

### §3 Forms on $\mathbb{R}^n$

#### Problem 1.

(a)

$$a^i e_i = a^1 e_1 + a^2 e_2 + a^3 e_3 = \begin{pmatrix} a^1 & a^2 & a^3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

(b)

$$\begin{aligned} e_i F_j^i a^j &= e_1 F_1^1 a^1 + e_1 F_2^1 a^2 + e_2 F_1^2 a^1 + e_2 F_2^2 a^2 = \\ &= \begin{pmatrix} a^1 & a^2 \end{pmatrix} \begin{pmatrix} F_1^1 & F_1^2 \\ F_2^1 & F_2^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} F_1^1 & F_2^1 \\ F_1^2 & F_2^2 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \end{aligned}$$

#### Problem 6.

$$\begin{aligned} d\sigma &= d \sum_{i=1}^n (-1)^{i+1} x^i dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n = \\ &= \sum_{i=1}^n (-1)^{i+1} dx^i \wedge dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n = \\ &= \sum_{i=1}^n (-1)^{i+1} (-1)^{i-1} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \dots \wedge dx^n = \sum_{i=1}^n dx^1 \wedge \dots \wedge dx^n = \\ &= n dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

(You can consider the cases  $n = 2$ ,  $n = 3$ ,  $n = 4$ ,  $\dots$ , to see what happens in general.)

#### Problem 7. First notice that in $\mathbb{R}^n$ ,

$$dr = d \left( \sum (x^i)^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left( \sum (x^i)^2 \right)^{\frac{1}{2}-1} (2x^1 dx^1 + \dots + 2x^n dx^n) = r^{-1} \sum x^i dx^i.$$

Now,

$$\begin{aligned} d\omega &= d \left( r^\alpha \sum (-1)^{i+1} x^i dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \right) = \\ &= \alpha r^{\alpha-1} r^{-1} (x^1 dx^1 + \dots + x^n dx^n) \wedge \sum (-1)^{i+1} x^i dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n + \\ &\quad r^\alpha n dx^1 \wedge \dots \wedge dx^n = \\ &= \alpha r^{\alpha-2} \left( \sum (x^i)^2 \right) dx^1 \wedge \dots \wedge dx^n + r^\alpha n dx^1 \wedge \dots \wedge dx^n = (\alpha + n) r^\alpha dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

(notice that when you multiply, say,  $dx^1$  with the product of the form  $dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n$  where  $i = 1, 2, \dots$ , the result is nonzero only if  $i = 1$ , i.e., the second factor is  $dx^2 \wedge dx^3 \wedge \dots \wedge dx^n$ ; the result will be  $dx^1 \wedge \dots \wedge dx^n$ ; in the same way, when you multiply  $dx^2$  with  $dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n$ , the result is nonzero only if  $i = 2$ , i.e., the second factor is  $dx^1 \wedge dx^3 \wedge \dots \wedge dx^n$ , and the result equals  $-dx^1 \wedge \dots \wedge dx^n$ , etc.). The differential vanishes for  $\alpha = -n$ .