28}$$

Assuming that $p_m(\xi_1)$ ($m=1,2,3,\ldots$) and $q_n(\eta_1)$ ($n=1,2,3,\ldots$) are two sets of normalized orthogonal series, respectively, with the weight functions $f(\xi_1)$ and $f(\eta_1)$ in the interval $[-1,1]$ as follows:

$$\begin{aligned}
&\int_{-1}^{1}f(\xi_1)p_m(\xi_1)p_{\bar{m}}(\xi_1)d\xi_1=\delta_{m\bar{m}}=\begin{cases}0, & m\neq\bar{m}\\ 1, & m=\bar{m}\end{cases}\\
&\int_{-1}^{1}f(\eta_1)q_n(\eta_1)q_{\bar{n}}(\eta_1)d\eta_1=\delta_{n\bar{n}}=\begin{cases}0, & n\neq\bar{n}\\ 1, & n=\bar{n}\end{cases}
\end{aligned} \tag{29}$$

Eq. (28) can be expanded into

$$\lambda^2\left(\sum_{i=1}^{I}\sum_{j=1}^{J}c_{ij}P_{im}Q_{jn}+d_{mn}\right)-\frac{K}{\varphi}d_{mn}=0,\quad m=1,2,3,\ldots,M,\ n=1,2,3,\ldots,N \tag{30}$$

where

$$P_{im} = \int_{-1}^{1} \varphi_i(\xi_0 + \alpha\xi_1) p_m(\xi_1) f(\xi_1) \mathrm{d}\xi_1, \quad Q_{jn} = \int_{-1}^{1} \psi_j(\eta_0 + \beta\eta_1) q_n(\eta_1) f(\eta_1) \mathrm{d}\eta_1 \tag{31}$$

Eqs. (25) and (30) compose of a group of linear eigenvalue equations, which can be directly solved by using a standard eigenvalue program. The back substitution of the eigenvalues and the corresponding eigenvectors into Eq. (23) gives the mode shapes. It can be seen that a combination of the Ritz method and the Galerkin method has been applied for the derivation of Eqs. (25) and (30). In other words, the Ritz method is applied to the plate vibration analysis through minimizing the energy functional of the plate under the spring-force and the Galerkin method is applied to the motion analysis of the distributed-mass. It should be mentioned that the mass matrix and the stiffness matrix, given by Eqs. (25) and (30), are asymmetric, however only the real eigenvalues and eigenvectors are obtained. This is a physical inevitability because the natural frequencies and modes of free vibration of an undamped structure should be real. In the analysis, the admissible functions are the product of the boundary functions and the Chebyshev polynomial series for the plate

$$\begin{aligned} \varphi_i(\xi) &= (1+\xi)^{r_x}(1-\xi)^{s_x} \cos[(i-1)\arccos(\xi)] \\ \psi_j(\eta) &= (1+\eta)^{r_y}(1-\eta)^{s_y} \cos[(j-1)\arccos(\eta)] \end{aligned} \tag{32}$$

and for the distributed-mass

$$\begin{aligned} p_m(\xi_1) &= \cos[(m-1)\arccos(\xi_1)]/P_m \\ q_n(\eta_1) &= \cos[(n-1)\arccos(\eta_1)]/Q_n \end{aligned} \tag{33}$$

In Eq. (32), the powers r_x, s_x and r_y, s_y are, respectively, determined by the boundary conditions of the plate at $\xi = -1$, $\xi = 1$ and $\eta = -1$, $\eta = 1$. Taking the boundary $\xi = -1$ as an example, $r_x = 2$ means the clamped edge, $r_x = 1$ means the simply supported edge and $r_x = 0$ means the free edge. It is obvious that Eq. (32) can exactly satisfy all the general geometric boundary conditions of the plates.

In Eq. (33),

$$P_m = \begin{cases} \pi/2, & m \neq 0 \\ \pi, & m = 0 \end{cases} \quad \text{and} \quad Q_n = \begin{cases} \pi/2, & n \neq 0 \\ \pi, & n = 0 \end{cases}.$$

It can be seen that $p_m(\xi_1)$ ($m = 1, 2, 3, \ldots$) and $q_n(\eta_1)$ ($n = 1, 2, 3, \ldots$) are two sets of normalised Chebyshev polynomial series, which are orthogonal about the weight functions $f(\xi_1) = 1/\sqrt{1-\xi_1^2}$ and $f(\eta_1) = 1/\sqrt{1-\eta_1^2}$ in the interval $[-1, 1]$, respectively. In other words, they exactly satisfy Eq. (29). The merits of using Chebyshev polynomials to construct the admissible functions in vibration analysis of structures have been discussed (Zhou, 2003).

Utilising Eq. (4), the distributed-mass displacement can be eliminated from the energy functional of the coupled system (the spring-forces are considered as the internal forces) and from the differential equation of the coupled system as given in Eq. (5). The only unknown function is the plate displacement which can be solved using only the Ritz method or only the Galerkin method. However, in such a case, either the Ritz solution or the Galerkin solution will result in a set of nonlinear eigenvalue equations. This will greatly increase the difficulty of solution. Moreover, it is not easy to construct the suitable admissible functions for the Galerkin solution because the admissible functions in the Galerkin method should not only satisfy the displacement boundary conditions but also the force boundary conditions of the plate. In the present study, the rank of the eigenvalue equation is extended from $I \times J$ to $I \times J + M \times N$ using a combination of the Ritz method and the Galerkin method. Therefore, a set of linear eigenvalue equations has been obtained. In the present Ritz–Galerkin method, the admissible functions used in the Ritz solution only need to satisfy the displacement boundary conditions of the plate whereas those used in the Galerkin solution are not constrained by boundary conditions.

4. Comparison and validation

Some papers studied the free vibration of rectangular plates with discrete elements such as concentrated sprung/rigid masses or elastic/rigid point-supports (Wu et al., 2003; Zhou and Ji, submitted for publication;

Wu and Luo, 1997). Moreover, some papers have studied the free vibration of rectangular plates with continuous elements such as elastic foundation and distributed solid mass (Avalos et al., 1989; Kopmaz and Telli, 2002; Wong, 2002). These cases can also be analysed using the present model. For distributed solid mass, a large number can be taken to represent the distributed stiffness k, such as $k = 10^8$. Similarly, for elastic foundation, a large number can be taken to represent the distributed-mass m such as $m = 10^8$. Moreover, for a concentrated spring-mass with mass M and stiffness K_m, a small rectangular region of distributed spring-mass can be taken to represent the concentrated elastic-mass by letting $m = M/(4a_1b_1)$ and $k = K_m/(4a_1b_1)$. In addition, for a concentrated rigid-mass, a large number can be taken to represent K_m and for an elastic point-support, a large number can be taken to represent M. In the following calculations, the numbers of terms of the admissible functions are fixed at $I = J = M = N = 12$ for all cases and $a_1 = 0.001a$, $b_1 = 0.001b$ are taken to approximately describe the region acted by the concentrated mass and/or stiffness. Poisson's ratio is fixed at $v = 0.3$.

The first comparison is for a simply supported square plate with a concentrated rigid-mass at $\xi_0 = \eta_0 = 0.5$. The side lengths of the plate are $2a = 2b = 2.0$ m. The plate thickness is $h = 0.005$ m and the mass per unit area of the plate is $\rho h = 39.25$ kg/m^2. Young's modulus of the plate are $E = 2.051 \times 10^{11}$ N/m^2. The concentrated rigid-mass is $M = 50$ kg. Taking $a_1 = b_1 = 0.001$ m, then $\mu = 318471$. In the numerical computations, the stiffness of the spring-mass is taken as $K_m = 10^6 D/b^2$ to approximately simulate the infinite stiffness. The first five natural frequencies are given in Table 1 and compared with the numerical solutions from Kopmaz and Telli (2002), Wong (2002), Wu and Luo (1997) and the exact solutions from Gershgorin (1933), respectively. It is shown from Table 1 that the present solutions closely agree with the solutions from the references.

The second comparison is for a simply supported rectangular plate with distributed solid-mass occupying a rectangular region of the plate. The aspect ratio of the plate is $\gamma = 2/3$. Two different rectangular regions of the distributed-mass are considered, respectively. The centre of the distributed-mass is at the centre of the plate for both cases. The sizes of one rectangular region are $\alpha = \beta = 0.5$ and the mass ratio is $\mu = 0.4$. The sizes of another rectangular region are $\alpha = \beta = 0.1$ and the mass ratio is $\mu = 10$. In the computations, $k = 10^8$ is used to approximately simulate the infinite stiffness of the distributed solid-mass. The first ten dimensionless frequencies are given in Table 2 and compared with those from Kopmaz and Telli (2002), Wong (2002) and those

Table 1
First five natural frequencies for a simply supported square plate with a concentrated rigid-mass

Method	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_4 (rad/s)	ω_5 (rad/s)
Gershgorin (1933)	31.8248	63.3182	95.4150	127.7414	180.6767
Kopmaz and Telli (2002)	31.7963	63.5739	95.4149	128.2167	181.0288
Wong (2002)	31.8536	63.5505	95.4149	128.0735	180.8910
Wu and Luo (1997)	31.8140	63.2319	95.4148	127.6160	180.5930
Present	31.7996	63.0607	95.3518	127.0902	179.8775

Table 2
First 10 dimensionless frequencies $\Omega_i = \omega_i(2a)^2\sqrt{\rho h/D}$ $(i = 1, 2, \ldots, 10)$ for a simply supported rectangular plate with a regular region occupied by the distributed solid-mass

Order of frequency	$\alpha = \beta = 0.5$			$\alpha = \beta = 0.1$			
	Wong (2002)	Finite element	Present	Wong (2002)	Kopmaz and Telli (2002)	Finite element	Present
1	12.6559	12.6482	12.6571	12.0753	12.0092	11.9991	12.0078
2	25.3867	25.3741	25.3903	26.8918	27.2403	27.2258	27.2395
3	40.6105	40.6087	40.6162	41.9978	43.2103	43.0643	43.0955
4	46.5741	46.5576	46.5777	45.3518	43.5832	43.5774	43.5820
5	54.2811	54.2841	54.2890	56.8956	57.0125	57.0061	57.0125
6	74.6482	74.6520	74.6497	73.7955	78.1580	77.9558	78.0286
7	76.0783	76.1020	76.0966	78.4533	78.4819	78.4653	78.4756
8	87.9530	87.9718	87.9724	86.4268	82.8148	81.5602	81.6536
9	102.5845	102.6260	102.6165	105.2100	105.7888	105.6665	105.7080
10	104.7244	104.8642	104.9153	108.9475	109.5762	109.5413	109.5767

from the finite element method. It is observed from Table 2 that the present solutions closely agree with the published data.

Table 3
First three dimensionless frequencies $\Omega_i = \omega_i(2a)^2\sqrt{\rho h/D}$ $(i = 1, 2, 3)$ of a simply supported rectangular plate partially supported on elastic foundation

Method	$\alpha = 0.25$			$\alpha = 0.5$			$\alpha = 0.75$		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
$\gamma = 2/3$, $K = 31.640625$									
Present	14.565	27.578	44.149	15.879	28.299	44.442	17.145	29.024	44.711
Avalos et al. (1989)	14.565	27.579	44.149	15.880	28.300	44.442	17.146	29.024	44.712
$\gamma = 1$, $K = 6.25$									
Present	19.964	49.439	49.600	20.945	49.847	49.860	21.918	50.102	50.260
Avalos et al. (1989)	19.965	49.439	49.600	20.946	49.848	49.860	21.918	50.102	50.260
$\gamma = 2$, $K = 0.390625$									
Present	49.439	79.115	128.423	49.847	79.275	128.499	50.260	79.430	128.576
Avalos et al. (1989)	49.439	79.115	128.423	49.848	79.275	128.500	50.260	79.431	128.576

Table 4
First four pairs of dimensionless frequencies $\lambda = \omega b^2\sqrt{\rho h/D}$ for a square plate with uniformly distributed spring-mass in a square region

Order of pairs	SSSS plate			CFFF plate		
	First	Second	$1^{st} \times 2^{nd}$	First	Second	$1^{st} \times 2^{nd}$
Independent spring-mass and bear plate						
1	2	4.9348	9.8696	2	0.86776	1.7355
2	(dimensionless frequency	12.337	24.674	(dimensionless frequency	2.1267	4.2533
3	of spring-mass)	12.337	24.674	of spring-mass)	5.3212	10.642
4		19.739	39.478		6.7997	13.599
$\alpha = \beta = 0.2$						
1	1.9696	5.0063	9.8602	0.85737	2.0089	1.7224
2	1.9991	12.340	24.670	1.9908	2.1356	4.2516
3	1.9991	12.340	24.670	1.9975	5.3389	10.664
4	1.9999	19.739	39.477	1.9999	6.8251	13.649
$\alpha = \beta = 0.4$						
1	1.9091	5.1641	9.8587	0.82386	2.0677	1.7035
2	1.9931	12.373	24.660	1.9099	2.2209	4.2417
3	1.9931	12.373	24.660	1.9837	5.3803	10.673
4	1.9984	19.749	39.466	1.9978	6.8774	13.740
$\alpha = \beta = 0.6$						
1	1.8597	5.3053	9.8664	0.76539	2.2220	1.7007
2	1.9828	12.441	24.667	1.7490	2.4194	4.2315
3	1.9828	12.441	24.667	1.9592	5.4368	10.652
4	1.9945	19.788	39.468	1.9904	6.9158	13.765
$\alpha = \beta = 0.8$						
1	1.8367	5.3736	9.8695	0.68705	2.5009	1.7182
2	1.9751	12.492	24.674	1.5346	2.7616	4.2380
3	1.9751	12.492	24.674	1.9224	5.5332	10.637
4	1.9906	19.832	39.478	1.9685	6.9517	13.685
$\alpha = \beta = 1$						
1	1.8330	5.3845	9.8696	0.59936	2.8956	1.7355
2	1.9736	12.502	24.674	1.2919	3.2924	4.2533
3	1.9736	12.502	24.674	1.8566	5.7321	10.642
4	1.9897	19.841	39.478	1.9125	7.1108	13.599

The third comparison is for a simply supported rectangular plate partially supported on elastic foundation. The elastic foundation occupies the left side of the plate, i.e. $\beta = 1$, $\eta_0 = 0$ and $\xi_0 = \alpha - 1$. Three aspect ratios of the plate and three different sizes of the elastic foundation are considered, respectively. In the calculations, the mass ratio is taken as $\mu = 10^8$ to approximately simulate an infinite distributed-mass as the earth. The first three dimensionless frequencies are given in Table 3 and compared with those from Avalos et al. (1989). It is shown from Table 3 that all the results are agreed with those of Avalos et al. (1989).

5. Pairs of frequencies and modes

First, we study a square plate with uniformly distributed spring-mass which is within a square region and symmetrically distributed about the centre of the plate, i.e. $\xi_0 = \eta_0 = 0$. The dimensionless mass and stiffness of the spring-mass are taken as $\mu = 1.0$ and $K = 4.0$, respectively. Two kinds of boundary condition, simply supported (SSSS) and cantilevered (CFFF), and five different sizes of the distributed spring-mass, $\alpha = \beta = 0.2$, 0.4, 0.6, 0.8 and 1, are considered. Numerical results show that the behaviour of the plate partially occupied by spring-mass is similar to that of the plate fully occupied by spring-mass. The natural frequencies of the coupled system can also be classified into two queues, as given in Table 4. It is seen that the product of two frequencies in a pair approximates to a constant. Namely, the size effect of the distributed spring-mass on the frequency

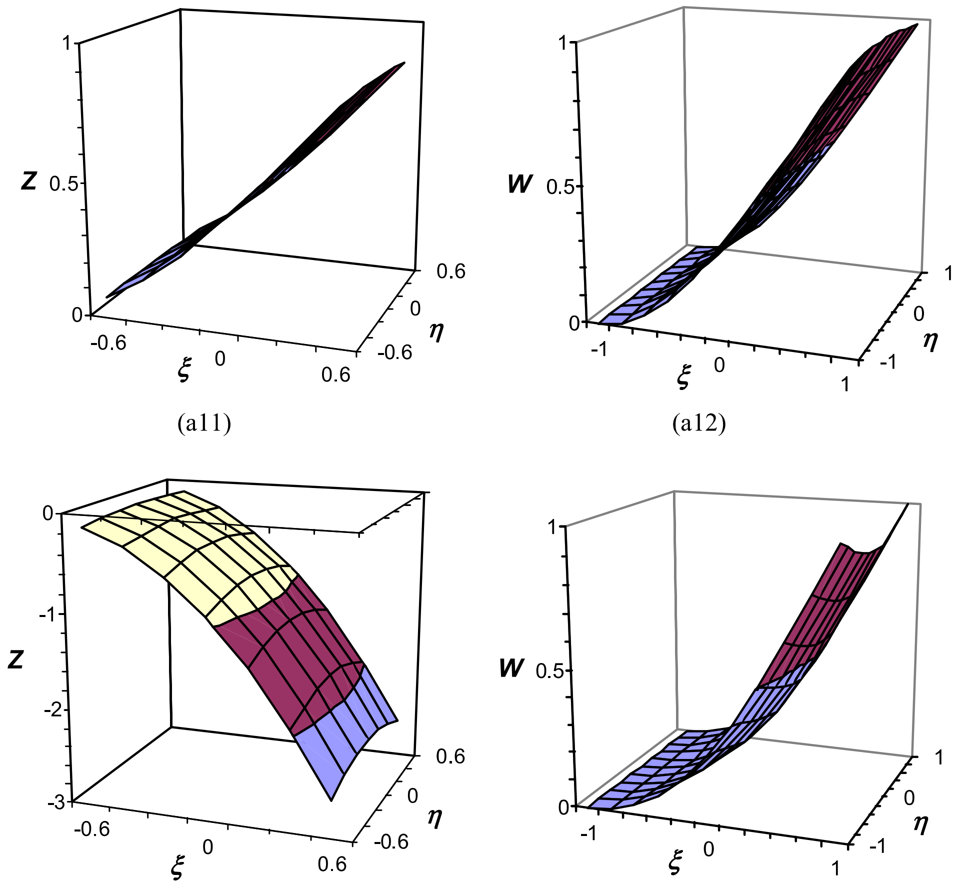


Fig. 4. The modes corresponding to the first pair of frequencies for a CFFF square plate with a square region of uniformly distributed spring-mass. (a11) and (a12) are the modes of the distributed-mass and the plate corresponding to $\lambda = 0.76539$, respectively; (b11) and (b12) are the modes of the distributed-mass and the plate corresponding to $\lambda = 2.2220$, respectively.

product is very small. The mode shapes of the distributed-mass and the cantilevered plate (CFFF), corresponding to the first two pairs of frequencies when $\alpha = \beta = 0.6$, are given in Figs. 4 and 5, respectively. The two-dimensional graph of the plate modes when $\eta = 0$, corresponding to the first pair of frequencies for the SSSS plate, with respect to different sizes of the distributed spring-mass is given in Fig. 6. It can be observed from Table 4 and Figs. 4–6 that

- The coupled natural frequencies appear in pairs. The mode shapes of the plate corresponding to the frequencies in a pair are similar and approximate to that of the bare plate. (If the spring-mass occupies the whole plate, the mode shapes of the coupled system are exactly the same as those of the bare plate.)
- The first frequency in a pair gradually approaches that of the spring-mass from the lower side while the second gradually approaches to those of the bear plate from the upper side, as the order of frequency pairs increases.
- The modes of the crowd and the plate corresponding to the first frequency in a pair are always in the same direction. However, the modes of the crowd and the plate corresponding to the second frequency in a pair are always in the inverse directions.

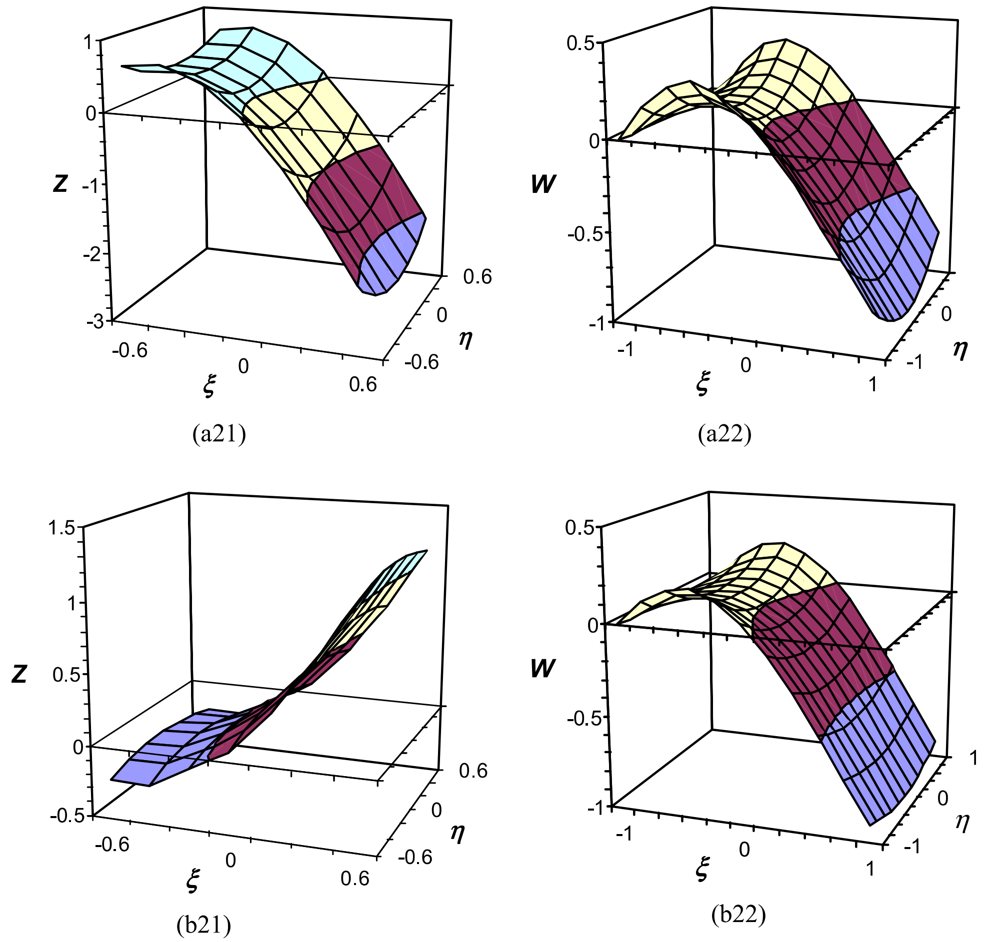


Fig. 5. The modes corresponding to the second pair of frequencies for a CFFF square plate with a square region of uniformly distributed spring-mass. (a21) and (a22) are the modes of the distributed-mass and the plate corresponding to $\lambda = 1.7490$, respectively; (b21) and (b22) are the modes of the distributed-mass and the plate corresponding to $\lambda = 2.4194$, respectively.

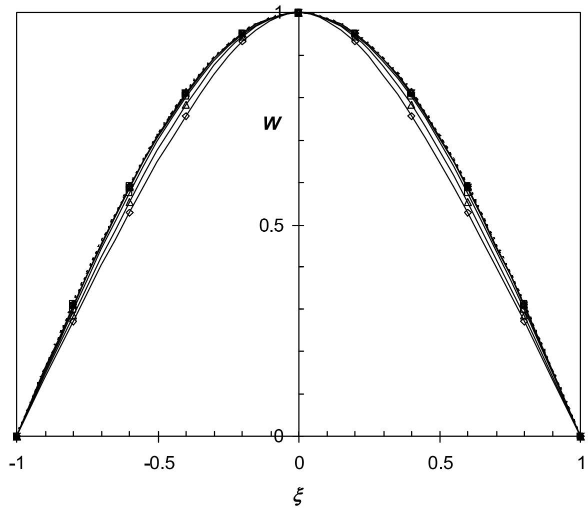


Fig. 6. The modes of the plate at $\eta = 0$, corresponding to the first pair of frequencies for a SSSS square plate with a square region of uniformly distributed spring-mass: (—) the mode corresponding to the first frequency, (-··-) the mode corresponding to the second frequency. ◊ $\alpha = \beta = 0.2$, △ $\alpha = \beta = 0.4$, □ $\alpha = \beta = 0.6$, ○ $\alpha = \beta = 0.0$ or 1.0.

- The frequency coupling between the spring-mass and the plate mainly appears in the first few pairs of frequencies, especially in the first pair of frequencies. With the increase of the order of frequency pairs, the degree of the frequency coupling between plate and spring-mass monotonically decreases.

6. Parametric study

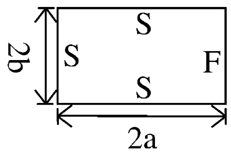
A rectangular plates with three edges simply supported and one edge free at $x = a$ (SSSF), partially occupied by uniformly distributed spring-mass in a rectangular domain, are taken as example to show the effect of various parameters on the coupled frequencies. Such a model can simulate a panel of the grandstand with a crowd of spectators. Three different aspect ratios of the plate are considered: $\gamma = 1$, 1.5 and 2.

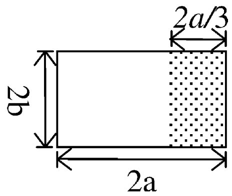
Table 5 presents the effect of the sizes of the spring-mass on the first three pairs of frequencies. The mass ratio and dimensionless stiffness of the spring-mass are $\mu = 1.0$ and $K = 10.0$, respectively. The spring-mass always occupies the whole length of the plate in the y direction ($\beta = 1$, $\eta_0 = 0$) but takes different lengths beginning from the free edge in the x direction ($\xi_0 = 1 - \alpha$). Three different lengths of spring-mass in the x direction, as shown in Table 5, are considered: $\alpha = 1/3$, 2/3 and 1 and the results are compared with those of the bare plate. It is shown in Table 5 that the degree of frequency coupling increases monotonically with the size increase of the spring-mass and the frequency coupling becomes strongest when the spring-mass occupies the whole plate.

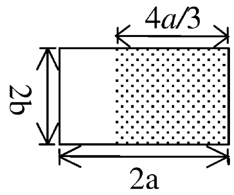
Table 6 shows the effect of the position of spring-mass on the first three pairs of coupled frequencies. The distributed spring-mass occupies a half of the plate. Three cases have been considered, the right half, the left half and the upper half, as shown in Table 6. The results in Table 6 show that the frequency coupling is strongest in the first case and lowest in the second case. In other words, the higher the contribution of spring-mass in free vibration the stronger the frequency coupling.

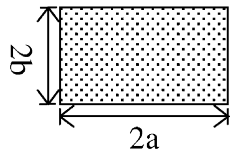
Table 7 gives the effect of the density of spring-mass on the first three pairs of frequencies, i.e. proportionally increasing the mass and stiffness of the distributed spring-mass. The dimensionless frequency of the spring-mass is fixed at $\bar{\lambda} = \sqrt{10}$ and the aspect ratio of the plate is $\gamma = 1.5$. Three different mass ratios and dimensionless stiffnesses are investigated: $\mu = 0.5$, $K = 5$; $\mu = 1$, $K = 10$ and $\mu = 2$, $K = 20$. The spring-mass occupies the whole length in the y direction ($\beta = 1$, $\eta_0 = 0$). Four different lengths of spring-mass in the x direction, as shown in Table 7, are considered: $\alpha = 1/4$, 1/2, 3/4 and 1. It is seen from Table 7 that the frequency coupling monotonically increases with the increase of distributing density of the spring-mass.

Table 5
First three pairs of frequencies $\lambda = \omega b^2\sqrt{\rho h/D}$ for SSSF rectangular plates with respect to the length of spring-mass beginning from the free end in the x direction ($\mu = 1.0$, $K = 10.0$, $\overline{\lambda} = 3.162$, $\beta = 1$, $\eta_0 = 0$)



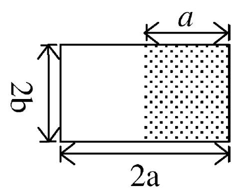


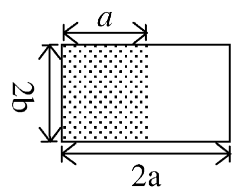


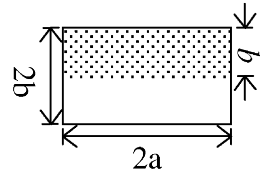


Order of pairs	λ_p of bare plate	$\alpha = 1/3,\ \xi_0 = 2/3$		$\alpha = 2/3,\ \xi_0 = 1/3$		$\alpha = 1,\ \xi_0 = 0$	
		First	Second	First	Second	First	Second
$\gamma = 1$							
1	2.921	1.986	4.492	1.861	4.954	1.843	5.013
2	6.939	3.040	7.782	2.901	7.546	2.830	7.754
3	10.30	3.090	10.64	3.016	10.792	3.011	10.82
$\gamma = 1.5$							
1	2.668	1.842	4.026	1.733	4.821	1.716	4.915
2	4.575	2.946	5.383	2.566	5.588	2.448	5.910
3	8.424	3.024	8.659	3.006	8.846	2.936	9.074
$\gamma = 2$							
1	2.575	1.775	3.590	1.683	4.646	1.668	4.882
2	3.692	2.759	4.952	2.297	5.041	2.172	5.376
3	5.905	3.015	6.285	2.869	6.603	2.709	6.893

Table 6
First three pairs of frequencies $\lambda = \omega b^2 \sqrt{\rho h/D}$ for SSSF rectangular plates with respect to different positions of the distributed spring-mass occupied a half of the plate

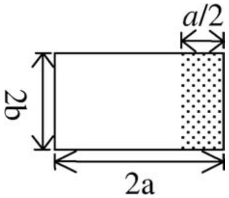
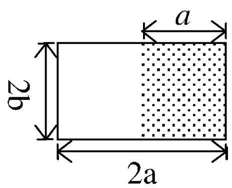
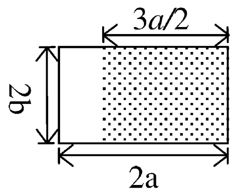
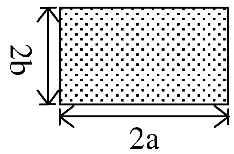






Order of pairs	$\alpha = 1/2,\ \xi_0 = 1/2,\ \beta = 1,\ \eta_0 = 0$		$\alpha = 1/2,\ \xi_0 = -1/2,\ \beta = 1,\ \eta_0 = 0$		$\beta = 1/2,\ \eta_0 = 1/2,\ \alpha = 1,\ \xi_0 = 0$	
	First	Second	First	Second	First	Second
$\gamma = 1$						
1	1.902	4.794	2.440	3.546	2.106	4.303
2	2.996	7.343	3.092	7.414	2.955	7.356
3	3.024	10.74	3.120	10.37	3.117	10.57
$\gamma = 1.5$						
1	1.769	4.479	2.219	3.245	1.969	4.198
2	2.738	5.398	2.987	5.442	2.682	5.320
3	3.012	8.823	3.062	8.726	3.021	8.751
$\gamma = 2$						
1	1.713	4.043	2.083	4.800	1.916	4.164
2	2.489	5.010	2.758	6.433	2.439	4.709
3	3.006	6.556	3.032	9.597	2.875	6.422

Table 7
First three pairs of frequencies $\lambda = \omega b^2\sqrt{\rho h/D}$ for a SSSF rectangular plate ($\gamma = 1.5$) when proportionally increasing the mass and stiffness densities of the distributed spring-mass ($\beta = 1$, $\eta_0 = 0$)

Order of pairs								
	$\alpha = 1/4$, $\xi_0 = 3/4$		$\alpha = 1/2$, $\xi_0 = 1/2$		$\alpha = 3/4$, $\xi_0 = 1/4$		$\alpha = 1$, $\xi_0 = 0$	
	First	Second	First	Second	First	Second	First	Second
$\mu = 0.5$, $K = 5$								
1	2.142	3.658	2.024	4.062	1.984	4.249	1.977	4.266
2	3.096	4.954	2.913	4.992	2.749	5.251	2.705	5.349
3	3.099	8.506	3.084	8.624	3.072	8.670	3.042	8.758
$\mu = 1$, $K = 10$								
1	1.905	3.827	1.769	4.479	1.724	4.885	1.716	4.915
2	3.035	5.344	2.738	5.398	2.507	5.745	2.448	5.910
3	3.042	8.591	3.012	8.823	2.990	8.906	2.936	9.074
$\mu = 2$, $K = 20$								
1	1.623	3.982	1.479	4.926	1.432	5.860	1.425	5.922
2	2.923	6.039	2.487	6.210	2.194	6.534	2.124	6.811
3	2.942	8.778	2.883	9.217	2.846	9.357	2.757	9.661

7. Conclusions

This paper investigates the characteristics of free vibration of rectangular plates with uniformly distributed spring-mass. The spring-mass is in a rectangular region, which can occupy a part or whole plate. This model represents a flat floor or a grandstand occupied by a crowd of people. The vibration interaction between a uniform rectangular plate and distributed spring-mass is solved by the combined use of the Ritz method and Galerkin method. The main conclusions obtained are summarized as follows:

- The mode shape of the uniformly distributed-mass is equal to that of the plate multiplied by a magnitude ratio in the attached region. The magnitude ratio corresponding to the first frequency in a pair is always positive and that corresponding to the second frequency is always negative.
- The natural frequencies and modes of the coupled system appear in pairs. The mode shapes of the plate corresponding to the frequencies in a pair are similar to the related mode of the bare plate. If the spring-mass occupies the whole plate, the mode shapes of the plate corresponding to the frequencies in a pair are the same as that of the bare plate.
- The vibratory characteristics of the coupled system can be approximately represented by a series of discrete TDOF systems. If the spring-mass occupies the whole plate, this representation is exact. This provides a theoretical background for the study of human–structure interaction using a TDOF system.
- The frequency coupling between spring-mass and plate mainly appears in the first several pairs of frequencies, especially in the first pair of frequencies.
- With the increase of the occupation of the spring-mass on a plate, the frequency coupling of the plate and spring-mass monotonically increases. The frequency coupling reaches to the maximum when the spring-mass occupies the whole plate.

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