Filtering Problems with Cox Jump Processes

Financial Mathematics Reading Group Talk hold by
Thomas Bernhardt

Introduction

Introduction

Two Topics

Cox processes

- counting processes
- jump rate correspondes to some intensity
- intensity can be discribed without the jumps
- goals: characterization, intensity manipulation

Filtering problem

- unknown intensity and known jumps
- exemplary calculation of a conditional expectation
- intensity is an Ornstein-Uhlenbeck process

Cox processes

Cox processes

Definition: On the given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

Let N counting process, X non-negative $\mathcal{F}_0 \otimes \mathbb{B}([0,\infty[)$-mb.
N is a Cox process with intensity X if for $t \geq r \geq 0$ and $n \in \mathbb{N}_0$

$$\mathbb{P}\Big[\int_{]0,t]} X_s \,\mathrm{d}s < \infty \quad \forall\, t \geq 0\Big] = 1$$

$$\mathbb{P}\Big[N_t - N_r = n \Big| \mathcal{F}_r\Big] = \mathrm{e}^{-\int_{]r,t]} X_s \,\mathrm{d}s} \cdot \frac{(\int_{]r,t]} X_s \,\mathrm{d}s)^n}{n!} \quad \mathbb{P}\text{-fs.}$$

- Intensity constantly one leads to a Poisson process with respect to the given filtration
- Conditional distribution of jump times can be computed: $\mathbb{P}[T_{n+1} > t|\mathcal{F}_r] = \mathbb{P}[N_t \leq n|\mathcal{F}_r] = \sum_{k=0}^{n-N_r} \mathbb{P}[N_t - N_r = k|\mathcal{F}_r]$

Cox processes

Theorem: Characterization of Cox processes

N finite counting process, X non-negative $\mathcal{F}_0 \otimes \mathbb{B}([0,\infty[)$-mb with $\mathbb{P}[\int_{]0,t]} X_s \,\mathrm{d}s < \infty \;\forall\, t \geq 0] = 1$. Then is equivalent that:

(i) N is a Cox process with intensity X,
(ii) $M = N - \int_0 X_s \,\mathrm{d}s$ is a local martingale,
(iii) $\varphi \geq 0$ predictable: $\mathbb{E}[\int_{]0,\infty[} \varphi_s \,\mathrm{d}N_s] = \mathbb{E}[\int_{]0,\infty[} \varphi_s X_s \,\mathrm{d}s]$.

Proof:

(i)$\Rightarrow$(ii) jump times for localization and cond. distribution,
(ii)$\Rightarrow$(iii) measures implied by N, $\int_0 X_s \,\mathrm{d}s$ coincide on a σ-finite $\cap$-stable generator,
(iii)$\Rightarrow$(i) stochastic exponential contains Laplace transform.

Cox processes

Theorem: Change of measure

N Cox with intensity X; Y non-negative $\mathcal{F}_0 \otimes \mathbb{B}([0,\infty[)$-mb with $\mathbb{P}[\int_{]0,t]} Y_s X_s \,\mathrm{d}s < \infty \;\forall\; t \geq 0] = 1$;

$W := \exp(-\int_0 (Y_s - 1) X_s \,\mathrm{d}s) \cdot \prod_{s \in]0,]} (1 + (Y_s - 1)\Delta N_s)$.

Then

- W is a non-negative right-continuous martingale,
- $W_T = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ it follows that $\mathbb{P} = \mathbb{Q}$ on $\mathcal{F}_0$ and N^T is again Cox under $\mathbb{Q}$ with intensity $\mathbb{1}_{[0,T]} YX$.

Proof: martingale property

W is a supermartingale (non-negative stochastic exponential).
W does not vary in expectation (using conditional distribution).

Cox processes

Theorem: Change of measure N Cox, intensity X; Y as X

$W := \exp(-\int_0 (Y_s - 1) X_s \,\mathrm{d}s) \cdot \prod_{s \in]0,]} (1 + (Y_s - 1)\Delta N_s)$.

Then

- N^T Cox process with intensity $\mathbb{1}_{[0,T]} YX$ under $\mathbb{Q}$.

Proof: Cox property $\varphi \geq 0$ predictable

$$\mathbb{E}_{\mathbb{Q}}[\int_{]0,T]} \varphi_s \,\mathrm{d}N_s] = \mathbb{E}[W_T \int_{]0,T]} \varphi_s \,\mathrm{d}N_s] = \mathbb{E}[\int_{]0,T]} \varphi_s W_s \,\mathrm{d}N_s]$$
$$= \mathbb{E}[\int_{]0,T]} \varphi_s W_{s-} Y_s \,\mathrm{d}N_s],$$

$$\mathbb{E}_{\mathbb{Q}}[\int_{]0,T]} \varphi_s Y_s X_s \,\mathrm{d}s] = \mathbb{E}_{\mathbb{Q}}[W_T \int_{]0,T]} \varphi_s Y_s X_s \,\mathrm{d}s]$$
$$= \mathbb{E}[\int_{]0,T]} \varphi_s W_{s-} Y_s X_s \,\mathrm{d}s].$$

Using characterization gives the claim.

Filtering problems with Cox processes

Example with an OU intensity

- model $\mathrm{d}X = \mathrm{d}J - \lambda X \mathrm{d}t$ where J is compound Poisson process and M its Poisson process
- aim the conditional Laplace transform with additional information about M

$$\mathbb{E}\big[\exp\big(\alpha X_R + \beta X_S + \gamma \textstyle\int_{]R,S]} X_s \,\mathrm{d}s\big) \big| \mathcal{F}_R^N \vee \mathcal{F}_R^M\big]$$

Equivalent to calculate (solving the SDE, using independency)

$$\mathbb{E}\big[\exp\big(\textstyle\int_{]R,S]} B_s \,\mathrm{d}J_s\big)\big] \cdot \mathbb{E}\big[\exp\big((\alpha + B_R) X_R\big) \big| \mathcal{F}_R^N \vee \mathcal{F}_R^M\big]$$

where $B_s = \gamma/\lambda + (\beta - \gamma/\lambda) \cdot \exp\big(\lambda(s - S)\big)$.

The left term $\mathbb{E}\Big[\exp\Big(\int_{]R,S]} B_s \,\mathrm{d}J_s\Big)\Big]$

M be a-Poisson process and ξ independent size of a jump of J.

$$\mathbb{E}\Big[\exp\Big(\int_{]R,T]} B_s \,\mathrm{d}J_s\Big)\Big|\mathcal{F}^M_\infty\Big] = \prod_{s\in]R,S]}\Big(1+\Big(\mathbb{E}\big[\exp(B_s\xi)\big]-1\Big)\Delta M_s\Big)$$

$$\Rightarrow \quad \mathbb{E}\Big[\exp\Big(\int_{]R,T]} B_s \,\mathrm{d}J_s\Big)\Big] \overset{!}{=} \exp\Big(a\int_{]R,S]}\mathbb{E}[\exp(B_s\xi)]-1\,\mathrm{d}s\Big)$$

cos following process is a martingale (measure change theorem)

$$t \mapsto \frac{\prod_{s\in]R,t]}(1+(\mathbb{E}[\exp(B_s\xi)]-1)\Delta M_s)}{\exp(a\int_{]R,t]}\mathbb{E}[\exp(B_s\xi)]-1\,\mathrm{d}s)}.$$

Filtering problems with Cox processes

The right term $\mathbb{E}\Big[\exp\big((\alpha + B_R)X_R\big)\Big|\mathcal{F}_R^N \vee \mathcal{F}_R^M\Big]$

General approach: X positive, choose $Y = 1/X$

Measure $\mathbb{Q}$ with N independent of X and untouched distribution of X.

Look at H $\mathcal{F}_0$-mb. Bayes, independency, measurability:

$$\mathbb{E}[H|\mathcal{F}_R^N \vee \mathcal{F}_R^M]$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}[H\,\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}|\mathcal{F}_R^N \vee \mathcal{F}_R^M]}{\mathbb{E}_{\mathbb{Q}}[\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}|\mathcal{F}_R^N \vee \mathcal{F}_R^M]} = \frac{\mathbb{E}[(H\,\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q})^{m,n}]}{\mathbb{E}[(\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q})^{m,n}]}\Bigg|_{N=n}^{M=m}.$$

Filtering Problems with Cox Process
Bernhardt/LSE
Introduction
Cox processes
Filtering problem

The formula (for $\Gamma(b,p)$ distributed jump sizes)

$$\mathbb{E}\left[\exp\left(\alpha X_R + \beta X_S + \gamma \int_{]R,S]} X_s \mathrm{d}s\right) \Big| \mathcal{F}_R^N \vee \mathcal{F}_R^M\right] = \exp\left(a \int_{]R,S]} \frac{b^p}{(b - B_s)^p} - 1 \,\mathrm{d}s\right) \cdot \frac{Z_R^{\alpha + B_R}}{Z_R^0}$$

$$\text{where} \quad Z_R^\alpha = \int_{M_R} \exp\left(\alpha X_R^{M,j}\right) \exp\left(\int_{]0,R]} 1 - X_s^{M,j} \,\mathrm{d}s\right) \prod_{s \in]0,R]} \left(1 + (X_s^{M,j} - 1)\Delta N_s\right) \,\mathrm{d}\Gamma_j$$

$$\text{and} \quad X_s^{M,j} = \mathrm{e}^{-\lambda s} x_0 + \mathrm{e}^{-\lambda s} \sum_{n=1}^{M_s} \mathrm{e}^{\lambda \theta_n} \cdot j_n$$

as well as M_R as subscript of the integrals the number of integrations over the positive real line means with respect to the multidimensional variable j.