§4 Integration of forms

Problem 1. **(a)** $\int_\gamma \omega = \int_{-2}^{2} \gamma^*\omega = \int_{-2}^{2}(2d(t^2)+d(t^3)) = 2t^2 \,|_{t=-2}^{t=2} + t^3 \,|_{t=-2}^{t=2} = 0 + 8 - (-8) = 16$

Problem 5. **(a)**

$$\oint_C A = \int_{KL} A + \int_{LM} A + \int_{MN} A + \int_{NK} A =$$

$$\int_{KL} (ax+by)\,dx + \int_{LM} (px+qy)\,dy + \int_{MN} (ax+by)\,dx + \int_{NK} (px+qy)\,dy =$$

$$\int_0^1 ax\,dx + \int_0^1 (p+qy)\,dy + \int_1^0 (ax+b)\,dx + \int_1^0 qy\,dy =$$

$$\frac{a}{2} + p + \frac{q}{2} - \frac{a}{2} - b - \frac{q}{2} = p - b.$$

(b) It follows that $\oint_C A = 0$ if $p = b$.
(c) $dA = d\left((ax+by)\,dx + (px+qy)\,dy\right) = b\,dy \wedge dx + p\,dx \wedge dy = (-b + p)\,dx \wedge dy$.
(d) $p = b$, the same as in part **(b)**.
(e) Setting $p = b$, we have:

$$\int_{C_1} A = \int_{KL} A + \int_{LM} A = \frac{a}{2} + b + \frac{q}{2}\,;$$

$$\int_{C_2} A = \int_{KN} A + \int_{NM} A = \frac{q}{2} + b + \frac{a}{2}$$

(we used the calculations from part **(a)**);

$$\int_{C_3} A = \int_{KM} A = \int_0^1 ((at+bt)\,dt + (bt+qt)\,dt) = \int_0^1 (a+2b+q)t\,dt = \frac{a}{2} + b + \frac{q}{2}$$

(we used the parametrization $x = y = t$);

$$\int_{C_4} A = \int_0^1 ((at+bt^2)\,dt + 2t(bt+qt^2)\,dt) = \int_0^1 (at+3bt^2+2qt^3)\,dt = \frac{a}{2} + b + \frac{q}{2}$$

(we used the parametrization $x = t, y = t^2$). As expected, all these integrals coincide.

Problem 9.
(a)

$$\int_\Gamma \omega = \int_D \Gamma^*\omega = \int_D (u + u^2 - v^2)\,du \wedge dv = \int_{-1}^{1}\int_{-1}^{1} (u + u^2 - v^2)\,du \wedge dv =$$

$$2\int_{-1}^{1} (u + u^2)\,du - 2\int_{-1}^{1} v^2\,dv = \frac{4}{3} - \frac{4}{3} = 0.$$

(b)

$$\int_\Gamma \omega = \int_D \Gamma^*\omega = \int_D (u+3uv)\,du\wedge dv = \int_{-1}^{1}\int_{-1}^{1}(u+3uv)\,du\wedge dv =$$
$$\int_{-1}^{1}\left(\int_{-1}^{1}(1+3v)u\,du\right)dv = 0$$

Problem 10. "Orientation by a normal" means that the cross-product of basis vectors defining a chosen orientation in a tangent plane gives this normal.

Considering the outward normal to the sphere we see that, e.g., at the point $(0,0,R)$ the normal points in the direction of $\boldsymbol{e}_3 = (0,0,1)$; hence at $(0,0,R)$ as such a basis in the tangent plane defining the orientation we can take $\boldsymbol{e}_1 = (1,0,0), \boldsymbol{e}_2 = (0,1,0)$. Hence, if we use x, y as parameters to calculate the integral and express z as $z = \sqrt{R^2 - x^2 - y^2}$ for the upper hemisphere, we do not have to insert an extra sign. For the lower hemisphere, where $z = -\sqrt{R^2 - x^2 - y^2}$, we have to take the integral over x, y with the minus sign, since there the orientation given by the outward normal and that defined by x, y are the opposite (e.g., at $(0,0,-R)$ the normal points in the direction of $-\boldsymbol{e}_3 = -\boldsymbol{e}_1 \times \boldsymbol{e}_2$).

The integral can be calculated using the parametrization by x, y or by spherical coordinates θ, φ. In the latter case, as a "test point" to compare orientations it is convenient to take $(R,0,0)$. At this point the outward normal points in the direction of $\boldsymbol{e}_1 = (1,0,0)$. At the same point the vectors $\boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$ corresponding to θ, φ coincide with $-\boldsymbol{e}_3, \boldsymbol{e}_2$, respectively, and their cross-product gives exactly $\boldsymbol{e}_1$. Therefore, the orientation by θ, φ coincides with the orientation by the outward normal. Hence the integral over θ, φ should be taken with the plus sign. (The advantage of using θ, φ is that they work for almost all points of the sphere and we do not have to split the integral over the sphere into the sum of two, as we do if we use x, y.)

The answer to the second question is: the integral changes sign if we take the inward normal instead.

We skip either of the direct calculations of the integral, because the shortest way is to apply the Stokes theorem. "Orientation by outward normal" in the above sense exactly coincides with the one induced on the sphere from the standard orientation of the ball in $\mathbb{R}^3$ if we consider the sphere as the boundary of the ball. Hence

$$\frac{1}{3}\int_{S^2_R}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) = \frac{1}{3}\int_{B_R} d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) =$$
$$\frac{1}{3}\int_{B_R} 3\,dx\wedge dy\wedge dz = \int_{B_R} dx\wedge dy\wedge dz = \mathrm{vol}(B_R) = \frac{4}{3}\pi R^3$$

(we used the formula for the volume of a ball; see also §5, Problem 3).