

Counting β -expansions and the absolute continuity of Bernoulli Convolutions

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Abstract

We study the typical growth rate of the number of words of length n which can be extended to β -expansions of x . In the general case we give a lower bound for the growth rate, while in the case that the Bernoulli convolution associated to parameter β is absolutely continuous we are able to give the growth rate precisely. This gives new necessary and sufficient conditions for the absolute continuity of Bernoulli convolutions.

1 Introduction

Given $\beta \in (1, 2]$ and $x \in I_\beta := \left[0, \frac{1}{\beta-1}\right]$, a β -expansion of x is a sequence $(x_n)_{n=1}^\infty \in \{0, 1\}^\mathbb{N}$ for which

$$x = \sum_{n=1}^{\infty} x_n \beta^{-n}.$$

For $\beta = 2$ these are the familiar binary expansions, almost every $x \in [0, 1]$ has a unique binary expansion but there are a countable number of x which have two expansions. The situation for $\beta \in (1, 2)$ is much more complicated; the set $\mathcal{E}_\beta(x)$ of β -expansions of $x \in I_\beta$ may be a singleton set [4], or have any positive integer cardinality [5], but it is typically uncountable [4, 16]. Since the work of Rényi [15] and Parry [11], there has been a great deal of interest in expansions of real numbers in non-integer bases.

Finer information on the structure of $\mathcal{E}_\beta(x)$ can be given by defining

$$\mathcal{E}_\beta^n(x) := \{(x_1, \dots, x_n) \in \{0, 1\}^n \mid \exists (x_{n+1}, x_{n+2}, \dots) : x = \sum_{k=1}^{\infty} x_k \beta^{-k}\}$$

and studying the growth of $\mathcal{N}_n(x; \beta) := |\mathcal{E}_\beta^n(x)|$ as n increases. The maximal growth rate of $\mathcal{N}_n(x; \beta)$ was studied in [7], while the growth rate for Lebesgue almost-every x was studied in [6], where precise information was given in the case that β is a Pisot number as well as some estimates for the general case.

This question is also linked to the question of absolute continuity of Bernoulli convolutions. For $\beta \in (1, 2)$, the Bernoulli convolution ν_β is defined as the weak star limit of the measures

$$\nu_{\beta,n} := \frac{1}{2^n} \sum_{a_1 \cdots a_n \in \{0,1\}^n} \delta_{\sum_{i=1}^n a_i \beta^{-i}},$$

where δ_x denotes the Dirac measure supported on $\{x\}$. It is a long standing open problem to determine for which parameters β the corresponding Bernoulli convolution is absolutely continuous; it is known that almost every $\beta \in (1, 2)$ admits an absolutely continuous Bernoulli convolution [17], but that if β is a Pisot number then the corresponding convolution is singular [3]. For a survey of Bernoulli convolutions, including various alternative definitions, see [12].

It was shown in [13] that the quantity

$$\limsup_{n \rightarrow \infty} \left(\frac{\beta}{2} \right)^n \mathcal{N}_n(x; \beta)$$

is uniformly bounded if and only if ν_β is absolutely continuous and has bounded density. This generalised a previous condition of Garsia and was termed the Garsia-Erdős condition.

The main focus of this article is to prove results on the typical growth rate of $\mathcal{N}_n(x; \beta)$ in the case that the Bernoulli convolution associated to β is absolutely continuous. A consequence of our results is that we are able to give new necessary and sufficient conditions for the absolute continuity of Bernoulli convolutions without imposing any further conditions on the density.

We also show how the growth rate of $\mathcal{N}_n(x; \beta)$ can be studied using the ergodic properties of the random β -transformation of Dajani and Kraaikamp [2]. Our techniques allow us to prove an almost everywhere lower bound on the growth rate of $\mathcal{N}_n(x; \beta)$ for $\beta \in (1, 2)$, extending a result of [6].

In section 2 we give our results for the case that ν_β is absolutely continuous and explain how these can be viewed as a generalisation of the techniques of Garsia. In section 3 we prove these results using an operator based on the self-similarity of the invariant density. In section 4 we give a result for general β using the random β -transformation.

2 Results

We find it more convenient to deal with a renormalised version of $\mathcal{N}_n(x; \beta)$. For $n \in \mathbb{N}$ we define functions $f_n : I_\beta \rightarrow \mathbb{R}$ by

$$f_n(x) := \left(\frac{(\beta - 1)\beta^n}{2^n} \right) \mathcal{N}_n(x; \beta).$$

Since $\sum_{i=n+1}^{\infty} x_i \beta^{-i}$ can take any value in $[0, \frac{1}{(\beta-1)\beta^n}]$, we have that a word x_1, \dots, x_n is an element of $\mathcal{E}_\beta^n(x)$ if and only if

$$\sum_{i=1}^n x_i \beta^{-i} \in \left[x - \frac{1}{(\beta-1)\beta^n}, x \right].$$

So for each n , the value of $f_n(x)$ is equal to $\frac{(\beta-1)\beta^n}{2^n}$ multiplied by the number of words x_1, \dots, x_n for which $\sum_{i=1}^n x_i \beta^{-i} \in \left[x - \frac{1}{(\beta-1)\beta^n}, x \right]$. Then since there are 2^n choices of x_1, \dots, x_n , each of which cover an interval of size $\frac{1}{(\beta-1)\beta^n}$, we see that

$$\int_{I_\beta} f_n(x) dx = 1.$$

We further define

$$\overline{f}(x) := \limsup_{n \rightarrow \infty} f_n(x)$$

and

$$\underline{f}(x) := \liminf_{n \rightarrow \infty} f_n(x).$$

The following is our main theorem.

Theorem 2.1. *The Bernoulli convolution ν_β is absolutely continuous if and only if*

$$0 < \int_{I_\beta} \overline{f}(x) dx < \infty.$$

In this case, one automatically has that $\int_{I_\beta} \overline{f}(y) dy \leq 2$ and that the density h_β of ν_β satisfies

$$h_\beta(x) = \frac{\overline{f}(x)}{\int_{I_\beta} \overline{f}(y) dy}.$$

We stress that the density function h_β is an \mathcal{L}^1 function and is defined only almost everywhere, thus whenever we write $h(x) = \dots$ this statement should be interpreted almost everywhere. We have a slightly weaker theorem for \underline{f} .

Theorem 2.2. *Suppose that*

$$0 < \int_{I_\beta} \underline{f}(x) dx < \infty.$$

Then ν_β is absolutely continuous with density function

$$h_\beta(x) = \frac{\underline{f}(x)}{\int_{I_\beta} \underline{f}(y) dy}.$$

Conversely, if ν_β is absolutely continuous with bounded density function h_β then \underline{f} satisfies

$$0 < \int_{I_\beta} \underline{f}(y) dy < \infty.$$

Theorems 2.1 and 2.2 give immediate consequences for the growth rate of $\mathcal{N}_n(x; \beta)$.

Corollary 2.1. *For almost every $\beta \in (1, 2)$, and for (Lebesgue) almost every $x \in I_\beta$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{N}_n(x; \beta)) = \log\left(\frac{2}{\beta}\right).$$

For almost every $\beta \in (1, \sqrt{2})$ and for (Lebesgue) almost every $x \in I_\beta$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{N}_n(x; \beta)) = \log\left(\frac{2}{\beta}\right).$$

The proof of this corollary is straightforward but we include it after the proof of Theorems 2.1 and 2.2 for completeness. These results contrast with the case that β is a Pisot number. It was shown in [6] that if β is a Pisot number then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{N}_n(x; \beta)) < \log\left(\frac{2}{\beta}\right).$$

2.1 Comparison with the Garsia entropy technique

In [8], Garsia approached the question of the potential singularity of Bernoulli convolutions by considering the entropy of the measures $\nu_{\beta,n}$, where the entropy of a discrete measure m supported on a finite set $\{x_1, \dots, x_k\}$ is defined as

$$H(m) = - \sum_{i=1}^k m(x_i) \log(m(x_i)).$$

Garsia showed that if

$$\lim_{n \rightarrow \infty} \frac{H(\nu_{\beta,n})}{n} < \log(\beta) \tag{1}$$

then ν_β is singular, but this has not been successful in determining whether ν_β is singular or absolutely continuous for any values of β .¹ Garsia's approach can be characterised as quantifying to what extent the finite sums $\sum_{i=1}^n x_i \beta^{-i}$ coincide for different choices of x_1, \dots, x_n .

Rather than asking for coincidence of these finite sums, our approach asks about clustering. The functions f_n have integral 1 and take values $f_n(x)$ proportional to the number of elements of

$$\mathcal{D}_n := \left\{ \sum_{i=1}^n x_i \beta^{-i} : x_i \in \{0, 1\} \right\}$$

which lie in $\left[x - \frac{1}{(\beta-1)\beta^n}, x \right]$. Then the regularity of the function f_n describes the degree of clustering in \mathcal{D}_n . In particular, if $\int_{I_\beta} \bar{f}(x) dx \in \{0, \infty\}$ then this corresponds to a high degree of clustering in the sets \mathcal{D}_n as n tends to infinity.

¹Garsia did show that if β is a Pisot number then inequality 1 holds, but his proof relied on the fact that the corresponding Bernoulli convolutions are singular. However using inequality 1 Garsia was able to prove that Bernoulli convolutions associated to Pisot numbers have Hausdorff dimension less than one, see also [14, 9].

Thus, our results can be interpreted as saying that the absolute continuity or singularity of the Bernoulli convolution can be determined by measuring the degree of clustering in the sets \mathcal{D}_n .²

3 An Operator on Densities

The results stated in the previous section come from simple analysis of a naturally defined operator related to Bernoulli convolutions. Bernoulli convolutions satisfy the self-similarity relation

$$\nu_\beta = \frac{1}{2}(\nu_\beta \circ T_0 + \nu_\beta \circ T_1)$$

where $T_i(x) : \beta x - i$. This implies that if ν_β is absolutely continuous with density h_β then h_β must also satisfy a self-similarity relation:

$$h_\beta(x) = \frac{\beta}{2}(h_\beta(\beta x) + h_\beta(\beta x - 1)).$$

We define the operator $P : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ by

$$P(f)(x) = \frac{\beta}{2}(f(\beta x) + f(\beta x - 1)).$$

P preserves the set of densities

$$D := \{f \in L^1 : f \geq 0, \int_{\mathbb{R}} f(x)dx = 1, x \notin I_\beta \implies f(x) = 0\}.$$

The absolute continuity of ν_β is equivalent to the existence of a function $h_\beta \in D$ satisfying $P(h_\beta) = h_\beta$.

Lemma 3.1. *The following are elementary properties of P*

1. $f \leq g \implies P(f) \leq P(g)$
2. $P(kf) = kP(f) \forall k \in \mathbb{R}$
3. If $f \geq 0$ then $\int_{\mathbb{R}} P(f) = \int_{\mathbb{R}} f$
4. If P has fixed point h_β then $\|P(f) - h_\beta\|_1 \leq \|f - h_\beta\|_1$ for all $f \in L^1$.

Proof. The first two statements follow immediately from the definition of P . The third statement follows from the fact that, for each i

$$\int_{\mathbb{R}} f(T_i(x))dx = \frac{1}{\beta} \int_{\mathbb{R}} f(x)dx.$$

²This perspective on the question of counting β -expansions arose out of conversations with Evgeny Verbitskiy, many thanks to Evgeny for these.

The fourth statement is just an application of the triangle inequality. Using that $P(h_\beta) = h_\beta$, we have that

$$\begin{aligned}
\int_{\mathbb{R}} |P(f)(x) - h_\beta(x)| dx &= \int_{\mathbb{R}} \left| \frac{\beta}{2} (f(\beta x) - h_\beta(\beta x) + f(\beta x - 1) - h_\beta(\beta x - 1)) \right| dx \\
&\leq \frac{\beta}{2} \left(\int_{\mathbb{R}} |f(\beta x) - h_\beta(\beta x)| dx + \int_{\mathbb{R}} |f(\beta x - 1) - h_\beta(\beta x - 1)| dx \right) \\
&= \frac{\beta}{2} \left(\frac{1}{\beta} \int_{\mathbb{R}} |f(x) - h_\beta(x)| dx + \frac{1}{\beta} \int_{\mathbb{R}} |f(x) - h_\beta(x)| dx \right) \\
&= \int_{\mathbb{R}} |f(x) - h_\beta(x)| dx
\end{aligned}$$

as required. \square

We now link P with the functions defined in the first section.

Lemma 3.2. *The functions f_n satisfy*

$$f_n = P^n((\beta - 1)\chi_{I_\beta})$$

where χ_{I_β} is the indicator function on the interval I_β .

Proof. Expanding P , we see that

$$P^n((\beta - 1)\chi_{I_\beta})(x) = (\beta - 1) \left(\frac{\beta}{2} \right)^n \sum_{x_1, \dots, x_n \in \{0,1\}^n} \chi_{I_\beta}(T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(x))$$

But a word $x_1, \dots, x_n \in \{0,1\}^n$ is in $\mathcal{E}_\beta^n(x)$ if and only if $T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(x) \in I_\beta$, see for example [2]. This gives that

$$\sum_{x_1, \dots, x_n \in \{0,1\}^n} \chi_{I_\beta}(T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(x)) = \mathcal{N}_n(x; \beta).$$

Then since $f_n := (\beta - 1) \left(\frac{\beta}{2} \right)^n \mathcal{N}_n(x; \beta)$ we have the required result. \square

The next four lemmas prove our main results.

Lemma 3.3. *If ν_β is absolutely continuous then $0 < \int_{I_\beta} \bar{f}(x) dx$.*

Proof. We suppose that ν_β is absolutely continuous with density h_β , and for a contradiction suppose that $\int_{I_\beta} \bar{f}(x) dx = 0$. Consequently we have that $f_n = P^n((\beta - 1)\chi_{I_\beta}) \rightarrow 0$ for Lebesgue almost every x . Now if $P^n((\beta - 1)\chi_{I_\beta}) \rightarrow 0$ almost everywhere, then linearity and monotonicity of P give that for any bounded $g \in D$, $P^n(g) \rightarrow 0$ almost everywhere. Since $h_\beta \in L^1$ we can, for any $0 < \epsilon < \frac{1}{2}$, take a bounded function $g_\epsilon \in D$ with

$$\|h_\beta - g_\epsilon\|_1 < \epsilon.$$

But $P^n(g_\epsilon) \rightarrow 0$ a.e. since g_ϵ is bounded. Then since h_β has integral 1, we see that eventually

$$\|h_\beta - P^n(g_\epsilon)\|_1 > \frac{1}{2} > \epsilon.$$

This contradicts part 4 of lemma 3.1, the non expansiveness of P in L^1 . \square

Lemma 3.4. *If ν_β is absolutely continuous then $\int_{I_\beta} \bar{f}(x)dx \leq 2$.*

Proof. In order to show that $\int_{I_\beta} \bar{f}(x)dx \leq 2$ we in fact show that $\bar{f}(x) \leq 2h_\beta(x)$. Given some word $x_1, \dots, x_n \in \{0, 1\}^n$, we see that all sequences \underline{x} starting with word x_1, \dots, x_n are β -expansions of points

$$x = \sum_{i=1}^{\infty} x_i \beta^{-i} \in \left[\sum_{i=1}^n x_i \beta^{-i}, \sum_{i=1}^n x_i \beta^{-i} + \frac{1}{(\beta-1)\beta^n} \right].$$

Now we suppose for each $j \in \{1, \dots, \mathcal{N}_n(x; \beta)\}$, $x_1^j, \dots, x_n^j \in \mathcal{E}_\beta^n(x)$. Then

$$\sum_{i=1}^n x_i^j \beta^{-i} \in \left[x - \frac{1}{(\beta-1)\beta^n}, x \right],$$

and so for any $m > n$, $j \in \{1, \dots, \mathcal{N}_n(x; \beta)\}$ and any word x_1, \dots, x_m starting with x_1^j, \dots, x_n^j we have that

$$\sum_{i=1}^m x_i \beta^{-i} \in \left[x - \frac{1}{(\beta-1)\beta^n}, x + \frac{1}{(\beta-1)\beta^n} \right].$$

There are at least $2^{m-n} \mathcal{N}_n(x; \beta)$ such words. Hence

$$\nu_{\beta, m} \left[x - \frac{1}{(\beta-1)\beta^n}, x + \frac{1}{(\beta-1)\beta^n} \right] \geq \frac{1}{2^m} (\mathcal{N}_n(x; \beta) \cdot 2^{m-n}) = \frac{\mathcal{N}_n(x; \beta)}{2^n}$$

Then using that $\nu_{\beta, m} \rightarrow \nu_\beta$, multiplying each side by β^n , and using the fact that

$$\lim_{n \rightarrow \infty} \frac{(\beta-1)\beta^n}{2} \nu_\beta \left[x - \frac{1}{(\beta-1)\beta^n}, x + \frac{1}{(\beta-1)\beta^n} \right] = h_\beta(x)$$

we get that

$$\bar{f}(x) = \limsup_{n \rightarrow \infty} (\beta-1) \left(\frac{\beta}{2} \right)^n \mathcal{N}_n(x; \beta) \leq 2h_\beta(x).$$

□

A similar argument was used in Appendix C of [13] to bound $\left(\frac{\beta}{2}\right)^n \mathcal{N}_n(x; \beta)$ in the case that $h_\beta(x)$ is bounded. The authors attributed the argument to Yuval Peres. We now relate \bar{f} to h_β .

Lemma 3.5. *If \bar{f} has positive finite integral then ν_β is absolutely continuous with density h_β satisfying $h_\beta(x) = \frac{\bar{f}(x)}{\int_{I_\beta} \bar{f}(y)dy}$. The same is true for \underline{f} .*

Proof. We proved in lemma 3.2 that $f_n(x) = P^n((\beta-1)\chi_{I_\beta})$, which gives in particular that

$$f_n(x) = P(f_{n-1}(x)) = \frac{\beta}{2} (f_{n-1}(\beta x) + f_{n-1}(\beta x - 1))$$

Then since $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$, we have that

$$\bar{f}(x) \leq \frac{\beta}{2} (\bar{f}(\beta x) + \bar{f}(\beta x - 1)),$$

i.e. $P(\bar{f}) \geq \bar{f}$. But since \bar{f} is non negative, part 3 of lemma 3.1 gives that $\int_{\mathbb{R}} P(\bar{f}) = \int_{\mathbb{R}} \bar{f}$. So we must have that $P(\bar{f}) = \bar{f}$ almost everywhere.

Then we see that $\frac{\bar{f}}{\int_{\mathbb{R}} \bar{f}}$ has integral one and is a fixed point of P , and therefore it must be equal to h_β almost everywhere. Similar arguments work for \underline{f} using that $P(\underline{f}) \leq \underline{f}$. \square

Lemmas 3.3, 3.4 and 3.5 complete the proof of Theorem 2.1. Finally, we prove the following lemma.

Lemma 3.6. *If ν_β is absolutely continuous and has bounded density function h_β then $0 < \int_{I_\beta} \underline{f}(x)dx \leq 1$.*

Proof. Fatou's lemma gives us that

$$\int_{I_\beta} \underline{f}(x)dx \leq \liminf_{n \rightarrow \infty} \int_{I_\beta} f_n(x)dx = 1$$

Now we assume that ν_β is absolutely continuous with bounded density $h_\beta < C$ for some $C > 0$. We suppose that $h_\beta(x) > 0$, this holds for almost every $x \in I_\beta$, see [10]. Applying the operator P n times to h_β we see that

$$\begin{aligned} h_\beta(x) &= \left(\frac{\beta}{2}\right)^n \sum_{x_1, \dots, x_n \in \mathcal{E}_\beta^n(x)} h_\beta(T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(x)) \\ &\leq \left(\frac{\beta}{2}\right)^n \mathcal{N}_n(x; \beta) \cdot C, \end{aligned}$$

where we have used that $|\mathcal{E}_\beta^n(x)| = \mathcal{N}_n(x; \beta)$ and that $h_\beta(T_{x_n} \circ T_{x_{n-1}} \circ \dots \circ T_{x_1}(x)) < C$ for each $x_1, \dots, x_n \in \{0, 1\}^n$.

Then rearranging and taking the lim inf we see that

$$\liminf_{n \rightarrow \infty} P^n(x) = \liminf_{n \rightarrow \infty} (\beta - 1) \left(\frac{\beta}{2}\right)^n \mathcal{N}_n(x; \beta) \geq \frac{(\beta - 1)h_\beta(x)}{C}$$

as required. \square

Lemmas 3.5 and 3.6 complete the proof of Theorem 2.2.

Finally we prove Corollary 2.1

Proof. If $\bar{f}(x) \in (0, \infty)$ then $\limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathcal{N}_n(x; \beta)) = \log\left(\frac{2}{\beta}\right)$. But it was proved in [10] that ν_β is equivalent to Lebesgue measure, and whenever this holds then Theorem 2.1 implies that $\bar{f}(x) = h_\beta(x) \cdot \int_{I_\beta} \bar{f}(y)dy$ is positive for almost every $x \in I_\beta$. Thus the first statement of the corollary holds.

To prove the second statement it is enough to prove that $\underline{f}(x) \in (0, \infty)$ for almost every $x \in I_\beta, \beta \in (1, \sqrt{2})$. But for almost every $\beta \in (1, \sqrt{2})$ we have that ν_β is absolutely continuous with bounded density (see [17]), and then Theorem 2.2 gives that $\underline{f}(x) = h_\beta(x) \cdot \int_{I_\beta} \underline{f}(y)dy$ is positive for almost every $x \in I_\beta$ (where we have used again that $h(x)$ is positive for almost every $x \in I_\beta$). \square

4 Counting expansions using the random β -transformation

In section 2 we gave results on the growth rate of $\mathcal{N}_n(x; \beta)$ which hold for almost all $\beta \in (1, 2)$. In this section we take an alternative approach, using the ergodic theory of the random β -transformation to study the growth rate of $\mathcal{N}_n(x; \beta)$ for all $\beta \in (1, 2)$.

In [6], Feng and Sidorov proved that for all $\beta \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ there exists a constant $c(\beta) > 0$ such that for all $x \in I_\beta$

$$\liminf_{n \rightarrow \infty} \frac{\log(\mathcal{N}_n(x; \beta))}{n} \geq c(\beta).$$

We extend this result beyond the case that $\beta \in (1, \frac{1+\sqrt{5}}{2})$.

Theorem 4.1. *For every β in $(1, 2)$, there exists $c(\beta) > 0$ such that for almost all $x \in (0, \frac{1}{\beta-1})$,*

$$\liminf_{n \rightarrow \infty} \frac{\log(\mathcal{N}_n(x; \beta))}{n} \geq c(\beta).$$

In fact the above theorem extends to all non integer $\beta > 1$, the proof for $\beta > 2$ follows that given below but has more complicated notation, and can be found in the author's thesis. We stress that, in the case that $\beta \geq \frac{1+\sqrt{5}}{2}$, the almost all $x \in (0, \frac{1}{\beta-1})$ of Theorem 4.1 cannot be extended to hold for all $x \in (0, \frac{1}{\beta-1})$. There are, for example, points x with unique β -expansion.

In [2], Dajani and Kraaikamp defined the random β -transformation $K_\beta : \{0, 1\}^{\mathbb{N}} \times [0, \frac{1}{\beta-1}] \rightarrow \{0, 1\}^{\mathbb{N}} \times [0, \frac{1}{\beta-1}]$ by

$$K_\beta(\omega, x) = \begin{cases} (\omega, T_0(x)) & x \in [0, \frac{1}{\beta}) \\ (\sigma(\omega), T_{\omega_1}(x)) & x \in [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}] \\ (\omega, T_1(x)) & x \in (\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}] \end{cases}.$$

Given a pair (ω, x) , a beta expansion of x is generated by iterating $K_\beta(\omega, x)$. If the n th iteration of $K_\beta(\omega, x)$ applies T_0 to the second coordinate we put $x_n = 0$, if it applies T_1 to the second coordinate we put $x_n = 1$. The sequence (x_n) is a β -expansion of x [2].

This allows the study of all β -expansions of x , each different choice of ω corresponds to a different β -expansion of x , up to a set of measure zero, and all β -expansions are given this way. We refer to the β -expansion $(x_n)_{n=1}^{\infty}$ obtained by iterating $K_\beta(\omega, x)$ as the β -expansion of x generated by ω .

We define the switch region $S := [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$. When the orbit of (ω, x) enters $\hat{S} := \Omega \times S$ we are allowed a choice about how to continue the β -expansion, this choice is made by looking at the first digit of the sequence ω . We define the hitting number

$$h(\omega, x, n) := \#\{i \in \{0, \dots, n-1\} : K_\beta^i(\omega, x) \in \hat{S}\}.$$

We note that the dependence of $h(\omega, x, n)$ on the sequence ω is really just a dependence on the finite word $\omega_1, \dots, \omega_{h(\omega, x, n)}$, since ω only influences the orbit of x when $K_\beta^n(\omega, x)$

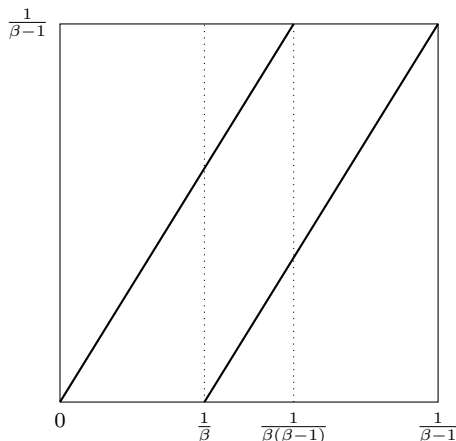


Figure 1: The projection onto the first coordinate of K_β for $\beta = \frac{1 + \sqrt{5}}{2}$

enters the switch region \hat{S} . We see that the β -expansions of x generated by ω and ω' agree to the first n places if and only if $\omega_1, \dots, \omega_{h(\omega, x, n)} = \omega'_1, \dots, \omega'_{h(\omega', x, n)}$.

In [1], Dajani and de Vries showed that K_β has invariant probability measure $\hat{\mu}_\beta = m \times \mu_\beta$, where μ_β is absolutely continuous with respect to λ (Lebesgue measure), and m is the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on $\{0, 1\}^\mathbb{N}$. They also showed that K_β is ergodic with respect to this measure.

We begin by describing $\mathcal{N}_n(x; \beta)$ in terms of $h(\omega, x, n)$ and m .

Lemma 4.1. *We have the equality*

$$\mathcal{N}_n(x; \beta) = \int_{\{0,1\}^\mathbb{N}} 2^{h(\omega, x, n)} dm.$$

Proof. The set $\mathcal{E}_\beta^n(x)$ of n -prefixes of β -expansions of x corresponds to the number of different words $\omega_1, \dots, \omega_{h(\omega, x, n)}$ for $\omega \in \{0, 1\}^\mathbb{N}$. So defining

$$\Omega(x, n) := \{\omega_1, \dots, \omega_{h(\omega, x, n)} : \omega \in \{0, 1\}^\mathbb{N}\},$$

we have $|\Omega(x, n)| = \mathcal{N}_n(x; \beta)$.

We see that $m[\omega_1, \dots, \omega_{h(\omega, x, n)}] = 2^{-h(\omega, x, n)}$. Then

$$\begin{aligned} |\Omega(x, n)| &= \sum_{k=0}^n |\{\omega_1, \dots, \omega_{h(\omega, x, n)} : \omega \in \{0, 1\}^\mathbb{N}, h(\omega, x, n) = k\}| \\ &= \sum_{k=0}^n 2^k m\{\omega \in \{0, 1\}^\mathbb{N} : h(\omega, x, n) = k\} \\ &= \int_{\{0,1\}^\mathbb{N}} 2^{h(\omega, x, n)} dm \end{aligned}$$

as required. □

From this point the proof of Theorem 4.1 is straightforward. By the ergodic theorem we have that for almost every (ω, x) (with respect to $\hat{\mu}_\beta$) in $[0, \frac{1}{\beta-1}] \times \{0, 1\}^{\mathbb{N}}$,

$$\lim_{n \rightarrow \infty} \frac{h(\omega, x, n)}{n} = \hat{\mu}_\beta(\hat{S}) = \mu_\beta(S).$$

Decomposing $\hat{\mu}_\beta = m \times \mu_\beta$, and recalling that μ_β is absolutely continuous with respect to λ , we get that for almost every x (w.r.t. λ), for almost every ω (w.r.t. m),

$$\lim_{n \rightarrow \infty} \frac{h(\omega, x, n)}{n} = \mu_\beta(S).$$

Then, since almost everywhere convergence implies convergence in probability, we have that for almost every x and for all $\epsilon, \delta > 0$ there exists a constant $N_{\epsilon\delta}$ such that for all $n > N_{\epsilon\delta}$,

$$m \left(\left\{ \omega \in \{0, 1\}^{\mathbb{N}} : \left| \frac{h(\omega, x, n)}{n} - \mu_\beta(S) \right| \geq \epsilon \right\} \right) < \delta.$$

We define the good set

$$\begin{aligned} G(n, x, \epsilon) &= \left\{ \omega \in \{0, 1\}^{\mathbb{N}} : \left| \frac{h(\omega, x, n)}{n} - \mu_\beta(S) \right| < \epsilon \right\} \\ &= \left\{ \omega \in \{0, 1\}^{\mathbb{N}} : n(\mu_\beta(S) - \epsilon) < h(\omega, x, n) < n(\mu_\beta(S) + \epsilon) \right\}. \end{aligned}$$

Now

$$m(G(n, x, \epsilon)) > 1 - \delta,$$

and so for almost every x ,

$$\begin{aligned} \int_{\{0,1\}^{\mathbb{N}}} 2^{h(\omega,x,n)} dm &\geq \int_{G(n,x,\epsilon)} 2^{h(\omega,x,n)} dm \\ &\geq (1 - \delta) 2^{n(\mu_\beta(S) - \epsilon)}. \end{aligned}$$

Then

$$\mathcal{N}_n(x; \beta) \geq (1 - \delta) (2^{n(\mu_\beta(S) - \epsilon)}),$$

and since ϵ and δ were arbitrary, we have

$$\liminf_{n \rightarrow \infty} \frac{\log(\mathcal{N}_n(x; \beta))}{n} \geq \log(2) \mu_\beta(S).$$

This completes the proof of Theorem 4.1.

Finally we demonstrate that the lower bound on $\mathcal{N}_n(x; \beta)$ need not be sharp. It was shown in [6] Table 1 that for $\beta = \frac{1+\sqrt{5}}{2}$ we have that

$$\lim_{n \rightarrow \infty} \frac{\log(\mathcal{N}_n(x; \beta))}{n} = \log(2)(0.302 \pm 0.01).$$

But from [1], Example 1, we have that

$$\mu_\beta(S) = \left(\frac{\sqrt{5}-1}{2} \right)^2 \left(\mu_\beta \left[0, \frac{1}{\beta} \right] + \mu_\beta \left[1, \frac{1}{\beta-1} \right] \right)$$

which yields $\mu_\beta(S) = \frac{5-\sqrt{5}}{10} \approx 0.276 < 0.302$. Thus in the case that β is equal to the golden mean we have that the inequality in Theorem 4.1 is strict. Thanks to Nikita Sidorov for pointing this out to us.

5 Further Questions

There are some further questions that arise naturally. The first is whether the absolute continuity of ν_β is equivalent to the convergence of f_n to the density of ν_β . We make the following conjecture.

Conjecture 1. *If ν_β is singular then $\lim_{n \rightarrow \infty} f_n(x) = 0$ almost everywhere. If ν_β is absolutely continuous then $\lim_{n \rightarrow \infty} f_n(x)$ exists almost everywhere and is equal to the density of ν_β .*

Bernoulli convolutions are often studied via the sequence of measures $\nu_{\beta,n}$, which converge weakly to ν_β . One could also define a sequence of measures $m_{\beta,n}$ to be the probability measures with density f_n . These measures are the measures obtained by letting $m_{\beta,0}$ be normalised Lebesgue measure on I_β and by defining

$$m_{\beta,n+1} = \frac{1}{2} (m_{\beta,n} \circ T_0 + m_{\beta,n} \circ T_1).$$

It is easy to see that the sequence $(m_{\beta,n})$ converges weak* to ν_β . A proof of the above conjecture would show that if m_β is absolutely continuous then $(m_{\beta,n})$ converges in the stronger sense that the densities converge almost everywhere to h_β . This would be useful in determining, for example, the multifractal properties of ν_β .

Our second question is whether one can calculate $\mathcal{N}_n(x; \beta)$ precisely in the case that K_β is a Markov map. Pisot numbers are the only known examples of β for which ν_β is singular, but Salem numbers are widely regarded as the most likely candidates for non-Pisot values of β which may yield singular Bernoulli convolutions, and there are many Salem numbers β for which K_β is Markov. Feng and Sidorov calculated the growth of $\mathcal{N}_n(x; \beta)$ for Pisot values of β , if one were able to extend their methods to include Salem numbers it would allow one to determine whether Salem numbers have absolutely continuous Bernoulli convolutions or not.

Finally, we ask whether it is possible to improve our proof of Theorem 4.1 in order to get precise lower bounds on $\mathcal{N}_n(x; \beta)$ using the K_β map. Lemma 4.1 describes $\mathcal{N}_n(x; \beta)$ accurately in terms of $h(\omega, x, n)$, but the ergodic theory that we use in our subsequent analysis of $h(\omega, x, n)$ is not strong enough to give a sharp lower bound. Perhaps a more delicate analysis of the ergodic theory of K_β could yield better understanding of the growth of $\mathcal{N}_n(x; \beta)$.

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