§6 Forms and vector fields on Euclidean space

Problem 1. For spherical coordinates we have the following expression of the basis vectors in the Cartesian coordinates x, y, z:

$$\begin{aligned}\boldsymbol{e}_r &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\\ \boldsymbol{e}_\theta &= (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta)\\ \boldsymbol{e}_\varphi &= (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)\end{aligned}$$

Using this, we can calculate the pairwise scalar products. First the scalar squares: $(\boldsymbol{e}_r, \boldsymbol{e}_r) = (\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = \sin^2\theta + \cos^2\theta = 1$; and, similarly, $(\boldsymbol{e}_\theta, \boldsymbol{e}_\theta) = r^2$, $(\boldsymbol{e}_\varphi, \boldsymbol{e}_\varphi) = r^2\sin^2\theta$. Now, $(\boldsymbol{e}_r, \boldsymbol{e}_\theta) = (\sin\theta\cos\varphi)(r\cos\theta\cos\varphi) + (\sin\theta\sin\varphi) - (\cos\theta)(r\sin\theta) = r\sin\theta\cos\theta - r\sin\theta\cos\theta = 0$. Similarly one can show that $(\boldsymbol{e}_r, \boldsymbol{e}_\varphi) = 0$ and $(\boldsymbol{e}_\theta, \boldsymbol{e}_\varphi) = 0$. Therefore we have the following matrix:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

where the coordinates are in the order r, θ, φ, and $g = \det(g_{ij}) = r^4\sin^2\theta$. Hence $\sqrt{g} = r^2\sin\theta$ and the volume form is $dV = r^2\sin\theta\, dr\, d\theta\, d\varphi$.

Problem 2.
(a) Using θ, φ as parameters we have for a point of the sphere:

$$\boldsymbol{x} = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta),$$

and the corresponding basis in the tangent plane is $\boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$, where

$$\begin{aligned}\boldsymbol{e}_\theta &= (R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta)\\ \boldsymbol{e}_\varphi &= (-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0).\end{aligned}$$

(Notice the difference with the previous problem; there coordinates in $\mathbb{R}^3$ are considered and r is a variable, while here R is a constant and we consider coordinates on the 2-dimensional sphere.) We obtain the matrix (h_{ij}) of the pairwise scalar products $(i, j = 1, 2)$:

$$(h_{ij}) = \begin{pmatrix} R^2 & 0\\ 0 & R^2\sin^2\theta \end{pmatrix}.$$

Hence its determinant equals $R^4\sin^2\theta$, and the area element for the sphere is

$$dS = R^2\sin\theta\, d\theta \wedge d\varphi.$$

(b) We denote the basis vectors corresponding to x, y in the tangent plane to the sphere at a point $\boldsymbol{x}$ as $\boldsymbol{e}_x, \boldsymbol{e}_y$. They should not be confused with the basis vectors in $\mathbb{R}^3$ corresponding to the Cartesian coordinates x, y, z. We have $\boldsymbol{x} = (x, y, \sqrt{R^2 - x^2 - y^2})$. (This parametrization is valid for the upper hemisphere.) Thus

$$\boldsymbol{e}_x = \frac{\partial \boldsymbol{x}}{\partial x} = (1, 0, -\frac{x}{\sqrt{R^2 - x^2 - y^2}})$$
$$\boldsymbol{e}_y = \frac{\partial \boldsymbol{x}}{\partial y} = (0, 1, -\frac{y}{\sqrt{R^2 - x^2 - y^2}})$$

and for the pairwise scalar products we have: $(\boldsymbol{e}_x, \boldsymbol{e}_x) = 1 + \frac{x^2}{R^2-x^2-y^2} = \frac{R^2-y^2}{R^2-x^2-y^2}$, $(\boldsymbol{e}_x, \boldsymbol{e}_y) = \frac{xy}{R^2-x^2-y^2}$, and $(\boldsymbol{e}_y, \boldsymbol{e}_y) = \frac{R^2-x^2}{R^2-x^2-y^2}$, giving a matrix (h_{ij}):

$$(h_{ij}) = \begin{pmatrix} \frac{R^2-y^2}{R^2-x^2-y^2} & \frac{xy}{R^2-x^2-y^2} \\ \frac{xy}{R^2-x^2-y^2} & \frac{R^2-x^2}{R^2-x^2-y^2} \end{pmatrix}.$$

Its determinant equals

$$(R^2 - x^2 - y^2)^{-2}\big((R^2 - x^2)(R^2 - y^2) - x^2y^2\big) =$$
$$(R^2 - x^2 - y^2)^{-2} R^2 (R^2 - x^2 - y^2) = R^2 (R^2 - x^2 - y^2)^{-1}$$

Hence the area element for the sphere is

$$dS = \frac{R\, dx \wedge dy}{\sqrt{R^2 - x^2 - y^2}} = \frac{R\, dx \wedge dy}{z}.$$

Problem 3. In Cartesian coordinates in $\mathbb{R}^n$ if $\boldsymbol{X} = \sum X^i \boldsymbol{e}_i$, then

$$\boldsymbol{X} \cdot d\boldsymbol{r} = \sum X^i \, dx^i$$
$$\boldsymbol{X} \cdot d\boldsymbol{S} = \sum (-1)^{i-1} X^i \, dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$$

(summation from 1 to n). In particular in $\mathbb{R}^3$, $\boldsymbol{X} \cdot d\boldsymbol{r} = X^1\, dx + X^2\, dy + X^3\, dz$ and $\boldsymbol{X} \cdot d\boldsymbol{S} = X^1\, dy \wedge dz + X^2\, dz \wedge dx + X^3\, dx \wedge dy$ (notice the order of factors in the second term). Hence we get the following answers:

(a) for $\boldsymbol{X} = 2\boldsymbol{e}_1 - 3\boldsymbol{e}_2 + 4\boldsymbol{e}_3$,

$$\boldsymbol{X} \cdot d\boldsymbol{r} = 2\, dx - 3\, dy + 4\, dz$$
$$\boldsymbol{X} \cdot d\boldsymbol{S} = 2\, dy \wedge dz - 3\, dz \wedge dx + 4\, dx \wedge dy$$

(b) for $\boldsymbol{X} = (2x + z)\boldsymbol{e}_1 + 5y\boldsymbol{e}_2 + (x - y + z)\boldsymbol{e}_3$,

$$\boldsymbol{X} \cdot d\boldsymbol{r} = (2x + z)\, dx + 5y\, dy + (x - y + z)\, dz$$
$$\boldsymbol{X} \cdot d\boldsymbol{S} = (2x + z)\, dy \wedge dz + 5y\, dz \wedge dx + (x - y + z)\, dx \wedge dy$$

(c) for $\boldsymbol{X} = \boldsymbol{r}$ (where $\boldsymbol{r}$ is the radius-vector),

$$\boldsymbol{X} \cdot d\boldsymbol{r} = x\,dx + y\,dy + z\,dz$$
$$\boldsymbol{X} \cdot d\boldsymbol{S} = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

Problem 4. The gradient $\operatorname{grad} f$ is the vector field corresponding to the 1-form df, i.e., $\operatorname{grad} f \cdot d\boldsymbol{r} = df$ or

$$(\operatorname{grad} f, \boldsymbol{Y}) = \langle df, \boldsymbol{Y} \rangle$$

for an arbitrary vector field $\boldsymbol{Y}$ (at the LHS stands the scalar product of two vectors, at the RHS the value of a 1-form on a vector). Let $\operatorname{grad} f = (\operatorname{grad} f)^i \boldsymbol{e}_i$. Taking as $\boldsymbol{Y}$ the basis vector fields $\boldsymbol{e}_j$ associated with an arbitrary coordinate system, $i = j, \ldots, n$, we obtain therefore

$$(\operatorname{grad} f)^i g_{ij} = \frac{\partial f}{\partial x^j}.$$

Here $g_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$. From this we get finally

$$(\operatorname{grad} f)^i = g^{ij} \frac{\partial f}{\partial x^j}$$

where g^{ij} with upper indices denote the elements of the inverse matrix for (g_{ij}).

Since for both polar coordinates in $\mathbb{R}^2$ and spherical coordinates in $\mathbb{R}^3$ the matrix (g_{ij}) is diagonal (see above for the case of spherical coordinates), multiplying by the inverse matrix reduces to the division by the diagonal entries of (g_{ij}). We get

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \boldsymbol{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \varphi} \boldsymbol{e}_\varphi$$

for polars in $\mathbb{R}^2$ and

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \boldsymbol{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \boldsymbol{e}_\theta + \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \varphi} \boldsymbol{e}_\varphi$$

for spherical coordinates in $\mathbb{R}^3$.

Problem 5.
(a) In Cartesian coordinates $\boldsymbol{r} = \sum x^i \boldsymbol{e}_i$ and $r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$, hence

$$\frac{\partial r}{\partial x^i} = \frac{2x^i}{2\sqrt{(x^1)^2 + \ldots + (x^n)^2}} = \frac{x^i}{r}.$$

It follows that

$$\operatorname{grad} r = \sum \frac{1}{r}\, x^i \boldsymbol{e}_i = \frac{\boldsymbol{r}}{r},$$

i.e., the unit vector in the direction of $\boldsymbol{r}$.

(b) In polar coordinates in $\mathbb{R}^2$, using the result of Problem 4, we have

$$\operatorname{grad} r = \frac{\partial r}{\partial r}\, \boldsymbol{e}_r + \frac{\partial r}{\partial \varphi}\, \boldsymbol{e}_\varphi = 1\boldsymbol{e}_r + 0\boldsymbol{e}_\varphi = \boldsymbol{e}_r.$$

(c) Similarly, in spherical coordinates in $\mathbb{R}^3$, we have

$$\operatorname{grad} r = \frac{\partial r}{\partial r}\, \boldsymbol{e}_r + 0 = \boldsymbol{e}_r.$$

The answers in parts (b) and (c) of course agree with the answer in part (a), since the basis vector $\boldsymbol{e}_r$ is exactly the unit vector in the direction of the radius-vector.

Problem 6.

First method. In Cartesian coordinates in $\mathbb{R}^n$ for $\boldsymbol{X} = X^i \boldsymbol{e}_i$

$$\operatorname{div} \boldsymbol{X} = \frac{\partial X^i}{\partial x^i}\,.$$

Hence for our vector field $\boldsymbol{X} = f(r)\boldsymbol{r} = f(r)x^i \boldsymbol{e}_i$ where $r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$ we have

$$\operatorname{div} \boldsymbol{X} = \frac{\partial (f x^i)}{\partial x^i} = \left(f'(r) \frac{\partial r}{\partial x^i}\, x^i + f(r)\, \frac{\partial x^i}{\partial x^i} \right) =$$
$$\left(f'(r) \frac{x^i}{r}\, x^i + f(r) \right) = f'(r) \frac{r^2}{r} + n f(r) = f'(r) r + n f(r).$$

Second method. Recall that in arbitrary coordinates in $\mathbb{R}^n$ for $\boldsymbol{X} = X^i \boldsymbol{e}_i$

$$\operatorname{div} \boldsymbol{X} = \frac{1}{\sqrt{g}} \frac{\partial (X^i \sqrt{g})}{\partial x^i}\,.$$

Here $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial x^i}$ is the basis associated with a given coordinate system and $g = \det(g_{ij})$, where $g_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$. We can apply this formula using our knowledge of polar coordinates in $\mathbb{R}^2$ and spherical coordinates in $\mathbb{R}^3$ (when $n = 2$ and $n = 3$). Our vector field equals $\boldsymbol{X} = f(r)\boldsymbol{r} = f(r) r \boldsymbol{e}_r$. In $\mathbb{R}^2$ in polar coordinates we have $\sqrt{g} = r$ and

$$\operatorname{div} \boldsymbol{X} = \frac{1}{r} \frac{\partial (f(r) r^2)}{\partial r} = \frac{f'(r) r^2 + 2 r f(r)}{r} = f'(r) r + 2 f(r);$$

in $\mathbb{R}^3$ in spherical coordinates we have $\sqrt{g} = r^2 \sin\theta$ and

$$\operatorname{div} \boldsymbol{X} = \frac{1}{r^2 \sin\theta} \frac{\partial(f(r)r^3 \sin\theta)}{\partial r} = \frac{f'(r)r^3 + 3r^2 f(r)}{r^2} = f'(r)r + 3f(r).$$

This agrees, of course, with the formula for any n obtained above using Cartesian coordinates.

Remark. Spherical coordinates $r, \theta_1, \ldots, \theta_{n-1}$ can be defined in $\mathbb{R}^n$ for arbitrary n, generalizing from $\mathbb{R}^2$ and $\mathbb{R}^3$. For them $\sqrt{g} = r^{n-1}\Omega(\theta_1, \ldots, \theta_{n-1})$, where $\Omega(\theta_1, \ldots, \theta_{n-1})$ is a certain function depending only on the angular coordinates $\theta_1, \ldots, \theta_{n-1}$ (for $n = 2$, $\Omega(\varphi) = 1$; for $n = 3$, $\Omega(\theta, \varphi) = \sin\theta$). This "angular factor" is irrelevant for calculating $\operatorname{div}(f(r)r\boldsymbol{e}_r)$, — we get

$$\operatorname{div} \boldsymbol{X} = \frac{1}{r^{n-1}\Omega(\theta_1, \ldots, \theta_{n-1})} \frac{\partial(f(r)r^n\Omega(\theta_1, \ldots, \theta_{n-1})}{\partial r} = \frac{1}{r^{n-1}} \frac{\partial(f(r)r^n)}{\partial r} = \frac{f'(r)r^n + nr^{n-1}f(r)}{r^{n-1}} = f'(r)r + nf(r).$$

Problem 7. If $\boldsymbol{X} = r^\alpha \boldsymbol{r}$ in $\mathbb{R}^n$, we can apply the result of the previous problem. We get

$$\operatorname{div}(r^\alpha \boldsymbol{r}) = \alpha r^{\alpha-1} r + n r^\alpha = (\alpha + n) r^\alpha.$$

It follows that $\operatorname{div}(r^\alpha \boldsymbol{r}) = 0$ in $\mathbb{R}^n$ for $\alpha = -n$. For example, we get the following vector fields with zero divergence: in $\mathbb{R}^2$, the field $\boldsymbol{X} = \frac{\boldsymbol{r}}{r^2}$; in $\mathbb{R}^3$, the field $\boldsymbol{X} = \frac{\boldsymbol{r}}{r^3}$. *Remark.* Notice that these fields are defined only for $\boldsymbol{r} \neq 0$. The equality $\operatorname{div} \boldsymbol{X} = 0$ is valid only for $\boldsymbol{r} \neq 0$.

Problem 8. When we calculate curl in $\mathbb{R}^3$, we use the following formula:

$$\operatorname{curl} \boldsymbol{X} = \boldsymbol{\nabla} \times \boldsymbol{X} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ X_1 & X_2 & X_3 \end{vmatrix}$$

with the right-hand side valid in Cartesian coordinates. Here $\partial_i = \frac{\partial}{\partial x^i}$,

$$\boldsymbol{X} = X_1 \boldsymbol{e}_1 + X_2 \boldsymbol{e}_2 + X_3 \boldsymbol{e}_3,$$

and $\boldsymbol{\nabla}$ denotes the symbolic vector field $\boldsymbol{\nabla} = \sum \boldsymbol{e}_i \partial_i$ (formula in Cartesian coordinates), the components of which are partial derivatives.

(a) We have

$$\operatorname{div} \boldsymbol{X} = \frac{\partial(x - y + 3z)}{\partial x} - \frac{\partial(2x + z)}{\partial y} + \frac{\partial(-x + y + z)}{\partial z} = 1 + 1 = 2;$$

$$\operatorname{curl} \boldsymbol{X} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_x & \partial_y & \partial_z \\ x - y + 3z & -(2x+z) & -x+y+z \end{vmatrix} =$$
$$\boldsymbol{e}_1(1+1) - \boldsymbol{e}_2(-1-3) + \boldsymbol{e}_3(-2+1) = 2\boldsymbol{e}_1 + 4\boldsymbol{e}_2 - \boldsymbol{e}_3.$$

(b)

$$\operatorname{div} \boldsymbol{X} = \frac{\partial(a_{11}x + a_{12}y + a_{13}z)}{\partial x} + \frac{\partial(a_{12}x + a_{22}y + a_{23}z)}{\partial y} + \frac{\partial(a_{13}x + a_{23}y + a_{33}z)}{\partial z} = a_{11} + a_{22} + a_{33};$$

$$\operatorname{curl} \boldsymbol{X} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_x & \partial_y & \partial_z \\ a_{11}x + a_{12}y + a_{13}z & a_{12}x + a_{22}y + a_{23}z & a_{13}x + a_{23}y + a_{33}z \end{vmatrix} =$$
$$\boldsymbol{e}_1(a_{23} - a_{23}) - \boldsymbol{e}_2(a_{13} - a_{13}) + \boldsymbol{e}_3(a_{12} - a_{12}) = 0.$$

(c)

$$\operatorname{div} \boldsymbol{X} = \frac{\partial(a_{12}y + a_{13}z)}{\partial x} + \frac{\partial(-a_{12}x + a_{23}z)}{\partial y} + \frac{\partial(-a_{13}x - a_{23}y)}{\partial z} = 0;$$

$$\operatorname{curl} \boldsymbol{X} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_x & \partial_y & \partial_z \\ a_{12}y + a_{13}z & -a_{12}x + a_{23}z & -a_{13}x - a_{23}y \end{vmatrix} =$$
$$\boldsymbol{e}_1(-a_{23} - a_{23}) - \boldsymbol{e}_2(-a_{13} - a_{13}) + \boldsymbol{e}_3(-a_{12} - a_{12}) =$$
$$-2\left(a_{23}\boldsymbol{e}_1 - a_{13}\boldsymbol{e}_2 + a_{12}\boldsymbol{e}_3\right).$$

Remark. Notice that in parts (b) and (c) the components (X_1, X_2, X_3) of the vector field $\boldsymbol{X} = X_1\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3$ can be written as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with a symmetric matrix A, or as

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with an antisymmetric matrix A, respectively. The divergence of $\boldsymbol{X}$ equals (in both cases) the trace of the matrix A, which is zero in the antisymmetric case, and the curl of $\boldsymbol{X}$ for an antisymmetric matrix A is equal up to a constant factor to the vector $\boldsymbol{\omega} \in \mathbb{R}^3$ such that $\boldsymbol{\omega} \times \boldsymbol{v} = A\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{R}^3$ (compare with Problem 9, part (b)).

Problem 9.
(a)
First method. One can see that the following identity holds:

$$\operatorname{curl}(f\boldsymbol{X}) = (\operatorname{grad} f) \times \boldsymbol{X} + f \operatorname{curl} \boldsymbol{X}$$

for arbitrary function f and vector field $\boldsymbol{X}$ in $\mathbb{R}^3$ (it immediately follows from the symbolic determinant formula for calculating curl in Cartesian coordinates). Applying it we get $\operatorname{curl}(f(r)\boldsymbol{r}) = (\operatorname{grad} f(r)) \times \boldsymbol{r} + f(r) \operatorname{curl} \boldsymbol{r}$. Notice now that $\operatorname{grad} f(r) = f'(r) \operatorname{grad} r = f'(r)\, \boldsymbol{r}/r$ (see Problem 5) and that $\operatorname{curl} \boldsymbol{r} = 0$ (check directly!). Hence $\operatorname{curl}(f(r)\boldsymbol{r}) = f'(r)r^{-1}\boldsymbol{r} \times \boldsymbol{r} + 0 = 0 + 0 = 0$.
Second method. We can use a version of the symbolic determinant formula valid for arbitrary coordinates: if $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial x^i}$ for arbitrary coordinates x^1, x^2, x^3 in an open domain in $\mathbb{R}^3$ and $\boldsymbol{X} = X^i \boldsymbol{e}_i$, then

$$\operatorname{curl} \boldsymbol{X} = \frac{1}{\sqrt{g}} \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ X_1 & X_2 & X_3 \end{vmatrix}$$

where $X_i = g_{ij}X^j = \boldsymbol{X} \cdot \boldsymbol{e}_i$. As above, $\partial_i = \frac{\partial}{\partial x^i}$ and $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, $g = \det(g_{ij})$. In particular, in spherical coordinates r, θ, φ we have

$$\operatorname{curl} \boldsymbol{X} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ X_r & X_\theta & X_\varphi \end{vmatrix}$$

where $X_r = \boldsymbol{X} \cdot \boldsymbol{e}_r$, etc. Hence for $\boldsymbol{X} = f(r)\boldsymbol{r} = f(r)r\boldsymbol{e}_r$ we immediately get

$$\operatorname{curl}(f(r)\boldsymbol{r}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ f(r)r & 0 & 0 \end{vmatrix} = 0$$

(recall that $\boldsymbol{e}_r \cdot \boldsymbol{e}_r = 1$, $\boldsymbol{e}_r \cdot \boldsymbol{e}_\theta = 0$ and $\boldsymbol{e}_r \cdot \boldsymbol{e}_\varphi = 0$).
(b) Denote $\boldsymbol{Y} = \Omega \times \boldsymbol{r}$; we have

$$\boldsymbol{Y} = \boldsymbol{e}_1(\omega_2 z - \omega_3 y) - \boldsymbol{e}_2(\omega_1 z - \omega_3 x) + \boldsymbol{e}_3(\omega_1 y - \omega_2 x),$$

where $\Omega = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$. Hence, in Cartesian coordinates

$$\operatorname{div} \boldsymbol{Y} = \frac{\partial(\omega_2 z - \omega_3 y)}{\partial x} + \frac{\partial(-\omega_1 z + \omega_3 x)}{\partial y} + \frac{\partial(\omega_1 y - \omega_2 x)}{\partial z} = 0$$

and

$$\operatorname{curl} \boldsymbol{Y} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_x & \partial_y & \partial_z \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix} =$$
$$\boldsymbol{e}_1(\omega_1 + \omega_1) + \boldsymbol{e}_2(\omega_2 + \omega_2) + \boldsymbol{e}_3(\omega_3 + \omega_3) = 2\Omega\,.$$

Problem 10. The simplest way is to use the formulae in Cartesian coordinates in $\mathbb{R}^n$:

$$\operatorname{div} \boldsymbol{X} = \frac{\partial X^i}{\partial x^i} \quad \text{and} \quad \operatorname{grad} f = \sum \frac{\partial f}{\partial x^i}\, \boldsymbol{e}_i\,.$$

We have

$$\operatorname{div}(f\boldsymbol{X}) = \frac{\partial (fX^i)}{\partial x^i} = \frac{\partial f}{\partial x^i}\, X^i + f\, \frac{\partial X^i}{\partial x^i} = \operatorname{grad} f \cdot \boldsymbol{X} + f \operatorname{div} \boldsymbol{X}$$

as claimed.

Problem 11. Compare with Problem **??** (the statements are almost identical). We either apply d to the form $\omega = \frac{1}{2}\, r^2\, d\varphi$ and get

$$d\omega = \frac{1}{2}\, d(r^2\, d\varphi) = \frac{1}{2}\, 2r dr \wedge d\varphi = r dr \wedge d\varphi = dS,$$

hence by the Stokes formula

$$\oint_{\partial D} \omega = \int_D d\omega = \int_D dS = \operatorname{area}(D)$$

(here we used the formula for the area element in polar coordinates $dS = r dr \wedge d\varphi$), or we can first rewrite ω in Cartesian coordinates: $\omega = \frac{1}{2}\,(x dy - y dx)$ and then $d\omega = dx \wedge dy = dS$, the rest being the same. (Notice that dS, in contrast with $d\omega$, does not have the meaning of 'd' of some 'S'; rather this traditional notation has the symbolic meaning of an 'infinitesimal element' of area, which is traditionally denoted by the letter S.)

Problem 12.
(a) Denote the sphere by S_R and the ball by B_R. Notice that $S_R = \partial B_R$. By the Stokes theorem we have

$$\oint_{S_R} \omega = \int_{B_R} d\omega = \int_{B_R} d(ax\, dy \wedge dz + by\, dz \wedge dx + cz\, dx \wedge dy) =$$
$$\int_{B_R} (a{+}b{+}c) dx{\wedge}dy{\wedge}dz = (a{+}b{+}c) \int_{B_R} dV = (a{+}b{+}c)\operatorname{vol}(B_R) = \frac{4}{3}\pi R^3 (a{+}b{+}c).$$

(b) Denote the cube by C; notice that $\operatorname{vol} C = 1$. By the Stokes formula we get

$$\oint_{\partial C} \omega = \int_C d\omega = \int_C (a{+}b{+}c) dx{\wedge}dy{\wedge}dz = (a{+}b{+}c)\operatorname{vol}(C) = a{+}b{+}c.$$