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International Journal of Solids and Structures xxx (2005) xxx–xxx

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Dynamic characteristics of a beam and distributed spring-mass system

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Received 4 May 2005

Abstract

This paper studies dynamic characteristics of a beam with continuously distributed spring-mass which may represent a structure occupied by a crowd of people. Dividing the coupled system into several segments and considering the distributed spring-mass and the beam in each segment being uniform, the equations of motion of the segment are established. The transfer matrix method is applied to derive the eigenvalue equation of the coupled system. It is interesting to note from the governing equations that the vibration mode shape of the uniformly distributed spring-mass is proportional to that of the beam at the attached regions and can be discontinuous if the natural frequencies of the spring-masses in two adjacent segments are different. Parametric studies demonstrate that the natural frequencies of the coupled system appear in groups. In a group of frequencies, all related modes have similar shapes. The number of natural frequencies in each group depends on the number of segments having different natural frequencies. With the increase of group order, the largest natural frequency in a group monotonically approaches the natural frequency of corresponding order of the bare beam from the upper side, whereas the others monotonically move towards those of the independent spring-mass systems from the lower side. Numerical results show that the frequency coupling between the beam and the distributed spring-mass mainly occurs in the low order of frequency groups, especially in the first group. In addition, vibratory characteristics of the coupled system can be approximately represented by a series of discrete multi-degrees-of-freedom system. It also demonstrates that a beam on Winkler elastic foundation and a beam with distributed solid mass are special cases of the proposed solution.

Keywords: Free vibration; Beam; Distributed spring-mass; Human–structure interaction; Transfer matrix method; Exact solution

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.09.002

1. Introduction

It has been experimentally identified that a person or a crowd acts at least as a spring-mass-damper on a structure when the person or the crowd is stationary, such as sitting or standing, where the interaction between human and the structure needs to be considered (Ellis and Ji, 1997). This finding has led to further research of human–structure interaction. The challenging and fascinating aspects of the study lie in that a slightly damped structure system and a highly damped human body system are combined to form a new system (Ji and Ellis, 1999; Ji, 2003; Ellis and Ji, 1996). When a crowd of stationary people occupies a cantilever grandstand, a long-span floor or a footbridge, the interaction between human and structure can be represented using either discrete or continuous models. The simplest discrete model is a two degrees-of-freedom (TDOF) system in which the crowd is modelled as a single degree-of-freedom (SDOF) system and the structure as another SDOF system. The advantage of the model is that the high damping ratio of the crowd can be considered without too much difficulty. On the other hand, it neglects the effect of all higher modes of the structure. The contribution of the higher modes may not be significant, but this needs to be validated. The simplest continuous model is to consider the structure as a continuous beam while the occupants are modelled as a continuously distributed spring-mass on the beam. Hence, all modes of vibration of the human-beam system can be considered. For such an engineering background, this paper investigates the dynamic characteristics of an Euler–Bernoulli beam with continuously distributed spring-mass.

A number of papers studied free vibration of beams attached by discrete spring-masses or rigid masses. The exact solutions for free vibration of shear beams, Euler–Bernoulli beams and Timoshenko beams with rigid or elastic concentrated masses were reported (Li, 2000; Chen, 1963; Goel, 1973; Rossit and Laura, 2001a,b). Wu and his co-workers (Wu et al., 1999; Wu and Chou, 1998, 1999; Chen and Wu, 2002; Wu and Chen, 2001) developed an analytical–numerical method to study the free vibration of uniform or non-uniform Euler–Bernoulli beams and Timoshenko beams with concentrated spring-masses. Low (2000, 2003) investigated approximate estimations of natural frequencies of a beam carrying concentrated masses. The vibratory characteristics of uniform or non-uniform beams carrying TDOF spring-mass systems were studied (Qiao et al., 2002; Wu, 2002, 2004; Wu and Whittaker, 1999). Drexel and Ginsberg (2001) investigated the effect of modal overlap and dissipation in a cantilevered beam attached by multiple spring-mass-damper systems. Chai et al. (1995) studied the tension effect of clamped beams carrying a concentrated mass on the natural frequencies. Chan and Zhang (1995) investigated the natural frequencies of a cantilever tube partially filled with liquid theoretically and experimentally. Chan et al. (1996) and Chan and Wang (1997) investigated the vibratory characteristics of a simply supported Euler–Bernoulli beam and of a cantilevered Timoshenko beam with distributed rigid mass, respectively.

In this paper, free vibration characteristics of a non-uniform beam with arbitrarily distributed spring-mass are studied. Dividing the system into several segments and approximately considering all the parameters in each segment being constant, the solution can be derived by using the transfer matrix method. The essence of the coupled vibration of a beam and distributed spring-mass is studied in detail. It is found that the natural frequencies of a beam with distributed spring-mass appear in groups and its vibratory characteristics can be equivalently represented by a series of discrete spring-mass system. It also shows that a beam on Winkler elastic foundation and a beam with distributed solid mass are the special cases of the proposed solution when the mass and stiffness of the distributed spring-mass become infinity, respectively.

2. Governing differential equation

Consider a beam with a varying cross-section guaranteeing the continuity of the neutral axis of the beam and attached by continuously distributed spring-mass, as shown in Fig. 1(a). Firstly, a segment with the length of l_i is isolated from the system, as shown in Fig. 1(b), and all parameters of the segment are assumed to be constants. Separating the attached spring-mass from the beam segment and remaining the actions of the spring-mass on the beam, the governing differential equations of the beam segment and the distributed spring-mass on the segment are given as follows, respectively:

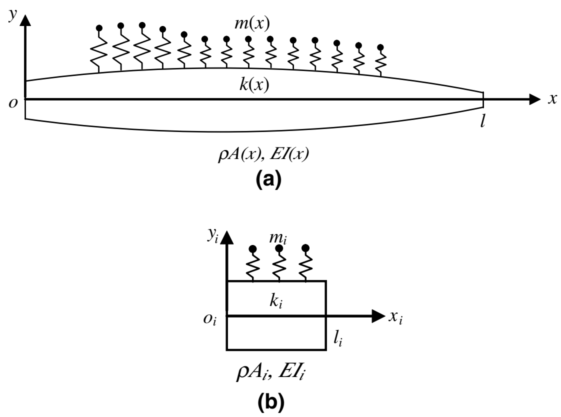

Fig. 1. (a) A beam with distributed spring-mass, and (b) a segment taken from the system and approximately considered to be uniform.

$$EI_i\frac{\partial^4 y_i}{\partial x_i^4}+\rho A_i\frac{\partial^2 y_i}{\partial t^2}=-k_i(y_i-z_i),\quad 0\leqslant x_i\leqslant l_i, \tag{1}$$

$$m_i\frac{\partial^2 z_i}{\partial t^2}=k_i(y_i-z_i),\quad 0\leqslant x_i\leqslant l_i, \tag{2}$$

where $y_i = y_i(x_i, t)$ is the displacement of the beam, EI_i is the flexural stiffness and ρA_i is the mass per unit length of the ith beam segment. $z_i = z_i(x_i, t)$ is the displacement of the distributed mass m_i, k_i is the stiffness of the distributed spring on the ith segment. It should be mentioned that for clamped thin beams carrying large distributed mass the tension effect should be considered in the vibration analysis (Chai et al., 1995). This situation is not included in the present study.

When the beam-spring-mass system experiences free vibration, the solutions of the above equations have the following form:

$$y_i(x_i,t)=Y_i(x_i)\mathrm{e}^{-\mathrm{j}\omega t},\qquad z_i(x_i,t)=Z_i(x_i)\mathrm{e}^{-\mathrm{j}\omega t}, \tag{3}$$

where ω is the natural frequency of the coupled system and $\mathrm{j}=\sqrt{-1}$. $Y_i(x_i)$ and $Z_i(x_i)$ are the vibration modes of the ith segment of the beam and the spring-mass on the segment, respectively. Substituting Eq. (3) into Eqs. (1) and (2) gives

$$EI_i\frac{\mathrm{d}^4 Y_i}{\mathrm{d}x_i^4}-\rho A_i\omega^2 Y_i+k_i(Y_i-Z_i)=0, \tag{4}$$

$$Z_i(x_i)=\frac{1}{1-(\omega/\bar{\omega}_i)^2}Y_i(x_i), \tag{5}$$

where $\bar{\omega}_i=\sqrt{k_i/m_i}$ is the natural frequency of the independent spring-mass system on the ith beam segment. Eq. (5) indicates that the beam and the spring-mass have the same mode shapes on a segment, as $1/[1-(\omega/\bar{\omega}_i)^2]$ is a constant. However, the mode shapes of the spring-masses on two adjacent segments, such as the ith and the $(i+1)$th segments, can be discontinuous if $\bar{\omega}_i$ and $\bar{\omega}_{i+1}$ are different. Substituting Eq. (5) into Eq. (4) gives

$$\frac{\mathrm{d}^4 Y_i}{\mathrm{d}\xi_i^4}-\lambda_i^4\left[1+\frac{\mu_i}{1-(\omega/\bar{\omega}_i)^2}\right]Y_i=0,\quad 0<\xi_i<1, \tag{6}$$

where $\xi_i = x_i/l_i$ is the dimensionless coordinate, $\mu_i = m_i/(\rho A_i)$ is the ratio of the spring-mass to the beam mass per unit length and $\lambda_i^2=\omega l_i^2\sqrt{\rho A_i/(EI_i)}$ is the dimensionless natural frequency of the system.

Eq. (6) is the governing differential equation of free vibration for the ith segment of the beam and distributed spring-mass system.

3. Solution

The solution of Eq. (6) depends on the sign of the coefficient $1 + \mu_i/[1 - (\omega/\bar{\omega}_i)^2]$ according to the theory of ordinary differential equation, which is shown in Fig. 2. To simplify the expression of the solution, the following parameter is introduced:

$$\alpha_i = \lambda_i \sqrt[4]{\left|1 + \frac{\mu_i}{1 - (\omega/\bar{\omega}_i)^2}\right|}. \tag{7}$$

There are three possible solutions for Eq. (6):

When $(\omega/\bar{\omega}_i)^2 > 1 + \mu_i$ or $(\omega/\bar{\omega}_i)^2 < 1$

$$Y_i(\xi_i) = c_{i1}\sin(\alpha_i\xi_i) + c_{i2}\cos(\alpha_i\xi_i) + c_{i3}\sinh(\alpha_i\xi_i) + c_{i4}\cosh(\alpha_i\xi_i). \tag{8}$$

When $1 < (\omega/\bar{\omega}_i)^2 < 1 + \mu_i$

$$Y_i(\xi_i) = c_{i1}\sin(\alpha_i\xi_i)\sinh(\alpha_i\xi_i) + c_{i2}\cos(\alpha_i\xi_i)\sinh(\alpha_i\xi_i) + c_{i3}\sin(\alpha_i\xi_i)\cosh(\alpha_i\xi_i) + c_{i4}\cos(\alpha_i\xi_i)\cosh(\alpha_i\xi_i). \tag{9}$$

When $(\omega/\bar{\omega}_i)^2 = 1 + \mu_i$

$$Y_i(\xi_i) = c_{i1} + c_{i2}\xi_i + c_{i3}\xi_i^2 + c_{i4}\xi_i^3, \tag{10}$$

where c_{ij} ($j = 1, 2, 3, 4$) are the unknown constants, which can be determined using boundary conditions. When $\mu_i = 0$, Eq. (8) still holds for a bare beam segment.

The relations between displacement Y_i, rotational angle $\theta_i = \mathrm{d}Y_i/\mathrm{d}x_i$, bending moment $M_i = EI_i\mathrm{d}^2Y_i/\mathrm{d}x_i^2$ and shear force $V_i = EI_i\mathrm{d}^3Y_i/\mathrm{d}x_i^3$ at the two ends ($x_i = 0$ and $x_i = l_i$) of the ith segment can be expressed in a matrix form as follows:

$$\{F\}_i^{\mathrm{R}} = [T]_i\{F\}_i^{\mathrm{L}}, \tag{11}$$

where the superscripts R and L mean the right end and left end of the beam-spring-mass segment, respectively, and

$$\{F\}_i^{\mathrm{R}} = [\,Y_i(1) \quad \theta_i(1) \quad M_i(1) \quad V_i(1)\,]^{\mathrm{T}},$$
$$\{F\}_i^{\mathrm{L}} = [\,Y_i(0) \quad \theta_i(0) \quad M_i(0) \quad V_i(0)\,]^{\mathrm{T}}, \tag{12}$$

$$[T]_i = \begin{bmatrix} t_{11}^i & t_{12}^i & t_{13}^i & t_{14}^i \\ t_{21}^i & t_{22}^i & t_{23}^i & t_{24}^i \\ t_{31}^i & t_{32}^i & t_{33}^i & t_{34}^i \\ t_{41}^i & t_{42}^i & t_{43}^i & t_{44}^i \end{bmatrix}. \tag{13}$$

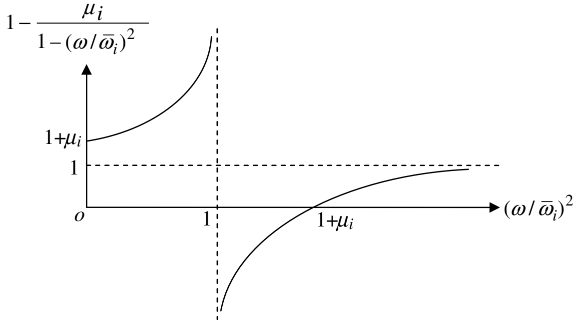

Fig. 2. The sign of the coefficient $1 + \mu_i/[1 - (\omega/\bar{\omega}_i)^2]$.

When $(\omega/\bar{\omega}_i)^2 > 1+\mu_i$ or $(\omega/\bar{\omega}_i)^2 < 1$

$$\begin{aligned}
&t^i_{11} = (\cos\alpha_i + \cosh\alpha_i)/2, \qquad t^i_{12} = l_i(\sin\alpha_i + \sinh\alpha_i)/(2\alpha_i),\\
&t^i_{13} = l_i^2(-\cos\alpha_i + \cosh\alpha_i)/(2EI_i\alpha_i^2), \qquad t^i_{14} = l_i^3(-\sin\alpha_i + \sinh\alpha_i)/(2EI_i\alpha_i^3),\\
&t^i_{21} = \alpha_i(-\sin\alpha_i + \sinh\alpha_i)/(2l_i), \qquad t^i_{22} = (\cos\alpha_i + \cosh\alpha_i)/2,\\
&t^i_{23} = l_i(\sin\alpha_i + \sinh\alpha_i)/(2EI_i\alpha_i), \qquad t^i_{24} = l_i^2(-\cos\alpha_i + \cosh\alpha_i)/(2EI_i\alpha_i^2),\\
&t^i_{31} = EI_i\alpha_i^2(-\cos\alpha_i + \cosh\alpha_i)/(2l_i^2), \qquad t^i_{32} = EI_i\alpha_i(-\sin\alpha_i + \sinh\alpha_i)/(2l_i),\\
&t^i_{33} = (\cos\alpha_i + \cosh\alpha_i)/2, \qquad t^i_{34} = l_i(\sin\alpha_i + \sinh\alpha_i)/(2\alpha_i),\\
&t^i_{41} = EI_i\alpha_i^3(\sin\alpha_i + \sinh\alpha_i)/(2l_i^3), \qquad t^i_{42} = EI_i\alpha_i^2(-\cos\alpha_i + \cosh\alpha_i)/(2l_i^2),\\
&t^i_{43} = \alpha_i(-\sin\alpha_i + \sinh\alpha_i)/(2l_i), \qquad t^i_{44} = (\cos\alpha_i + \cosh\alpha_i)/2.
\end{aligned} \tag{14}$$

When $1 < (\omega/\bar{\omega}_i)^2 < 1+\mu_i$, there are

$$\begin{aligned}
&t^i_{11} = \cos\bar{\alpha}_i\cosh\bar{\alpha}_i, \qquad t^i_{12} = l_i(\sin\bar{\alpha}_i\cosh\bar{\alpha}_i + \cos\bar{\alpha}_i\sinh\bar{\alpha}_i)/(2\bar{\alpha}_i),\\
&t^i_{13} = l_i^2\sin\bar{\alpha}_i\sinh\bar{\alpha}_i/(2EI_i\bar{\alpha}_i^2), \qquad t^i_{14} = l_i^3(\sin\bar{\alpha}_i\cosh\bar{\alpha}_i - \cos\bar{\alpha}_i\sinh\bar{\alpha}_i)/(4EI_i\bar{\alpha}_i^3),\\
&t^i_{21} = \bar{\alpha}_i(-\sin\bar{\alpha}_i\cosh\bar{\alpha}_i + \cos\bar{\alpha}_i\sinh\bar{\alpha}_i)/l_i, \qquad t^i_{22} = \cos\bar{\alpha}_i\cosh\bar{\alpha}_i,\\
&t^i_{23} = l_i(\cos\bar{\alpha}_i\sinh\bar{\alpha}_i + \sin\bar{\alpha}_i\cosh\bar{\alpha}_i)/(2EI_i\bar{\alpha}_i), \qquad t^i_{24} = l_i^2\sin\bar{\alpha}_i\sinh\bar{\alpha}_i/(2EI_i\bar{\alpha}_i^2),\\
&t^i_{31} = -2EI_i\bar{\alpha}_i^2\sin\bar{\alpha}_i\sinh\bar{\alpha}_i/l_i^2, \qquad t^i_{32} = EI_i\bar{\alpha}_i(-\sin\bar{\alpha}_i\cosh\bar{\alpha}_i + \cos\bar{\alpha}_i\sinh\bar{\alpha}_i)/l_i,\\
&t^i_{33} = \cos\bar{\alpha}_i\cosh\bar{\alpha}_i, \qquad t^i_{34} = l_i(\cos\bar{\alpha}_i\sinh\bar{\alpha}_i + \sin\bar{\alpha}_i\cosh\bar{\alpha}_i)/(2\bar{\alpha}_i),\\
&t^i_{41} = -2EI_i\bar{\alpha}_i^3(\cos\bar{\alpha}_i\sinh\bar{\alpha}_i + \sin\bar{\alpha}_i\cosh\bar{\alpha}_i)/l_i^3, \qquad t^i_{42} = -2EI_i\bar{\alpha}_i^2\sin\bar{\alpha}_i\sinh\bar{\alpha}_i/l_i^2,\\
&t^i_{43} = \bar{\alpha}_i(-\sin\bar{\alpha}_i\cosh\bar{\alpha}_i + \cos\bar{\alpha}_i\sinh\bar{\alpha}_i)/l_i, \qquad t^i_{44} = \cos\bar{\alpha}_i\cosh\bar{\alpha}_i,
\end{aligned} \tag{15}$$

where $\bar{\alpha}_i = \alpha_i/\sqrt{2}$.

When $(\omega/\bar{\omega}_i)^2 = 1+\mu_i$, it gives

$$\begin{aligned}
&t^i_{11} = 1, \quad t^i_{12} = l_i, \quad t^i_{13} = l_i^2/(2EI_i), \quad t^i_{14} = l_i^3/(6EI_i), \quad t^i_{21} = 0, \quad t^i_{22} = 1, \quad t^i_{23} = l_i/(EI_i),\\
&t^i_{24} = l_i^2/(2EI_i), \quad t^i_{31} = t^i_{32} = 0, \quad t^i_{33} = 1, \quad t^i_{34} = l_i, \quad t^i_{41} = t^i_{42} = t^i_{43} = 0, \quad t^i_{44} = 1.
\end{aligned} \tag{16}$$

The present solutions are applicable for the following special cases:

- When the mass density of the distributed spring-mass on the ith segment becomes infinity i.e., $\mu_i = \infty$, it means the ith segment of the beam on Winkler elastic foundation.
- When the stiffness of the distributed spring-mass on the ith segment approaches infinity, it represents a distributed rigid mass on the segment of the beam.

4. Transfer matrix

The displacements and forces on the right end of the ith segment should be equal to those on the left end of the $(i+1)$th segment, i.e.,

$$\{F\}^{\mathrm{L}}_{i+1} = \{F\}^{\mathrm{R}}_i. \tag{17}$$

Substituting Eq. (11) into the above equation gives

$$\{F\}^{\mathrm{L}}_{i+1} = [T]_i\{F\}^{\mathrm{L}}_i, \tag{18}$$

where $[T]_i$ is called as the transfer matrix. If the structure is divided into I segments, the following successive formula can be obtained:

$$\{F\}^{\mathrm{L}}_I = [T]_{I-1}[T]_{I-2}\cdots[T]_1\{F\}^{\mathrm{L}}_1. \tag{19}$$

From Eq. (11), one has

$$\{F\}_I^{\mathrm{R}} = [T]_I \{F\}_I^{\mathrm{L}}. \tag{20}$$

Substituting Eq. (19) into Eq. (20) gives

$$\{F\}_I^{\mathrm{R}} = [T]_I [T]_{I-1} [T]_{I-2} \cdots [T]_1 \{F\}_1^{\mathrm{L}} = [T]\{F\}_1^{\mathrm{L}}, \tag{21}$$

where $[T]$ is a 4×4 matrix as follows:

$$[T] = [T]_I [T]_{I-1} [T]_{I-2} \cdots [T]_1 = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}. \tag{22}$$

Considering the boundary conditions of the two ends of the beam, a 2×2 homogeneous matrix equation can be obtained from Eq. (22). The determinant of the coefficient matrix should to zero, which gives the eigenvalue equation dependent on the boundary conditions of the beam.

For a simply supported (S–S) beam, $t_{12}t_{34} - t_{14}t_{32} = 0.$ (23)

For a clamped–clamped (C–C) beam, $t_{13}t_{24} - t_{14}t_{23} = 0.$ (24)

For a simply supported–clamped (S–C) beam, $t_{12}t_{24} - t_{14}t_{22} = 0.$ (25)

For a cantilevered (F–C) beam, $t_{11}t_{22} - t_{12}t_{21} = 0.$ (26)

For a free–free (F–F) beam, $t_{31}t_{42} - t_{41}t_{32} = 0.$ (27)

The back substitution of eigenvalues obtained from Eqs. (23)–(27) gives the corresponding mode shapes.

5. Basic characteristics of solutions

Eq. (5) shows that the mode of the uniformly distributed spring-mass is proportional to the mode of the beam at the attached regions. The amplitude ratio of the spring-mass to the beam is equal to $1/[1-(\omega/\bar{\omega}_i)^2]$. This means that in a segment if the natural frequency of the coupled system is smaller than that of the independent spring-mass, the movements of the beam and the spring-mass are in the same direction. However, if the natural frequency of the coupled system is larger than that of the independent spring-mass, the ratio becomes negative and the movements of the beam and the spring-mass are in the opposite directions. The closer the natural frequency of the coupled system to that of the independent spring-mass, the larger the absolute value of the amplitude ratio. Moreover, if the natural frequencies of the independent spring-masses on two adjacent segments are different, the mode shape of the spring-mass on two adjacent segments could be discontinuous because of different amplitude ratios.

It can be noted from Fig. 2 that when $(\omega/\bar{\omega}_i)^2 > 1 + \mu_i$ or $(\omega/\bar{\omega}_i)^2 < 1$, the form of the solution is the same as that of the solution of free vibration of a bare beam, as seen in Eq. (8). When $1 < (\omega/\bar{\omega}_i)^2 < 1 + \mu_i$, the form of the solution is the same as that of the homogeneous solution of a static beam on elastic foundation, as seen in Eq. (9). When $(\omega/\bar{\omega}_i)^2 = 1 + \mu_i$, the solution is the homogeneous solution of a static beam, as seen in Eq. (10). In this case the inertia force of the distributed mass just counteracts the inertia force of the beam.

A uniform beam with uniformly distributed spring-mass along its full length is now considered. There is only one segment and for simplicity the subscript $i = 1$ is omitted. For this special case, some general characteristics of the beam and spring-mass system can be deduced from Eqs. (5) and (8) that

- The mode shapes of the beam $Y(\xi)$ and the spring-mass $Z(\xi)$ are identical.
- The parameters c_j $(j = 1, 2, 3, 4)$ and α in Eq. (8) are uniquely determined by the four boundary conditions of the beam as shown in Eqs. (23)–(27), which are independent of the spring-mass on the beam. Therefore α^2 is the dimensionless natural frequency of the bare beam and can be expressed as $\omega_{\mathrm{b}} l^2 \sqrt{\rho A/EI}$ where ω_{b} is the natural frequency of the bare beam.

- Substituting c_j ($j = 1, 2, 3, 4$) and α into Eq. (8) gives the mode shape $Y(\xi)$ which is valid for both bare beam and the beam with spring-mass. In other words, the mode shapes of the bare beam and the coupled beam-spring-mass system are identical.
- Substituting α into Eq. (7) leads to the solution of the natural frequencies of the beam and spring-mass system.

In this case one natural frequency of the bare beam corresponds to a pair of natural frequencies of the coupled system, which can be obtained by solving Eq. (7) as follows:

$$\begin{aligned}\omega_{1j}^2 &= \frac{1}{2}\left\{(1+\mu)\bar{\omega}^2 + \omega_{\mathrm{b}j}^2 - \sqrt{[(1+\mu)\bar{\omega}^2 + \omega_{\mathrm{b}j}^2]^2 - 4\omega_{\mathrm{b}j}^2\bar{\omega}^2}\right\},\\ \omega_{2j}^2 &= \frac{1}{2}\left\{(1+\mu)\bar{\omega}^2 + \omega_{\mathrm{b}j}^2 + \sqrt{[(1+\mu)\bar{\omega}^2 + \omega_{\mathrm{b}j}^2]^2 - 4\omega_{\mathrm{b}j}^2\bar{\omega}^2}\right\}, \quad j = 1, 2, 3, \ldots\infty.\end{aligned} \tag{28}$$

Eq. (28) indicates that the jth pair of natural frequencies of the coupled system corresponds to the jth mode of vibration of the bare beam. From Eq. (28), the following relationships can be easily demonstrated, as given by Ellis and Ji (1997)

$$\omega_{1j}\omega_{2j} = \omega_{\mathrm{b}j}\bar{\omega}, \quad \omega_{1j} < (\omega_{\mathrm{b}j}, \bar{\omega}) < \omega_{2j}, \qquad j = 1, 2, 3, \ldots\infty. \tag{29}$$

Eq. (29) indicates that in the jth pair of natural frequencies of the coupled system, the first frequency is always lower than the natural frequency of the independent spring-mass and the jth natural frequency of the bare beam; the second is always higher than the two natural frequencies of the independent spring-mass and the bare beam.

It can be noted that Eq. (28) is actually the solution of a discrete TDOF system, as shown in Fig. 3, with the following parameters:

$$M_a = ml, \quad M_b = \rho A l, \quad K_a = kl, \quad K_{\mathrm{b}j} = M_s\omega_{\mathrm{b}j}^2. \tag{30}$$

From the above analysis, it can be concluded that for a uniform beam with uniformly distributed spring-mass along its length, its vibration characteristics can be exactly represented by a series of discrete TDOF systems. It can be seen from Eq. (30) and Fig. 3 that the two lumped masses, M_a and M_b, and the stiffness of the upper mass, K_a are constants. However, the stiffness K_{bj} is proportional to the square of the natural frequency ω_{bj} of the bare beam. With the increase of the mode order, ω_{bj} monotonically increases. This means that the stiffness K_{bj} also monotonically increases. Eq. (28) can be rewritten as follows:

$$\begin{aligned}\omega_{1j}^2 &= \frac{\omega_{\mathrm{b}j}^2}{2}\left\{(1+\mu)\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2} + 1 - \sqrt{\left[(1+\mu)\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2} + 1\right]^2 - 4\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2}}\right\},\\ \omega_{2j}^2 &= \frac{\omega_{\mathrm{b}j}^2}{2}\left\{(1+\mu)\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2} + 1 + \sqrt{\left[(1+\mu)\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2} + 1\right]^2 - 4\frac{\bar{\omega}^2}{\omega_{\mathrm{b}j}^2}}\right\}.\end{aligned} \tag{31}$$

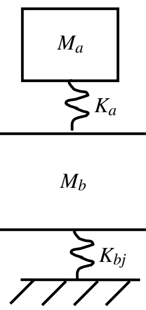

Fig. 3. The equivalent TDOF system of a uniform beam with uniformly distributed spring-mass along its full length.

As $\bar{\omega}^2$ is a constant, $\bar{\omega}^2/\omega_{bj}^2$ is a small value when j is large. Thus, the following approximation can be obtained:

$$\sqrt{\left[(1+\mu)\frac{\bar{\omega}^2}{\omega_{bj}^2}+1\right]^2-4\frac{\bar{\omega}^2}{\omega_{bj}^2}} \approx 1+(\mu-1)\frac{\bar{\omega}^2}{\omega_{bj}^2}. \tag{32}$$

Substituting the above equation into Eq. (31) gives

$$\omega_{1j} \approx \bar{\omega}, \qquad \omega_{2j} \approx \omega_{bj} \tag{33}$$

Eq. (33) indicates that the first natural frequency of the TDOF system is close to that of the independent spring-mass and the second is close to that of corresponding order of the bare beam when j increases. This demonstrates that

- For a high order natural frequency of the bare beam, the upper mass M_a and the lower mass M_b are vibrating independently at their own natural frequencies. In other words, there is little coupling between the free vibrations of the two SDOF system.
- As $\omega_{1j} < \bar{\omega}$ and $\omega_{1j} \approx \bar{\omega}$ for a large j, ω_{1j} gradually and monotonically approaches $\bar{\omega}$ from the lower side with the increase of the order j of the frequency pair.
- As $\omega_{2j} > \omega_{bj}$ and $\omega_{2j} \approx \omega_{bj}$ for a large j, ω_{2j} gradually and monotonically approaches ω_{bj} from the upper side with the increase of the order j of the frequency pair.
- The coupled vibration of a beam and uniformly distributed spring-mass mainly occurs in the low order of frequency pairs, especially in the first pair of frequencies.

6. Parametric studies

In the following study, a uniform beam with up to three segments of uniformly distributed spring-mass, as shown in Fig. 4, will be investigated in detail. The mass and stiffness of the spring-mass are constants in each segment, but vary from one segment to other. Three kinds of boundary condition are considered: the two ends are clamped (C–C), the two ends are simply supported (S–S) and one end is free and the other clamped (F–C). The following dimensionless parameters are used:

$$\eta_i = l_i/l, \quad \beta_i = k_i l^4/(EI), \quad \mu_i = m_i/(\rho A), \qquad (i = 1, 2, 3) \tag{34}$$

The coupled natural frequencies would be obtained by solving Eqs. (23)–(27).

6.1. Uniformly distributed spring-mass on one segment

6.1.1. Coupled natural frequencies

A uniform cantilevered beam (free at the left end and clamped at the right end) is first considered with partly uniformly distributed spring-mass starting from its free end. Thus, $\eta_2 = \mu_2 = \beta_2 = \eta_3 = \mu_3 = \beta_3 = 0$. Three length ratios, $\eta_1 = 0.25$, 0.5 and 1.0, are considered. Namely, the spring-mass occupies the first quarter, the first half and the full of the span of the beam, respectively. Assume that the other two parameters in Eq. (34) are $\beta_1 = 60$ and $\mu_1 = 5.0$. The eigenvalues can be obtained by solving Eq. (26). As the same as the beam with fully

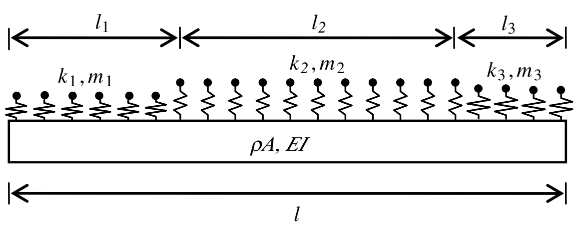

Fig. 4. A uniform beam with three segments of uniformly distributed spring-mass.

Table 1
The pairs of dimensionless natural frequencies $\lambda^2 = \omega l^2 \sqrt{\rho A/EI}$ for a cantilevered beam with a part of uniformly distributed spring-mass from its free end when $\beta_1 = 60$ and $\mu_1 = 5$

Order of pairs	Spring-mass $\bar{\lambda}^2$	Bare beam γ^2	$\eta_1 = 0.25$		$\eta_1 = 0.5$		η_1=1.0	
			First	Second	First	Second	First	Second
1	3.46410	3.51602	1.52178	7.89247	1.36617	8.90074	1.34042	9.08656
2		22.0345	3.43897	22.4493	3.34382	22.7998	3.26403	23.3851
3		61.6972	3.46344	61.8128	3.45624	61.9385	3.43704	62.1831
4		120.902	3.46400	120.913	3.46273	121.026	3.45701	121.150
5		199.860	3.46407	199.901	3.46371	199.935	3.46150	200.010
6		298.556	3.46409	298.578	3.46395	298.606	3.46294	298.656
∞		∞	3.46410	∞	3.46410	∞	3.46410	∞

uniformly distributed spring-mass, the dimensionless coupled natural frequencies can be classified into two queues, as given in Table 1. The mode shapes of the beam, corresponding to the first two pairs of natural frequencies, are given in Figs. 5 and 6, respectively. It can be observed from Table 1 and Figs. 5 and 6 that

- The coupled natural frequencies appear in pairs. Each pair of frequencies has two mode shapes which are similar to that of the bare beam (when $\eta_1 = 1$, the mode shapes of the coupled system are the same as those of the bare beam).
- The first natural frequency in a pair gradually approaches that of the independent spring-mass from the lower side while the second gradually approaches the natural frequencies of the bare beam from the upper side as the order of frequency pair increases.
- The frequency coupling between spring-mass and beam mainly appears in the first pair of natural frequencies. As the order of frequency pairs increases, the degree of the frequency coupling between the beam and the spring-mass decreases quickly.

6.1.2. Effect of the distribution of spring-mass

Table 2 gives the first four pairs of dimensionless natural frequencies for S–S and C–C beams with uniform spring-mass symmetrically distributed about the midpoint of the beam when $\beta_2 = 500$ and $\mu_2 = 5.0$. In such a case, $\eta_1 = \eta_3$ and $\beta_1 = \mu_1 = \beta_3 = \mu_3 = 0$. It is seen from Table 2 that, which enhances the observations in the last subsection

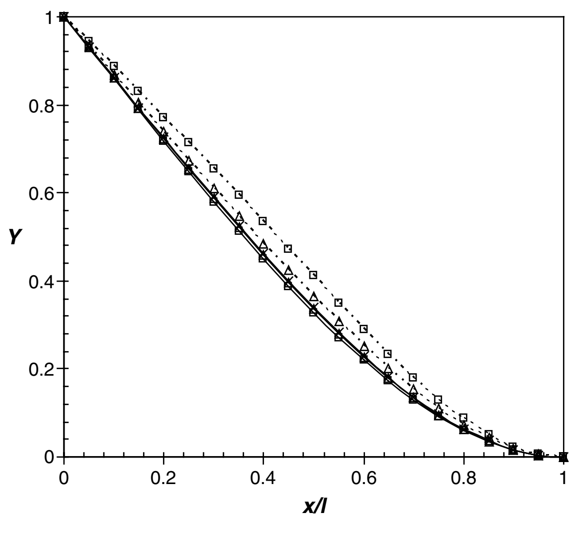

Fig. 5. The modes corresponding to the first pair of natural frequencies for a cantilevered beam (F–C) with a part of uniformly distributed mass. — the first mode, --- the second mode, (□) $\eta_2 = 0.25$, (△) $\eta_2 = 0.5$, (×) $\eta_2 = 1.0$ and bear beam.

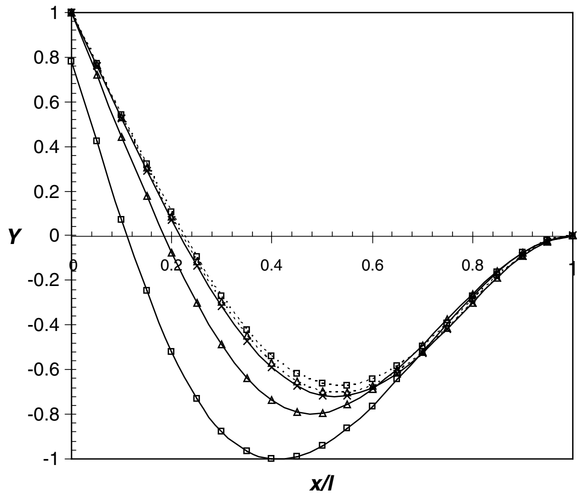

Fig. 6. The modes corresponding to the second pair of natural frequencies for a cantilevered beam (F–C) with a part of uniformly distributed mass. — the first mode, - - - the second mode, (□) $\eta_2 = 0.25$, (△) $\eta_2 = 0.5$, (×) $\eta_2 = 1.0$ and the bear beam.

Table 2
The first four pairs of dimensionless natural frequencies $\lambda^2 = \omega l^2\sqrt{\rho A/EI}$ for S–S and C–C beams with uniform spring-mass symmetrically and partly distributed on the midpoint of the beam when $\beta_2 = 500$ and $\mu_2 = 5.0$

Order of pairs	S–S beam			C–C beam		
	First	Second	1st × 2nd	First	Second	1st × 2nd
Independent spring-mass $\bar{\lambda}^2$ and bear beam γ^2						
1	10 (dimensionless	9.86960	98.696	10 (dimensionless	22.3733	223.73
2	frequency of	39.4784	394.78	frequency of	61.6728	616.73
3	spring-mass)	88.8264	888.26	spring-mass)	120.903	1209.0
4		157.914	1579.1		199.859	1998.6
$\eta_1 = 0.4$, $\eta_2 = 0.2$, $\eta_3 = 0.4$						
1	5.16059	18.9277	97.678	8.01623	27.7719	222.63
2	9.89679	39.8042	393.93	9.93279	61.9929	615.76
3	9.99465	89.6993	896.51	9.99528	121.471	1214.1
4	9.99911	158.157	1581.4	9.99916	200.082	2000.7
$\eta_1 = 0.3$, $\eta_2 = 0.4$, $\eta_3 = 0.3$						
1	4.28070	22.8853	97.965	7.23428	30.8536	223.20
2	9.47629	41.4676	392.96	9.69560	63.4528	615.21
3	9.94460	89.8115	893.14	9.95610	121.557	1210.2
4	9.98857	158.672	1584.9	9.98985	200.410	2002.1
$\eta_1 = 0.2$, $\eta_2 = 0.6$, $\eta_3 = 0.2$						
1	3.92073	25.1297	98.527	6.95551	32.1582	223.68
2	8.98464	43.8484	393.96	9.47398	65.0584	616.36
3	9.83190	90.3497	888.31	9.88338	122.272	1208.5
4	9.95724	158.749	1580.7	9.96534	200.546	1998.5
$\eta_1 = 0.1$, $\eta_2 = 0.8$, $\eta_3 = 0.1$						
1	3.79440	26.0095	98.690	6.89721	32.4381	223.73
2	8.70053	45.3696	394.74	9.39335	65.6554	616.72
3	9.72213	91.3563	888.18	9.83667	122.909	1209.0
4	9.91528	159.254	1579.0	9.94134	201.034	1998.5
$\eta_1 = 0.0$, $\eta_2 = 1.0$, $\eta_3 = 0.0$						
1	3.77608	26.1371	98.696	6.89461	32.4504	223.73
2	8.64842	45.6481	394.78	9.38822	65.6917	616.73
3	9.69397	91.6306	888.26	9.83216	122.967	1209.0
4	9.90085	159.495	1579.1	9.93784	201.110	1998.6

- The products of pairs of natural frequencies of the system are close (or equal) to that of the natural frequencies of the bare beam and the independent spring-masses for all cases. This is just the unique property of a TDOF system, as given in Eq. (29).
- With the increase of the occupation of the spring-mass on the beam, the first natural frequency in a pair monotonically decreases and the second monotonically increases, i.e., the more the occupation of the spring-mass on the beam, the stronger the frequency coupling.
- The free vibration of the coupled system can be approximately simulated by a series of TDOF systems.

6.1.3. Effect of stiffness ratio and mass ratio

Table 3 provides the results showing the effect of the structural parameters on the first pair of natural frequencies for a uniform beam with partly uniformly distributed mass for three types of boundary conditions. In this case, $\beta_1 = \mu_1 = \beta_3 = \mu_3 = 0$. Three length ratios and eight groups of spring-mass parameters (mass ratio and stiffness ratio of the spring-mass to the beam) are considered. It is shown in Table 3 that

- With the increase of the occupation of the spring-mass on the beam, the first natural frequency in a pair monotonically decreases and the second monotonically increases, as the same as observed in Table 2.
- Increasing the stiffness ratio will result in the increase of both natural frequencies in a pair. However, increasing the mass ratio will result in the decrease of both frequencies.
- Proportionally increasing the mass ratio and the stiffness ratio (i.e., the natural frequency of the independent spring-mass remains constant) will result in the decrease of the first natural frequency and the increase of the second natural frequency in the pair. In other words, this enlarges the frequency coupling of the spring-mass and the beam, vice versa.
- The closer the natural frequency of a bare beam to that of the independent spring-mass, the stronger the frequency coupling between the beam and the spring-mass.

Table 3
The first pair of dimensionless natural frequencies $\lambda^2 = \omega l^2 \sqrt{\rho A/EI}$ for beams with partly uniformly distributed spring-mass

Parameters		S–S		C–C		F–C	
η_1, η_2	μ_2, β_2	First	Second	First	Second	First	Second
0.0, 1.0	1, 5	2.17808	10.1324	2.22487	22.4859	1.79747	4.37395
	1, 10	2.99733	10.4127	3.13054	22.6001	2.10358	5.28558
	1, 50	5.37623	12.9810	6.71226	23.5693	2.40844	10.3229
	1, 100	6.12164	16.1225	8.98676	24.8958	2.44751	14.3657
	5, 25	1.98616	11.1114	2.18174	22.9304	1.23011	6.39134
	5, 50	2.54024	12.2864	3.01274	23.4838	1.32316	8.40312
	5, 250	3.55791	19.6151	5.70876	27.7123	1.41122	17.6173
	5, 500	3.77608	26.1371	6.89461	32.4504	1.42320	24.7050
0.2, 0.6	1, 5	2.18347	10.1072	2.22526	22.2819	2.00547	3.91267
	1, 10	3.01202	10.3616	3.13164	22.5921	2.46661	4.49006
	1, 50	5.47269	12.7500	6.72381	23.5282	2.88740	8.44332
	1, 100	6.26057	15.7594	9.01616	24.8134	2.93013	11.5485
	5, 25	2.00656	10.9976	2.18357	22.9109	1.57025	4.95842
	5, 50	2.58388	12.0769	3.01761	23.4453	1.76786	6.16863
	5, 250	3.67977	18.9495	5.74306	27.5434	1.96584	11.5503
	5, 500	3.92073	25.1297	6.95551	32.1582	1.99296	15.0505
0.4, 0.2	1, 5	2.21282	9.97229	2.23060	22.4370	2.16477	3.62485
	1, 10	3.09422	10.0847	3.14672	22.4817	2.84565	3.89217
	1, 50	6.14018	11.3543	6.88889	22.9537	3.31414	7.35192
	1, 100	7.29386	13.5036	9.45784	23.6326	3.33710	10.1348
	5, 25	2.12699	10.3705	2.20913	22.6406	1.95869	3.97581
	5, 50	2.86417	10.8858	3.08677	22.9094	2.38184	4.57964
	5, 250	4.67212	14.8610	6.29080	25.0866	2.78452	8.12615
	5, 500	5.16059	18.9277	8.01623	27.7719	2.82880	10.5329

6.2. Uniformly distributed spring-mass on several segments

To extend the studies in the last subsection, we consider the spring-mass to be distributed over two, three or more parts of the span. The mass density and the stiffness of the distributed spring-mass are constants in each segment but vary from segment to segment.

6.2.1. Different uniformly distributed spring-masses on two segments

Consider a uniform cantilevered beam (free at the left end) with two segments of distributed spring-mass over the span of the beam. In such a case, $\eta_3 = 0$ and $\beta_3 = \mu_3 = 0$. The spring-masses have the parameters $\beta_1 = k_1 l^4/(EI) = 60$ and $\mu_1 = m_1/\rho A = 5$ on the first segment and $\beta_2 = 20$ and $\mu_2 = 5$ on the second segment. Two sets of distribution length, ($\eta_1 = 0.25$, $\eta_2 = 0.75$) and ($\eta_1 = \eta_2 = 0.5$), are considered. The calculated results show that the coupled natural frequencies can still be clarified into groups and there are three natural frequencies in each group. The first six groups of dimensionless natural frequencies are given in Table 4 and the mode shapes corresponding to the first group of natural frequencies are shown in Fig. 7. It can be seen from Table 4 and Fig. 7 that

- The coupled natural frequencies appear in groups. The mode shapes corresponding to the first group of natural frequencies are similar to the first mode of the bare beam and the modes corresponding to the second group of natural frequencies are similar to the second mode of the bare beam, and so on.
- The frequency coupling between the spring-mass and the beam mainly appear in the first group of natural frequencies. From the second group or higher, the first two natural frequencies in the groups gradually approach those of the independent spring-mass systems from the lower side while the third natural frequency in the groups gradually approaches those of the bare beam from the upper side, as the order of vibration mode increases.
- The free vibration of the coupled system can be approximately represented by a series of three degrees-of-freedom systems, where two independent SDOF spring-masses systems are placed in parallel on the SDOF structure system as shown in Fig. 8.

6.2.2. Different uniformly distributed spring-masses on three or more segments

A uniform simply–simply supported beam with three segments of uniformly distributed spring-mass is investigated. The spring-mass occupies the full length of the beam and the lengths of the three segments are the same. The spring-masses have the parameters $\beta_1 = k_1 l^4/(EI) = 500$ and $\mu_1 = m_1/\rho A = 2.5$ on the first segment, $\beta_2 = 500$ and $\mu_2 = 5$ on the second, $\beta_3 = 500$ and $\mu_3 = 10$ on the third. The calculated results show once again that the coupled natural frequencies can be grouped and each group has four natural frequencies in the studied case. The first six groups of dimensionless natural frequency are given in Table 5. It can be noted from the table that with the increase of the group order, the first three natural frequencies in a group are, respectively, close to those of the three different distributed spring-mass systems and the fourth is close to that

Table 4

The groups of dimensionless natural frequencies $\lambda^2 = \omega l^2 \sqrt{\rho A/EI}$ for a cantilevered beam (F–C) with two segments of uniformly distributed masses when $\beta_1 = 60$, $\beta_2 = 20$, $\mu_1 = \mu_2 = 5$

Order of groups	Spring-mass $\bar{\lambda}^2$	Bare beam γ^2	$\eta_1 = 0.25$, $\eta_2 = 0.75$			$\eta_1 = 0.5$, $\eta_2 = 0.5$		
			First	Second	Third	First	Second	Third
1	2	3.51602	1.27595	2.25441	8.28170	1.32729	2.02667	8.95941
2	3.46410	22.0345	1.98562	3.44016	22.7553	1.99623	3.34652	22.9937
3		61.6972	1.99830	3.46344	61.9361	1.99964	3.45628	62.0197
4		120.902	1.99956	3.46400	121.032	1.99991	3.46273	121.067
5		199.860	1.99984	3.46408	199.937	1.99997	3.46371	199.960
6		298.556	1.99993	3.46409	298.604	1.99999	3.46395	298.623
∞		∞	2	3.46410	∞	2	3.46410	∞

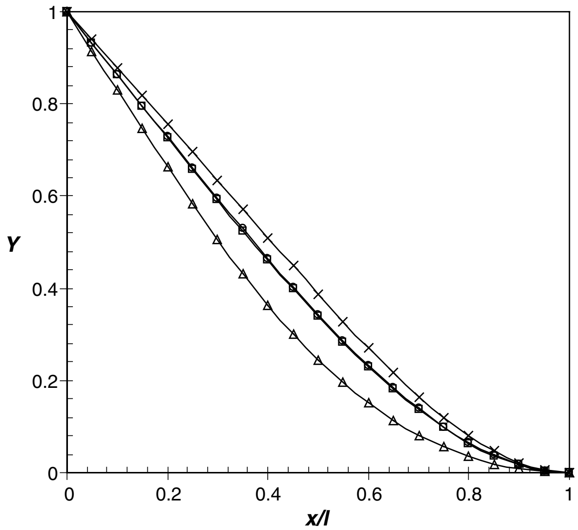

Fig. 7. The modes corresponding to the first group of natural frequencies for a uniform cantilevered beam (F–C) with two segments of uniformly distributed mass ($\beta_1 = 60$, $\eta_1 = 0.25$, $\beta_2 = 20$, $\eta_2 = 0.75$, $\mu_1 = \mu_2 = 5$), (○) the first mode, (△) the second mode, (×) the third mode, (□) the bare beam.

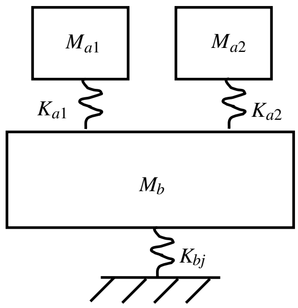

Fig. 8. The approximately equivalent model of a uniform beam with two segments of uniformly distributed spring-mass.

Table 5
The groups of dimensionless natural frequencies $\lambda^2 = \omega l^2\sqrt{\rho A/EI}$ for a simply supported beam with three segments of uniformly distributed masses when $\beta_1 = \beta_2 = \beta_3 = 500$, $\mu_1 = 2.5$, $\mu_2 = 5$, $\mu_3 = 10$, $\eta_1 = \eta_2 = \eta_3 = 1/3$

Order of groups	Spring-mass $\bar{\lambda}^2$	Bare beam γ^2	Dimensionless frequencies $\lambda^2 = \omega l^2\sqrt{\rho A/EI}$			
			First	Second	Third	Fourth
1	7.07107	9.86960	3.55615	7.38182	12.4735	26.3509
2	10	39.4784	6.96491	9.67852	14.0360	45.7136
3	14.1421	88.8264	7.06222	9.96957	14.1276	91.6363
4		157.914	7.06904	9.99408	14.1385	159.496
5		246.740	7.07038	9.99823	14.1408	247.753
6		355.306	7.07077	9.99931	14.1416	356.009
∞		∞	7.07107	10	14.1421	∞

of the bare beam. The dynamic characteristics are equivalent to a discrete four degrees of freedom system in which the three SDOF systems formed by the spring-masses on the three segments act in parallel on the SDOF system converted from the beam.

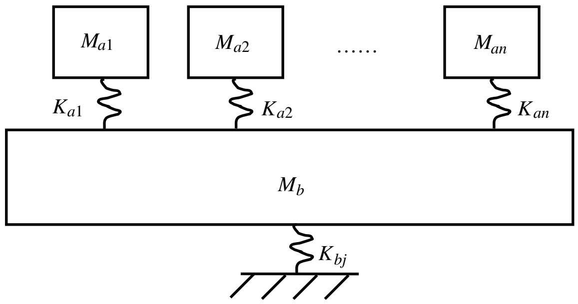

Fig. 9. The approximately equivalent model of a beam with n segments of uniformly distributed spring-mass.

Based on the above analysis, it is a physical deduction that if n segments of uniformly distributed spring-mass with different natural frequencies are applied on a beam, the vibration of the coupled spring-mass and beam system can be approximately represented by a series of $n + 1$ DOF systems, as shown in Fig. 9, in which n SDOF spring-mass systems act in parallel on the SDOF structure system. The conclusions from the foregoing analysis for beams with one, two and three segments of uniformly distributed spring-mass still hold for the general case.

7. Conclusions

This paper provides an exact analytical solution to investigate the characteristics of free vibration of a beam and distributed spring-mass system. The spring-mass acts over parts of the span of the beam and has the constant mass and stiffness on a segment but may vary from segment to segment. This model represents a structure occupied by a crowd of people. The study of the combined beam and distributed spring-mass system allows to examining the relationship between the continuous model and the corresponding discrete models, and assessing the effect of higher order modes of free vibration.

The main conclusions obtained are summarised as follows:

1. In each segment, the mode shape of uniformly distributed spring-mass is the same as that of the beam. However, the mode shape of the spring-mass can be discontinuous between two adjacent segments if the natural frequencies of the spring-masses on the two segments are different.
2. In a segment, when the natural frequency of the coupled system is smaller than that of the spring-mass, the motions of the spring-mass and the beam are in the same direction; when the natural frequency of the coupled system is larger than that of the spring-mass, the motions of the spring-mass and the beam are in the opposite directions.
3. A beam attached by n segments of distributed spring-mass with different n frequencies can be approximately represented by a series of $n + 1$ DOF systems. The n discrete spring-masses representing the distributed spring-masses connect in parallel to the base spring-mass representing the beam. This provides a theoretical basis for converting a continuous system with distributed spring-mass into several discrete systems. This conclusion is useful for developing simplified methods and studying human–structure interaction in engineering practice.
4. The coupled natural frequencies appear in groups. The number of frequencies in each group is equal to $n + 1$, if n segments of uniformly distributed spring-mass with different natural frequencies act on the beam. With the increase of the group order, the first n natural frequencies in a group approach those of the independent spring-masses from the lower side and the other approaches that of the bare beam from the upper side.
5. The degree of frequency coupling between a beam and distributed spring-mass is dependent not only on the structural parameters, but also on the order of natural frequencies. The coupled free vibration mainly occurs in the low order of natural frequency groups, especially in the first group of natural frequencies.

6. The solution of a beam and distributed spring-mass system can be applicable to some other problems, such as a beam on Winkler elastic foundation and a beam with distributed rigid mass.

Acknowledgement

The work reported in this paper has been conducted as part of research project, Human–Structure Interaction—Applying Body Biodynamics into Structural Dynamics, funded by The Leverhulme Trust, whose support is gratefully acknowledged.

References

Chai, G.B., Low, K.H., Lim, T.M., 1995. Tension effects on the natural frequencies of central-loaded clamped beams. Journal of Sound and Vibration 181, 727–736.

Chan, K.T., Wang, X.Q., 1997. Free vibration of a Timoshenko beam partially loaded with distributed mass. Journal of Sound and Vibration 206, 353–369.

Chan, K.T., Zhang, J.Z., 1995. Free vibration of a cantilever tube partially filled with liquid. Journal of Sound and Vibration 182, 185–190.

Chan, K.T., Leung, T.P., Wong, W.O., 1996. Free vibrations of simply supported beam partially loaded with distributed mass. Journal of Sound and Vibration 191, 590–597.

Chen, Y., 1963. On the vibration of beams or rods carrying a concentrated mass. Journal of Applied Mechanics—ASME 30, 310–312.

Chen, D.W., Wu, J.S., 2002. The exact solutions for the natural frequencies and mode shapes of non-uniform beams with multiple spring-mass systems. Journal of Sound and Vibration 255, 299–322.

Drexel, M.V., Ginsberg, J.H., 2001. Modal overlap and dissipation effects of a cantilever beam with multiple attached oscillators. Journal of Vibration and Acoustics—ASME 123, 181–187.

Ellis, B.R., Ji, T., 1996. The effect of human–structure interaction on structural vibration. In: The Third European Conference on Structural Dynamics, Florence, Italy, 5–8 June.

Ellis, B.R., Ji, T., 1997. Human–structure interaction in vertical vibrations. Proceedings of the Institution of Civil Engineers—Structures and Buildings 122, 1–9.

Goel, R.P., 1973. Vibrations of a beam carrying a concentrated mass. Journal of Applied Mechanics—ASME 40, 821–822.

Ji, T., 2003. Understanding the interactions between people and structures. The Structural Engineers 81, 12–13.

Ji, T., Ellis, B.R., 1999. Human whole-body models in structural vibration. In: The 13th ASCE Engineering Mechanics Conference, Baltimore, USA, 13–16 June.

Li, Q.S., 2000. A new exact approach for determining natural frequencies and mode shapes of non-uniform shear beams with arbitrary distribution of mass or stiffness. International Journal of Solids and Structures 37, 5123–5141.

Low, K.H., 2000. A modified Dunkerley formula for eigenfrequencies of beams carrying concentrated masses. International Journal of Mechanical Sciences 42, 1287–1305.

Low, K.H., 2003. Natural frequencies of a beam-mass system in transverse vibration: Rayleigh estimation versus eigenanalysis solutions. International Journal of Mechanical Sciences 45, 981–993.

Qiao, H., Li, Q.S., Li, G.Q., 2002. Vibratory characteristics of flexural non-uniform Euler–Bernoulli beams carrying an arbitrary number of spring-mass systems. International Journal of Mechanical Sciences 44, 725–743.

Rossit, C.A., Laura, P.A.A., 2001a. Free vibrations of a cantilever beam with a spring-mass system attached to the free end. Ocean Engineering 28, 933–939.

Rossit, C.A., Laura, P.A.A., 2001b. Transverse, normal modes of vibration of a cantilever Timoshenko beam with a mass elastically mounted at the free end. Journal of the Acoustical Society of America 110, 2837–2840.

Wu, J.J., 2002. Alternative approach for free vibration of beams carrying a number of two-degree of freedom spring-mass systems. Journal of Structural Engineering—ASCE 128, 1604–1616.

Wu, J.J., 2004. Free vibration analysis of beams carrying a number of two-degree-of-freedom spring-damper-mass systems. Finite Element Analysis and Design 40, 363–381.

Wu, J.S., Chen, D.W., 2001. Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems by using the numerical assembly technique. International Journal for Numerical Methods in Engineering 50, 1039–1058.

Wu, J.S., Chou, H.M., 1998. Free vibration analysis of a cantilever beam carrying any number of elastically mounted point masses with the analytical-and-numerical-combined method. Journal of Sound and Vibration 213, 317–332.

Wu, J.S., Chou, H.M., 1999. A new approach for determining the natural frequencies and mode shapes of a uniform beam carrying any number of sprung masses. Journal of Sound and Vibration 220, 451–468.

Wu, J.J., Whittaker, A.R., 1999. The natural frequencies and mode shapes of a uniform cantilever beam with multiple two-dof spring-mass systems. Journal of Sound and Vibration 227, 361–381.

Wu, J.S., Chen, D.W., Chou, H.M., 1999. On the eigenvalues of a uniform cantilever beam carrying any number of spring-damper-mass systems. International Journal for Numerical Methods in Engineering 45, 1277–1295.