§5 Stokes theorem

Problem 1.

$$\frac{1}{2}\int_{\partial D}(x\,dy - y\,dx) = \frac{1}{2}\int_D d(x\,dy - y\,dx) = \frac{1}{2}\int_D (dx\wedge dy - dy\wedge dx) =$$
$$\frac{1}{2}\int_D 2dx\wedge dy = \int_D dx\wedge dy = \mathrm{area}(D)$$

Problem 3. Denote the sphere by S_R. Consider the ball B_R of radius R such that the sphere S_R is its boundary. Then by Stokes's theorem

$$\int_{S_R}\omega = \int_{\partial B_R}\omega = \int_{B_R} d\omega.$$

We need to calculate $d\omega$. We have

$$d\omega = d(r^\alpha)\wedge(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) + r^\alpha d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy);$$

it is possible to use the answer obtained earlier (see coursework):

$$d\omega = (\alpha+3)r^\alpha\,dx\wedge dy\wedge dz.$$

Thus (using spherical coordinates and recalling that $dx\wedge dy\wedge dz = r^2\sin\theta\,dr\wedge d\theta\wedge d\varphi$)

$$\int_{B_R} d\omega = \int_{B_R}(\alpha+3)r^\alpha\,dx\wedge dy\wedge dz = \int_{B_R}(\alpha+3)r^\alpha\,r^2\sin\theta\,dr\wedge d\theta\wedge d\varphi =$$
$$\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\varphi\int_0^R(\alpha+3)r^{\alpha+2}\,dr = (-\cos\theta)|_0^\pi\,2\pi\,r^{\alpha+3}|_0^R = 4\pi R^{\alpha+3}.$$

Notice that we used the condition $\alpha\geqslant 0$ to be able to apply the Stokes theorem to the form ω: we have to be sure that ω is defined everywhere in B_R, including the origin. (If α is negative, ω is not defined at $r=0$.)
Another method. Notice that

$$\int_{S_R}\omega = \int_{S_R}\omega_0$$

where $\omega_0 = R^\alpha(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$, since on the surface of the sphere ω equals ω_0. Applying the Stokes theorem we get

$$\int_{S_R}\omega = \int_{S_R}\omega_0 = \int_{B_R}d\omega_0 = R^\alpha\int_{B_R}d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) =$$
$$R^\alpha\int_{B_R}3\,dx\wedge dy\wedge dz = 3R^\alpha\,\mathrm{vol}(B_R) = 4\pi R^{\alpha+3},$$

where we used the formula $\mathrm{vol}(B_R) = \frac{4}{3}\pi R^3$ for the volume of a ball of radius R. The trick is that it is easier to calculate $d\omega_0$ than $d\omega$ and the integral over B_R is reduced to a familiar formula for the volume. (The formula $\mathrm{vol}(B_R) = \frac{4}{3}\pi R^3$ can be obtained, e.g., by using spherical coordinates.) An extra advantage is that now we can drop the restriction $\alpha \geqslant 0$, since the Stokes formula is applied not to ω, but to ω_0, which is defined everywhere regardless of α. The answer is valid for any $\alpha \in \mathbb{R}$.
Solution without Stokes's theorem. Introduce parametrization by x, y so that $z = \pm\sqrt{R^2 - x^2 - y^2}$ (plus sign in the upper hemisphere and minus, in the lower). Then

$$dz = \pm\frac{-xdx - ydy}{\sqrt{R^2 - x^2 - y^2}} = -\frac{xdx + ydy}{z}.$$

From here it follows that the restriction of ω on the sphere will be

$$R^\alpha\Big(-\frac{x^2 dy\wedge dx}{z} + \frac{y^2 dx\wedge dy}{z} + zdx\wedge dy\Big) = \\ R^\alpha\,\frac{(x^2+y^2+z^2)dx\wedge dy}{z} = \frac{R^{\alpha+2}\,dx\wedge dy}{z} = \pm\frac{R^{\alpha+2}\,dx\wedge dy}{\sqrt{R^2-x^2-y^2}}\,.$$

Here $x^2 + y^2 \leqslant R^2$, and plus sign should be taken for the upper hemisphere, while minus, for the lower. Now, the orientation induced by x, y in the upper hemisphere coincides with the orientation given by the outward normal, and the lower, is the opposite. Hence the integral for the lower hemisphere should be taken with the negative sign. It cancels the negative sign in the above formula. Therefore

$$\int_{S^2_R}\omega = 2\int\limits_{x^2+y^2\leqslant R^2}\frac{R^{\alpha+2}\,dx\wedge dy}{\sqrt{R^2-x^2-y^2}}\,.$$

We use substitution: $x = R\rho\cos\varphi$, $y = R\rho\sin\varphi$, where $0 \leqslant \rho \leqslant 1$, $0 \leqslant \varphi \leqslant 2\pi$. We have $dx\wedge dy = R^2\rho d\rho\wedge d\varphi$, and the denominator will be $R\sqrt{1-\rho^2}$. We have

$$\int_{S^2_R}\omega = 2\int_0^1\int_0^{2\pi}\frac{R^{4+\alpha}\rho d\rho\wedge d\varphi}{R\sqrt{1-\rho^2}} = 2R^{\alpha+3}\int_0^1\frac{\rho d\rho}{\sqrt{1-\rho^2}}\int_0^{2\pi}d\varphi = \\ 2\pi R^{\alpha+3}\int_0^1\frac{d(\rho^2)}{\sqrt{1-\rho^2}} = 2\pi R^{\alpha+3}\int_0^1\frac{u}{\sqrt{u}} = 4\pi R^{\alpha+3}\,.$$

Problem 6.

$$\partial\partial(ABCD) = \partial\left([AB] + [BC] + [CD] + [DA]\right) = \\ \partial[AB] + \partial[BC] + \partial[CD] + \partial[DA] = \\ B - A + C - B + D - C + D - A = 0$$